Arch. Rational Mech. Anal.
Digital Object Identifier (DOI) https://doi.org/10.1007/s00205-020-01521-7

Global Small Solutions of Heat Conductive Compressible Navier–Stokes Equations with Vacuum: Smallness on Scaling Invariant Quantity

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Communicated by P.-L. Lions

Abstract

In this paper, we consider the Cauchy problem to the heat conductive compressible Navier–Stokes equations in the presence a of vacuum and with a vacuum far field. Global well-posedness of strong solutions is established under the assumption, among some other regularity and compatibility conditions: that the scaling invariant quantity $\|\rho_0\|_\infty(\|\rho_0\|_3 + \|\rho_0\|_\infty^2\|\sqrt{\rho_0}u_0\|_2^2)(\|\nabla u_0\|_2^2 + \|\rho_0\|_\infty\|\sqrt{\rho_0}E_0\|_2^2)$ is sufficiently small, with the smallness depending only on the parameters R, γ, μ, λ, and κ in the system. Notably, the smallness assumption is imposed on the above scaling invariant quantity exclusively, and it is independent of any norms of the initial data, which is different from the existing papers. The total mass can be either finite or infinite. An equation for the density-more precisely for its cubic, derived from combining the continuity and momentum equations-is employed to get the $L_t^\infty(L^3)$ type estimate of the density.

1. Introduction

In this paper, we consider the following heat conductive compressible Navier–Stokes equations for the ideal gas:

$$\partial_t\rho + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$\rho(\partial_t u + (u\cdot\nabla)u) - \mu\Delta u - (\mu+\lambda)\nabla\operatorname{div} u + \nabla p = 0, \tag{1.2}$$

$$c_v\rho(\partial_t\theta + u\cdot\nabla\theta) + p\operatorname{div} u - \kappa\Delta\theta = \mathcal{Q}(\nabla u), \tag{1.3}$$

in $\mathbb{R}^3\times(0,\infty)$, where the unknowns $\rho \geqq 0$, $u \in \mathbb{R}^3$, and $\theta \geqq 0$, respectively, represent the density, velocity, and absolute temperature. Here $p = R\rho\theta$, with positive constant R, is the pressure, $c_v > 0$ is a constant, constants μ and λ are

Published online: 18 April 2020

the bulk and shear viscous coefficients, respectively, positive constant κ is the heat conductive coefficient, and

$$\mathcal{Q}(\nabla u) = \frac{\mu}{2}|\nabla u + (\nabla u)^t|^2 + \lambda(\operatorname{div} u)^2,$$

$(\nabla u)^t$ is the transpose of ∇u. The viscous coefficients μ and λ satisfy the physical constraints

$$\mu > 0, \quad 2\mu + 3\lambda \geqq 0.$$

The additional assumption $2\mu > \lambda$ will also be used in this paper.

Due to their fundamental importance in fluid dynamics, extensive studies have been carried out and many developments have been achieved on the compressible Navier–Stokes equations over the last seventy years. Mathematical studies on the compressible Navier–Stokes equations started with the uniqueness results by Graffi [15] in 1953 for barotropic fluids and by Serrin [43] in 1959 for general fluids, and include the local existence result by Nash [41] in 1962 for the Cauchy problem. Since that time, comprehensive mathematical theories have been established for the compressible Navier–Stokes equations.

The mathematical theory for the compressible Navier–Stokes equations in 1D is satisfactory and, in particular, the corresponding global well-posedness for arbitrary large initial data and the initial density can either be uniformly positive or only nonnegative (that is, it can vanish on some subset of the domain). For the case in which the initial density is uniformly positive, the global well-posedness of strong solutions with large initial data was first proved in [23] for the isentropic case, and later in [25] for the general case, and the asymptotic behavior of the solutions was recently proved in [29], (see also [2,22,24,49,50]) for some related results. For the case in which the initial density contains a vacuum, the corresponding global well-posedness of strong solutions was recently proved by the author and his collaborator; see [27,28,31,32].

Compared with the one dimensional case, the mathematical theory for the multi-dimensional case is far from satisfactory and, in particular, some basic problems such as the global existence of strong solutions and the uniqueness of the weak solution are still unknown. In the case that the initial density is uniformly positive, the local well-posedness was proved a long time ago, (see [20,36,41,44,45,47] and, in particular, that the inflow and outflow were allowed in [36]), however, the general global well-posedness is still unknown. Global well-posedness of strong solutions with small initial data was first proved in [37–40], and later further developed in many papers; see, e.g., [3,4,8–11,16,26,42,46]. In the case that the initial density allows a vacuum, the global existence of weak solutions was first proved in [34,35] (see [1,12–14,21] for further developments), but the uniqueness is still an open problem. The local well-posedness of strong solutions was proved in [5–7], and the global well-posedness with small initial data but allowing large oscillations, was proved in [19] (see [18,30,48] for further developments).

The aim of this paper is to establish the global existence of strong solutions to the Cauchy problem of (1.1)–(1.3), under some smallness assumptions on the initial data, in the presence of initial vacuum, and with a vacuum far field. The main novelty of this paper is that the smallness assumption is imposed exclusively on some quantities that are scaling invariant with respect to the following scaling transform:

$$(\rho_{0\lambda}(x), u_{0\lambda}(x), \theta_{0\lambda}(x)) = (\rho_0(\lambda x), \lambda u_0(\lambda x), \lambda^2\theta_0(\lambda x)), \quad \forall \lambda \neq 0. \tag{1.4}$$

This scaling transform on the initial data inheres in the following natural scaling invariant property of system (1.1)–(1.3):

$$\rho_\lambda(x,t) = \rho(\lambda x, \lambda^2 t), \quad u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \quad \theta_\lambda(x,t) = \lambda^2\theta(\lambda x, \lambda^2 t);$$

that is, if (ρ, u, θ) is a solution with initial data (ρ_0, u_0, θ_0), then $(\rho_\lambda, u_\lambda, \theta_\lambda)$ is also a solution for any nonzero λ, but with initial data $(\rho_{0\lambda}, u_{0\lambda}, \theta_{0\lambda})$.

The reason for us to impose the smallness assumptions on scaling invariant quantities, rather than on those that have not, is the following fact: if assuming that $\mathscr{M}$ is a functional such that

$$\mathscr{M}(\rho_{0\lambda}, u_{0\lambda}, \theta_{0\lambda}) = \lambda^\ell(\rho_0, u_0, \theta_0), \quad \forall \lambda \neq 0, \quad \text{for some constant } \ell \neq 0,$$

and that the global well-posedness holds, for any initial data (ρ_0, u_0, θ_0) satisfying $\mathscr{M}(\rho_0, u_0, \theta_0) \leqq \varepsilon_0$, and for some $\varepsilon_0 > 0$ depending only on the parameters of the system, then, by suitably choosing the scaling parameter λ, one can show that the system is actually globally well-posed for arbitrary large initial data. This global well-posedness for arbitrary large initial data is, however, far from what we have already known.

Before stating the main results, we first clarify some necessary notations used throughout this paper. For $1 \leqq q \leqq \infty$ and positive integer m, we use $L^q = L^q(\mathbb{R}^3)$ and $W^{m,q} = W^{m,q}(\mathbb{R}^3)$ to denote the standard Lebesgue and Sobolev spaces, respectively, and in the case that $q = 2$, we use H^m to replace $W^{m,2}$. For simplicity, we also use notations L^q and H^m to denote the N product spaces $(L^q)^N$ and $(H^m)^N$, respectively. We always use $\|u\|_q$ to denote the L^q norm of u. For shortening the expressions, we sometimes use $\|(f_1, f_2, \ldots, f_n)\|_X$ to denote the norm $\sum_{i=1}^N \|f_i\|_X$ or its equivalent, $\left(\sum_{i=1}^N \|f_i\|_X^2\right)^{\frac{1}{2}}$. We denote

$$D^{k,r} = \left\{u \in L^1_{loc}(\mathbb{R}^3) \,\middle|\, \|\nabla^k u\|_r < \infty\right\}, \quad D^k = D^{k,2},$$
$$D^1_0 = \left\{u \in L^6(\mathbb{R}^3) \,\middle|\, \|\nabla u\|_2 < \infty\right\}.$$

For simplicity of notations, we adopt

$$\int f\,\mathrm{d}x = \int_{\mathbb{R}^3} f\,\mathrm{d}x.$$

Definition 1.1. Let T be a positive time and assume that

$$\rho_0, \theta_0 \geqq 0, \quad \rho_0 \in H^1 \cap W^{1,q}, \quad (u_0, \theta_0) \in D^1_0 \cap D^2.$$

A triple (ρ, u, θ) is called a strong solution to system (1.1)–(1.3), on $\mathbb{R}^3 \times (0, T)$, with initial data (ρ_0, u_0, θ_0), if it enjoys the regularities

$$\rho \in C([0, T]; H^1 \cap W^{1,q}), \quad (u, \theta) \in C([0, T]; D_0^1 \cap D^2) \cap L^2(0, T; D^{2,q}),$$
$$\rho_t \in C([0, T]; L^2 \cap L^q), \quad (u_t, \theta_t) \in L^2(0, T; D_0^1), \quad (\sqrt{\rho} u_t, \sqrt{\rho}\theta_t) \in L^\infty_{\mathrm{loc}}(0, T; L^2),$$

satisfies (1.1)–(1.3) a.e. in $\mathbb{R}^3 \times (0, T)$, and fulfills the initial condition

$$(\rho, u, \theta)|_{t=0} = (\rho_0, u_0, \theta_0).$$

Remark 1.1. Note that the solutions considered in this paper have the regularities $(u, \theta) \in C([0, T]; D_0^1 \cap D^2)$, and the initial values of ρ, u, and θ are well-defined. Therefore, one can impose the initial conditions on u and θ, even if the initial vacuum is involved.

Definition 1.2. Assume that (ρ_0, u_0, θ_0) satisfies the conditions in Definition 1.1. A triple (ρ, u, θ) is called a global strong solution to system (1.1)–(1.3), with initial data (ρ_0, u_0, θ_0), if it is a solution to the same system with the same initial data on $\mathbb{R}^3 \times (0, T)$ for any positive time T.

We are now ready to state the main result of this paper.

Theorem 1.1. *Assume $2\mu > \lambda$ and let $q \in (3, 6]$ be a fixed constant. Assume that ρ_0, u_0, and θ_0 satisfy*

$$\rho_0, \theta_0 \geqq 0, \quad \rho \leqq \bar\rho, \quad \rho_0 \in H^1 \cap W^{1,q}, \quad \sqrt{\rho_0}\theta_0 \in L^2, \quad (u_0, \theta_0) \in D_0^1 \cap D^2,$$
$$-\mu\Delta u_0 - (\mu + \lambda)\nabla div\, u_0 + \nabla p_0 = \sqrt{\rho_0} g_1, \quad \kappa\Delta\theta_0 + \mathcal{Q}(\nabla u_0) = \sqrt{\rho_0} g_2$$

for a positive constant $\bar\rho$ and some $(g_1, g_2) \in L^2$, where $p_0 = R\rho_0\theta_0$.

Then, there is a positive number ε_0 depending only on R, γ, μ, λ, and κ, such that system (1.1)–(1.3), with initial data (ρ_0, u_0, θ_0), has a unique global strong solution provided that

$$\mathscr{N}_0 := \bar\rho(\|\rho_0\|_3 + \bar\rho^2\|\sqrt{\rho_0}u_0\|_2^2)(\|\nabla u_0\|_2^2 + \bar\rho\|\sqrt{\rho_0}E_0\|_2^2) \leqq \varepsilon_0.$$

Remark 1.2. (i) One can easily check that the quantity $\mathscr{N}_0$ in Theorem 1.1 is scaling invariant with respect to the scaling transform (1.4). Therefore, Theorem 1.1 provides the global well-posedness of system (1.1)–(1.3) under some smallness assumption on a scaling invariant quantity, in the case that the vacuum is allowed. We were not aware of this kind of result before for the compressible Navier–Stokes equations, even for the isentropic case.

(ii) Global well-posedness of strong solutions to the Cauchy problem of system (1.1)–(1.3) in the presence of vacuum has been proved in [18,48], with a non-vacuum far field and a vacuum far field, respectively. The assumptions concerning the smallness in [18,48] are imposed as

$$C_0 = \int \left(\frac{\rho_0}{2}|u_0|^2 + R(\rho_0\log\rho_0 - \rho_0 + 1) + \frac{R}{\gamma - 1}\rho_0(\theta_0 - \log\theta_0 + 1)\right) \mathrm{d}x$$
$$\leqq \varepsilon_0 = \varepsilon_0(\|\rho_0\|_\infty, \|\theta_0\|_\infty, \|\nabla u_0\|_2, R, \gamma, \mu, \lambda, \kappa)$$

and

$$\int \rho_0 \,\mathrm{d}x \leqq \varepsilon_0 = \varepsilon_0(\|\rho_0\|_\infty, \|\sqrt{\rho_0}\theta_0\|_2, \|\nabla u_0\|_2, R, \gamma, \mu, \lambda, \kappa),$$

respectively. Since the explicit dependence of ε_0 on $\|\rho_0\|_\infty$, $\|\theta_0\|_\infty$, $\|\sqrt{\rho_0}\theta_0\|_2$ and $\|\nabla u_0\|_2$ are not derived in [18,48], the scaling invariant quantities, on which the smallness guarantees the global well-posedness, cannot be identified there.

(iii) Compared with the global well-posedness result in [48], our result, Theorem 1.1, allows the initial mass to be infinite. This will be crucial for obtaining the global entropy-bounded solutions in our forthcoming paper [33].

Compared with the isentropic case considered in [19], the additional difficulty for studying the global well-posedness of the full compressible Navier–Stokes equations is that the following basic energy inequality does not provide any dissipation estimates:

$$\int \rho\left(\frac{|u|^2}{2} + c_v\theta\right) \mathrm{d}x = \int \rho_0\left(\frac{|u_0|^2}{2} + c_v\theta_0\right) \mathrm{d}x.$$

Note that the dissipation estimates of the form $\int_0^T \|\nabla u\|_2^2 \leqq C$, which can be guaranteed by the basic energy estimates for the isentropic case, is crucial in the arguments of [19]. To overcome this difficulty, some types of dissipation estimates were recovered in [18,48], in the cases with a non-vacuum and a vacuum far field, respectively, by using the entropy inequality and the conservation of mass. Noticing that the entropy inequality (a crucial tool in [18]) holds only in the case of having a non-vacuum far field and the finiteness of mass is crucial in [48], and recalling that we consider the case with a vacuum far field and allowing possible infinite mass, the arguments in [18,48] do not apply to the current paper.

A crucial ingredient for obtaining the dissipation estimates in this paper is the following equation for ρ^3 (see the proof in Proposition 2.4):

$$\frac{2\mu+\lambda}{2}(\partial_t\rho^3 + \operatorname{div}(u\rho^3)) + \rho^3 p + \rho^3\Delta^{-1}\operatorname{div}(\rho u)_t + \rho^3\Delta^{-1}\operatorname{div}\operatorname{div}(\rho u\otimes u)$$
$$= 0,$$

which is derived in the same spirit of (5.42) in Chapter 5 of [35]. Note that the temperature equation plays no role in deriving the above equation. This equation is employed to get the $L^\infty(0,T;L^3)$ estimate of ρ. Compared with the continuity equation, the main advantage of the above is that it enables us to get the time independent $L^\infty(0,T;L^3)$ estimate of ρ without appealing to the $L^1_t(L^\infty)$ of $\operatorname{div} u$. In fact, the above equation leads to the inequality

$$\sup_{0\leqq t\leqq T}\|\rho\|_3^3 + \int_0^T\int \rho^3 p \,\mathrm{d}x\,\mathrm{d}t \leqq C\sup_{0\leqq t\leqq T}(\|\rho\|_\infty^{\frac{2}{3}}\|\sqrt{\rho}u\|_2^{\frac{1}{3}}\|\sqrt{\rho}|u|^2\|_2^{\frac{1}{3}}\|\rho\|_3^3)$$
$$+C\int_0^T\|\rho\|_\infty^2\|\rho\|_3^2\|\nabla u\|_2^2\,\mathrm{d}t + C\|\rho_0\|_3^3; \tag{1.5}$$

see Proposition 2.4. A key point of the above estimate is that the time integral $\int_0^T \|\rho\|_\infty^2 \|\rho\|_3^2 \|\nabla u\|_2^2\, \mathrm{d}t$ is quadratic with respect to ∇u and, thus, can be expected to be time independent, due to the presence of viscosity in the system. Moreover, the above inequality also provides a time independent estimate for $\int_0^T \int \rho^3 p\, \mathrm{d}x\, \mathrm{d}t$, which, though not used in this paper, is expected to be will useful for studying the large time behavior of global solutions. Note that if using the density equation, i.e., (1.1), to perform the same type of estimate, then the corresponding inequality reads as

$$\sup_{0\leqq t\leqq T} \|\rho\|_3^3 \leqq \|\rho_0\|_3^3 + 2\int_0^T \int |\mathrm{div} u|\rho^3\, \mathrm{d}x\, \mathrm{d}t,$$

which requires some decay property of $\mathrm{div}\, u$ for getting the desired estimate for $\|\rho\|_3$.

Inequality (1.5) motivates us to impose a smallness condition on the quantity $\|\rho_0\|_\infty^2 \|\sqrt{\rho_0} u_0\|_2 \|\sqrt{\rho_0}|u_0|^2\|_2$, which is one of the terms of $\mathscr{N}_0$ in Theorem 1.1, to get the bound of $\|\rho\|_3$. Inequality (1.5) also suggests that we should carry out the estimates on $\|\sqrt{\rho} u\|_{L^\infty(0,T;L^2)}$, $\|\sqrt{\rho} E\|_{L^\infty(0,T;L^2)}$ and $\|\rho\|_{L^\infty(0,T;L^\infty)}$ which are performed in Propositions 2.2, 2.3, and 2.6, respectively. Higher order estimates are required in the estimate for $\|\rho\|_{L^\infty(0,T;L^\infty)}$, and they are carried out with the help of $\omega = \nabla \times u$ and $G = (2\mu + \lambda)\mathrm{div}\, u - p$; see Proposition 2.5. Combining Propositions 2.2–2.6 and by continuity arguments, we are able to get a time-independent estimate on a scaling invariant quantity $\mathscr{N}_T$ (its expression is given in Proposition 2.7) as long as it is small initially. With this a priori estimate for $\mathscr{N}_T$, one can further get the time-independent a priori estimates of $\|\nabla u\|_{L^\infty(0,T;L^2)}$ and $\|\rho\|_{L^\infty(0,T;L^\infty)}$, based on which the blow-up criteria apply, and thus the global well-posedness follows.

Throughout this paper, we use C to denote a general positive constant which may vary from line to line. $A \lesssim B$ means $A \leqq CB$ for some positive constant C.

2. A Priori Estimates

This section is devoted to deriving some a priori estimates for the solutions to the Cauchy problem of system (1.1)–(1.3). The local well-posedness of strong solutions is guaranteed by the following proposition, which is proved in [7]:

Proposition 2.1. *Under the conditions in Theorem 1.1, system (1.1)–(1.3) has a unique local strong solution with initial data* (ρ_0, u_0, θ_0).

In the rest of this section, we always assume that (ρ, u, θ) is a strong solution to system (1.1)–(1.3) on $\mathbb{R}^3 \times (0, T)$ for some positive time T, with initial data (ρ_0, u_0, θ_0). By definition, (ρ, u, θ) has the regularities stated in Definition 1.1.

2.1. Energy Inequalities

Proposition 2.2. *The following estimate holds:*

$$\sup_{0\leqq t\leqq T}\|\sqrt{\rho}u\|_2^2+\int_0^T\|\nabla u\|_2^2\,\mathrm{d}t\leqq C\|\sqrt{\rho_0}u_0\|_2^2+C\int_0^T\|\rho\|_3^2\|\nabla\theta\|_2^2\,\mathrm{d}t,$$

for a positive constant C depending only on R,γ,μ,λ, and κ.

Proof. Multiplying (1.2) by u and integrating the result over $\mathbb{R}^3$, and noticing that $\mu+\lambda>0$, it follows from integration by parts and the Cauchy inequality that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}u\|_2^2+\mu\|\nabla u\|_2^2&\leqq R\|\rho\|_3\|\theta\|_6\|\mathrm{div} u\|_2\leqq C\|\rho\|_3\|\nabla\theta\|_2\|\mathrm{div}\,u\|_2\\&\leqq(\mu+\lambda)\|\mathrm{div}\,u\|_2^2+C\|\rho\|_3^2\|\nabla\theta\|_2^2,\end{aligned}$$

from which the conclusion follows by integrating in t. □

Proposition 2.3. *Assume that $2\mu>\lambda$. Then, the following estimate holds:*

$$\begin{aligned}&\sup_{0\leqq t\leqq T}\|\sqrt{\rho}E\|_2^2+\int_0^T(\|\nabla\theta\|_2^2+\||u|\nabla u\|_2^2)\,\mathrm{d}t\\&\leqq C\|\sqrt{\rho_0}E_0\|_2^2+C\int_0^T\|\rho\|_\infty\|\rho\|_3^{\frac12}\|\sqrt{\rho}\theta\|_2\|(\nabla\theta,|u|\nabla u)\|_2^2\,\mathrm{d}t,\end{aligned}$$

for a positive constant C depending only on R,γ,μ,λ, and κ, where $E=\frac{|u|^2}{2}+c_v\theta$.

Proof. One can verify that

$$\rho(\partial_tE+u\cdot\nabla E)+\mathrm{div}\,(up)-\kappa\Delta\theta=\mathrm{div}\,(\mathcal{S}\cdot u),\tag{2.1}$$

where $\mathcal{S}=\mu(\nabla u+(\nabla u)^t)+\lambda\mathrm{div}\,uI$. Multiplying (2.1) by E and integrating the result over $\mathbb{R}^3$, it follows from integration by parts that

$$\begin{aligned}&\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}E\|_2^2+\kappa c_v\|\nabla\theta\|_2^2\\&=\int\left[-\frac{\kappa}{2}\nabla\theta\cdot\nabla|u|^2+(up-\mathcal{S}\cdot u)\cdot\left(c_v\nabla\theta+\frac{\nabla|u|^2}{2}\right)\right]\mathrm{d}x\\&\leqq\frac{c_v\kappa}{2}\|\nabla\theta\|_2^2+C\||u|\nabla u\|_2^2+C\int\rho^2\theta^2|u|^2\,\mathrm{d}x,\end{aligned}$$

which yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}E\|_2^2+\kappa c_v\|\nabla\theta\|_2^2\lesssim\||u|\nabla u\|_2^2+\int\rho^2\theta^2|u|^2\,\mathrm{d}x.\tag{2.2}$$

Multiplying (1.2) by $|u|^2u$, it follows from integration by parts that

$$\begin{aligned}&\frac14\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}|u|^2\|_2^2-\int(\mu\Delta u+(\mu+\lambda)\nabla\mathrm{div}\,u)\cdot|u|^2u\,\mathrm{d}x\\&=-\int p\mathrm{div}\,(|u|^2u)\,\mathrm{d}x\\&\leqq\left(\mu-\frac{\lambda}{2}\right)\int|u|^2|\nabla u|^2\,\mathrm{d}x+C\int\rho^2\theta^2|u|^2\,\mathrm{d}x.\end{aligned}$$

Some elementary calculations show that

$$-\int (\mu\Delta u + (\mu+\lambda)\nabla \mathrm{div}\, u)\cdot |u|^2 u\,\mathrm{d}x \geqq (2\mu-\lambda)\int |u|^2|\nabla u|^2\,\mathrm{d}x.$$

Combining the above two inequalities leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\rho}|u|^2\|_2^2 + 2(2\mu-\lambda)\||u|\nabla u\|_2^2 \lesssim \int \rho^2\theta^2|u|^2\,\mathrm{d}x. \tag{2.3}$$

Multiplying (2.3) by a sufficiently large number K depending only on R, γ, μ, λ and κ and summing the result with (2.2), one obtains

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|\sqrt{\rho}E\|_2^2 + K\|\sqrt{\rho}|u|^2\|_2^2) + \kappa c_v\|\nabla\theta\|_2^2 + (2\mu-\lambda)K\||u|\nabla u\|_2^2\\ &\lesssim \int \rho^2\theta^2|u|^2\,\mathrm{d}x,\end{aligned}$$

from which, noticing that the Hölder and Sobolev inequalities yield

$$\begin{aligned}\int \rho^2\theta^2|u|^2\,\mathrm{d}x &\leqq \|\sqrt{\rho}\theta\|_2\|\theta\|_6\||u|^2\|_6\|\rho\|_9^{\frac{3}{2}}\\ &\lesssim \|\sqrt{\rho}\theta\|_2\|\nabla\theta\|_2\|\nabla|u|^2\|_2\|\rho\|_\infty\|\rho\|_3^{\frac{1}{2}},\end{aligned} \tag{2.4}$$

one obtains

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|\sqrt{\rho}E\|_2^2 + K\|\sqrt{\rho}|u|^2\|_2^2) + \kappa c_v\|\nabla\theta\|_2^2 + (2\mu-\lambda)K\||u|\nabla u\|_2^2\\ &\lesssim \|\rho\|_\infty\|\rho\|_3^{\frac{1}{2}}\|\sqrt{\rho}\theta\|_2\|\nabla\theta\|_2\|\nabla|u|^2\|_2.\end{aligned}$$

Integrating this in t and using the Cauchy inequality, the conclusion follows. □

Proposition 2.4. *The following estimate holds:*

$$\begin{aligned}\sup_{0\leqq t\leqq T}\|\rho\|_3^3 + \int_0^T\int \rho^3 p\,\mathrm{d}x\,\mathrm{d}t \leqq{}& C\sup_{0\leqq t\leqq T}(\|\rho\|_\infty^{\frac{2}{3}}\|\sqrt{\rho}u\|_2^{\frac{1}{3}}\|\sqrt{\rho}|u|^2\|_2^{\frac{1}{3}}\|\rho\|_3^3)\\ &+C\int_0^T\|\rho\|_\infty^2\|\rho\|_3^2\|\nabla u\|_2^2\,\mathrm{d}t + C\|\rho_0\|_3^3,\end{aligned}$$

for a positive constant C depending only on R, γ, μ, λ, and κ.

Proof. Applying the operator $\Delta^{-1}\mathrm{div}$ to (1.2) yields

$$\Delta^{-1}\mathrm{div}\,(\rho u)_t + \Delta^{-1}\mathrm{div}\,\mathrm{div}\,(\rho u\otimes u) - (2\mu+\lambda)\mathrm{div}\, u + p = 0. \tag{2.5}$$

Multiplying the above equation by ρ^3 and noticing that

$$\partial_t\rho^3 + \mathrm{div}\,(u\rho^3) + 2\mathrm{div}\, u\rho^3 = 0,$$

one obtains

$$\frac{2\mu+\lambda}{2}(\partial_t\rho^3+\mathrm{div}\,(u\rho^3))+\rho^3p+\rho^3\Delta^{-1}\mathrm{div}\,(\rho u)_t+\rho^3\Delta^{-1}\mathrm{div}\,\mathrm{div}\,(\rho u\otimes u)$$
$$=0. \tag{2.6}$$

Integrating the above equation over $\mathbb{R}^3$ yields

$$\begin{aligned}\frac{2\mu+\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\rho\|_3^3&+\int\rho^3p\,\mathrm{d}x+\int\rho^3\Delta^{-1}\mathrm{div}\,(\rho u)_t\,\mathrm{d}x\\&=-\int\rho^3\Delta^{-1}\mathrm{div}\,\mathrm{div}\,(\rho u\otimes u)\,\mathrm{d}x.\end{aligned}\tag{2.7}$$

Using (1.1), one deduces

$$\begin{aligned}&\int\rho^3\Delta^{-1}\mathrm{div}\,(\rho u)_t\,\mathrm{d}x\\&=\frac{\mathrm{d}}{\mathrm{d}t}\int\rho^3\Delta^{-1}\mathrm{div}\,(\rho u)\,\mathrm{d}x+\int[\mathrm{div}\,(\rho^3u)+2\mathrm{div}\,u\rho^3]\Delta^{-1}\mathrm{div}\,(\rho u)\,\mathrm{d}x\\&=\int[2\mathrm{div}\,u\rho^3\Delta^{-1}\mathrm{div}\,(\rho u)-\rho^3u\cdot\nabla\Delta^{-1}\mathrm{div}\,(\rho u)]\,\mathrm{d}x\\&\quad+\frac{\mathrm{d}}{\mathrm{d}t}\int\rho^3\Delta^{-1}\mathrm{div}\,(\rho u)\,\mathrm{d}x.\end{aligned}$$

Therefore, it follows from (2.7) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\frac{2\mu+\lambda}{2}+\Delta^{-1}\mathrm{div}\,(\rho u)\right)\rho^3\,\mathrm{d}x+\int\rho^3p\,\mathrm{d}x\\&=\int\Big[\rho^3(u\cdot\nabla\Delta^{-1}\mathrm{div}\,(\rho u)-\Delta^{-1}\mathrm{divdiv}\,(\rho u\otimes u))-2\mathrm{div}\,u\rho^3\Delta^{-1}\mathrm{div}\,(\rho u)\Big]\,\mathrm{d}x.\end{aligned}\tag{2.8}$$

Noticing that

$$\begin{aligned}\|\nabla\Delta^{-1}\mathrm{div}\,(\rho u)\|_2&\lesssim\|\rho u\|_2\lesssim\|\rho\|_3\|u\|_6,\\\|\Delta^{-1}\mathrm{div}\,\mathrm{div}\,(\rho u\otimes u)\|_{\frac32}&\lesssim\|\rho|u|^2\|_{\frac32}\lesssim\|\rho\|_3\|u\|_6^2,\end{aligned}$$

it follows from the Hölder and Sobolev embedding inequality that

$$\begin{aligned}&\int\rho^3(u\cdot\nabla\Delta^{-1}\mathrm{div}\,(\rho u)-\Delta^{-1}\mathrm{divdiv}\,(\rho u\otimes u))\,\mathrm{d}x\\&\lesssim\|\rho\|_9^3\|\rho\|_3\|u\|_6^2\lesssim\|\rho\|_\infty^2\|\rho\|_3^2\|\nabla u\|_2^2.\end{aligned}\tag{2.9}$$

By the Sobolev embedding and elliptic estimates we have that

$$\begin{aligned}\|\Delta^{-1}\mathrm{div}\,(\rho u)\|_6\lesssim\|\nabla\Delta^{-1}\mathrm{div}\,(\rho u)\|_2&\lesssim\|\rho u\|_2\\&\lesssim\|\rho\|_3\|u\|_6\lesssim\|\rho\|_3\|\nabla u\|_2,\end{aligned}$$

and, thus, the Hölder inequality yields

$$\left|\int \operatorname{div} u\rho^3\Delta^{-1}\operatorname{div}(\rho u)\,\mathrm{d}x\right| \lesssim \|\operatorname{div} u\|_2\|\rho\|_9^3\|\rho\|_3\|\nabla u\|_2 \lesssim \|\rho\|_\infty^2\|\rho\|_3^2\|\nabla u\|_2^2. \tag{2.10}$$

By the Gagliardo-Nirenberg inequality and using the elliptic estimates, it follows that

$$\begin{aligned}\|\Delta^{-1}\operatorname{div}(\rho u)\|_\infty &\lesssim \|\Delta^{-1}\operatorname{div}(\rho u)\|_6^{\frac{1}{3}}\|\nabla\Delta^{-1}\operatorname{div}(\rho u)\|_4^{\frac{2}{3}}\\ &\lesssim \|\rho u\|_2^{\frac{1}{3}}\|\rho u\|_4^{\frac{2}{3}} \lesssim \|\rho\|_\infty^{\frac{2}{3}}\|\sqrt{\rho}u\|_2^{\frac{1}{3}}\|\sqrt{\rho}|u|^2\|_2^{\frac{1}{3}}.\end{aligned} \tag{2.11}$$

Integrating (2.8) in t, using (2.9)–(2.11), and by some straightforward calculations, the conclusion follows. □

Proposition 2.5. *Assume that*

$$\sup_{0\leqq t\leqq T}\|\rho\|_\infty \leqq 4\bar{\rho}.$$

Then, there is a positive constant C depending only on R, γ, μ, λ and κ, such that

$$\begin{aligned}&\sup_{0\leqq t\leqq T}\|\nabla u\|_2^2 + \int_0^T\left\|\left(\sqrt{\rho}u_t, \frac{\nabla G}{\sqrt{\bar{\rho}}}, \frac{\nabla\omega}{\sqrt{\bar{\rho}}}\right)\right\|_2^2\mathrm{d}t\\ &\leqq C\|\nabla u_0\|_2^2 + C\bar{\rho}\sup_{0\leqq t\leqq T}\|\sqrt{\rho}\theta\|_2^2 + C\bar{\rho}^3\int_0^T\|\nabla u\|_2^4\|(\nabla u, \sqrt{\bar{\rho}}\sqrt{\rho}\theta)\|_2^2\,\mathrm{d}t\\ &+C\int_0^T(\bar{\rho}+\bar{\rho}^2\|\rho\|_3^{\frac{1}{2}}\|\sqrt{\rho}\theta\|_2)\|(\nabla\theta, |u|\nabla u)\|_2^2\,\mathrm{d}t,\end{aligned}$$

where $G = (2\mu+\lambda)\operatorname{div} u - p$ and $\omega = \nabla\times u$.

Proof. Multiplying (1.2) by u_t, it follows from integration by parts that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\mu\|\nabla u\|_2^2 + (\mu+\lambda)\|\operatorname{div} u\|_2^2) - \int p\operatorname{div} u_t\,\mathrm{d}x + \|\sqrt{\rho}u_t\|_2^2\\ &= -\int\rho(u\cdot\nabla)u\cdot u_t\,\mathrm{d}x.\end{aligned} \tag{2.12}$$

Noticing that $\operatorname{div} u = \frac{G+p}{2\mu+\lambda}$, it follows that

$$\begin{aligned}-\int p\operatorname{div} u_t\,\mathrm{d}x &= -\frac{\mathrm{d}}{\mathrm{d}t}\int p\operatorname{div} u\,\mathrm{d}x + \int p_t\operatorname{div} u\,\mathrm{d}x\\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\int p\operatorname{div} u\,\mathrm{d}x + \frac{1}{2(2\mu+\lambda)}\frac{\mathrm{d}}{\mathrm{d}t}\|p\|_2^2 + \frac{1}{2\mu+\lambda}\int p_t G\,\mathrm{d}x.\end{aligned} \tag{2.13}$$

Note that (1.3) implies

$$p_t = (\gamma-1)(\mathcal{Q}(\nabla u) - p\operatorname{div} u + \kappa\Delta\theta) - \operatorname{div}(up),$$

and thus, integration by parts gives

$$\int p_t G \,\mathrm{d}x = \int [(\gamma-1)(\mathcal{Q}(\nabla u) - p\mathrm{div}\, u)G + (up - \kappa(\gamma-1)\nabla\theta)\cdot\nabla G]\,\mathrm{d}x. \tag{2.14}$$

Substituting (2.14) into (2.13), then the result into (2.12), and noticing that $\|\nabla u\|_2^2 = \|\omega\|_2^2 + \|\mathrm{div}\, u\|_2^2$, by some straightforward calculations that one obtains that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu\|\omega\|_2^2 + \frac{\|G\|_2^2}{2\mu+\lambda}\right) + \|\sqrt{\rho}u_t\|_2^2 \\
&= -\int \rho(u\cdot\nabla)u\cdot u_t\,\mathrm{d}x + \frac{1}{2\mu+\lambda}\int(\kappa(\gamma-1)\nabla\theta - up)\cdot\nabla G\,\mathrm{d}x \\
&\quad -\frac{\gamma-1}{2\mu+\lambda}\int(\mathcal{Q}(\nabla u) - p\mathrm{div}\, u)G\,\mathrm{d}x.
\end{aligned} \tag{2.15}$$

Use $\Delta u = \nabla\mathrm{div}\, u - \nabla\times\nabla\times u$ to rewrite (1.2) as

$$\rho(u_t + u\cdot\nabla u) = \nabla G - \mu\nabla\times\omega. \tag{2.16}$$

Testing this by ∇G, noticing that $\int \nabla G\cdot\nabla\times\omega\,\mathrm{d}x = 0$, and recalling $\|\rho\|_\infty \leqq 4\bar{\rho}$, yields

$$\begin{aligned}
\|\nabla G\|_2^2 &= \int \rho(u_t + u\cdot\nabla u)\cdot\nabla G\,\mathrm{d}x \\
&\leqq \int\left(\frac{|\nabla G|^2}{2} + 2\bar{\rho}\rho|u_t|^2\right)\mathrm{d}x + \int \rho u\cdot\nabla u\cdot\nabla G\,\mathrm{d}x,
\end{aligned}$$

which gives

$$\frac{\|\nabla G\|_2^2}{16\bar{\rho}} \leqq \frac{1}{4}\|\sqrt{\rho}u_t\|_2^2 + \frac{1}{8\bar{\rho}}\int \rho(u\cdot\nabla)u\cdot\nabla G\,\mathrm{d}x. \tag{2.17}$$

Similarly,

$$\frac{\mu^2\|\nabla\omega\|_2^2}{16\bar{\rho}} \leqq \frac{1}{4}\|\sqrt{\rho}u_t\|_2^2 + \frac{1}{8\bar{\rho}}\int \rho(u\cdot\nabla)u\cdot\nabla\times\omega\,\mathrm{d}x. \tag{2.18}$$

Thanks to (2.17) and (2.18), one obtains from (2.15) that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu\|\omega\|_2^2 + \frac{\|G\|_2^2}{2\mu+\lambda}\right) + \frac{1}{2}\|\sqrt{\rho}u_t\|_2^2 + \frac{1}{16\bar{\rho}}(\|\nabla G\|_2^2 + \mu^2\|\nabla\omega\|_2^2) \\
&\leqq C\int \rho|u||\nabla u|\left[|u_t| + \frac{1}{\bar{\rho}}(\nabla G| + |\nabla\omega|)\right]\mathrm{d}x + C\int(|\nabla\theta| + \rho\theta|u|)|\nabla G|\,\mathrm{d}x \\
&\quad + C\int(|\nabla u|^2 + \rho\theta|\nabla u|)|G|\,\mathrm{d}x =: I_1 + I_2 + I_3.
\end{aligned} \tag{2.19}$$

The terms I_1, I_2, and I_3 are estimated as follows: for I_1, by the Hölder and Young inequalities, one obtains

$$
\begin{aligned}
I_1 &\lesssim \sqrt{\bar\rho}\||u|\nabla u\|_2\|\sqrt{\rho}u_t\|_2 + \||u|\nabla u\|_2(\|\nabla G\|_2 + \|\nabla\omega\|_2)\\
&\leqq \frac{1}{6}\left[\frac{1}{2}\|\sqrt{\rho}u_t\|_2^2 + \frac{1}{16\bar\rho}(\|\nabla G\|_2^2 + \mu^2\|\nabla\omega\|_2^2)\right] + C\bar\rho\||u|\nabla u\|_2^2.
\end{aligned}
$$

Recalling (2.4), it follows from the Hölder and Young inequalities that

$$
\begin{aligned}
I_2 &\lesssim \|\nabla\theta\|_2\|\nabla G\|_2 + \|\rho\theta u\|_2\|\nabla G\|_2\\
&\lesssim \|\nabla\theta\|_2\|\nabla G\|_2 + \sqrt{\bar\rho}\|\rho\|_3^{\frac14}\|\sqrt{\rho}\theta\|_2^{\frac12}\|\nabla\theta\|_2^{\frac12}\|\nabla|u|^2\|_2^{\frac12}\|\nabla G\|_2\\
&\leqq \frac{\|\nabla G\|_2^2}{96\bar\rho} + C\left(\bar\rho^2\|\rho\|_3^{\frac12}\|\sqrt{\rho}\theta\|_2 + \bar\rho\right)(\|\nabla\theta\|_2^2 + \||u|\nabla u\|_2^2).
\end{aligned}
$$

The elliptic estimates and Sobolev embedding inequality yield that

$$
\begin{aligned}
\|\nabla u\|_6 &\lesssim \|\nabla\times u\|_6 + \|\mathrm{div}\, u\|_6 \lesssim \|\omega\|_6 + \|G\|_6 + \|\rho\theta\|_6\\
&\lesssim \|\nabla\omega\|_2 + \|\nabla G\|_2 + \bar\rho\|\nabla\theta\|_2.
\end{aligned}
\tag{2.20}
$$

Using (2.20), by the Hölder, Sobolev, and Young inequalities, one deduces that

$$
\begin{aligned}
I_3 &\lesssim \|\nabla u\|_2\|\nabla u\|_6\|G\|_3 + \|\nabla u\|_2\|\rho\theta\|_6\|G\|_3\\
&\lesssim C\|\nabla u\|_2(\|\nabla G\|_2 + \|\nabla\omega\|_2 + \bar\rho\|\nabla\theta\|_2)\|G\|_2^{\frac12}\|\nabla G\|_2^{\frac12}\\
&\quad +\bar\rho\|\nabla u\|_2\|\nabla\theta\|_2\|G\|_2^{\frac12}\|\nabla G\|_2^{\frac12}\\
&\leqq \frac{1}{96\bar\rho}(\|\nabla G\|_2^2 + \mu^2\|\nabla\omega\|_2^2) + C\bar\rho^3\|\nabla u\|_2^4\|G\|_2^2 + C\bar\rho\|\nabla\theta\|_2^2.
\end{aligned}
$$

Substituting the estimates for $I_i, i = 1, 2, 3$ that into (2.19) yields

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu\|\omega\|_2^2 + \frac{\|G\|_2^2}{2\mu+\lambda}\right) + \frac{1}{2}\|\sqrt{\rho}u_t\|_2^2 + \frac{1}{16\bar\rho}(\|\nabla G\|_2^2 + \mu^2\|\nabla\omega\|_2^2)\\
&\quad\lesssim (\bar\rho + \bar\rho^2\|\rho\|_3^{\frac12}\|\sqrt{\rho}\theta\|_2)(\|\nabla\theta\|_2^2 + \||u|\nabla u\|_2^2) + \bar\rho^3\|\nabla u\|_2^4\|G\|_2^2,
\end{aligned}
$$

from which, integrating in t and using

$$
\|\nabla u\|_2 \lesssim \|\omega\|_2 + \|G\|_2 + \|\rho\theta\|_2 \lesssim \|\omega\|_2 + \|G\|_2 + \sqrt{\bar\rho}\|\sqrt{\rho}\theta\|_2,
$$

the conclusion follows by straightforward calculations. □

Proposition 2.6. *Assume that*

$$
\sup_{0\leqq t\leqq T}\|\rho\|_\infty \leqq 4\bar\rho.
$$

Then, there is a positive constant C depending only on R, γ, μ, λ, and κ, such that

$$
\sup_{0\leqq t\leqq T}\|\rho\|_\infty \leqq \|\rho_0\|_\infty e^{C\bar\rho^{\frac23}\sup_{0\leqq t\leqq T}\|\sqrt{\rho}u\|_2^{\frac13}\|\sqrt{\rho}|u|^2\|_2^{\frac13} + C\bar\rho\int_0^T\|\nabla u\|_2\|(\nabla G,\nabla\omega,\bar\rho\nabla\theta)\|_2\,\mathrm{d}t}.
$$

Proof. Denote $\mathcal{O} = \{x \in \mathbb{R}^3 | \rho_0(x) = 0\}$ and $\Omega = \{x \in \mathbb{R}^3 | \rho_0(x) > 0\}$. Define X as

$$\partial_t X(x,t) = u(X(x,t),t), \quad X(x,0) = x.$$

Then $\rho(X(x,t),t) \equiv 0$ for any $x \in \mathcal{O}$, and $\rho(X(x,t),t) > 0$ for any $x \in \Omega$. One can verify that $\{X(x,t) | x \in \mathbb{R}^3\} = \mathbb{R}^3$ for any $t \in (0,T)$. Therefore

$$\sup_{x\in\mathbb{R}^3} \rho(x,t) = \sup_{x\in\mathbb{R}^3} \|\rho(X(x,t),t)\|_\infty = \sup_{x\in\Omega} \rho(X(x,t),t). \tag{2.21}$$

Rewrite (2.5) as

$$\begin{aligned}&\partial_t \Delta^{-1}\mathrm{div}\,(\rho u) + u\cdot\nabla\Delta^{-1}\mathrm{div}\,(\rho u) - (2\mu+\lambda)\mathrm{div}\,u + p \\ &\quad = u\cdot\nabla\Delta^{-1}\mathrm{div}\,(\rho u) - \Delta^{-1}\mathrm{div}\,\mathrm{div}\,(\rho u\otimes u) = [u, \mathcal{R}\otimes\mathcal{R}](\rho u),\end{aligned} \tag{2.22}$$

where $\mathcal{R}$ is the Riesz transform on $\mathbb{R}^3$. Using the fact that $\frac{\mathrm{d}}{\mathrm{d}t}(f(X(x,t),t)) = (\partial_t f + u\cdot\nabla f)(X(x,t),t)$, it follows from (1.1) that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\log\rho(X(x,t),t)) = -\mathrm{div}\,u(X(x,t),t), \quad \forall x\in\Omega.$$

Therefore, for any $x\in\Omega$, it follows from (2.22) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big((2\mu+\lambda)\log\rho(X(x,t),t) + (\Delta^{-1}\mathrm{div}\,(\rho u))(X(x,t),t)\Big) \\ &\qquad + p(X(x,t),t) = \Big([u,\mathcal{R}\otimes\mathcal{R}](\rho u)\Big)(X(x,t),t).\end{aligned}$$

Due to $p \geqq 0$ and (2.21), one can easily derive from the above equality that

$$\|\rho\|_\infty \leqq \|\rho_0\|_\infty e^{C\left(\sup_{0\leqq t\leqq T}\|\Delta^{-1}\mathrm{div}\,(\rho u)\|_\infty + \int_0^T \|[u,\mathcal{R}\otimes\mathcal{R}](\rho u)\|_\infty\,\mathrm{d}t\right)}. \tag{2.23}$$

Using the Gagliardo-Nirenberg inequality and the commutator estimates, one deduces

$$\begin{aligned}\|[u,\mathcal{R}\otimes\mathcal{R}](\rho u)\|_\infty &\lesssim \|[u,\mathcal{R}\otimes\mathcal{R}](\rho u)\|_3^{\frac15}\|\nabla[u,\mathcal{R}\otimes\mathcal{R}](\rho u)\|_4^{\frac45} \\ &\lesssim \|u\|_6^{\frac15}\|\rho u\|_6^{\frac15}\|\nabla u\|_6^{\frac45}\|\rho u\|_{12}^{\frac45} \lesssim \bar\rho\|u\|_6^{\frac15}\|u\|_6^{\frac15}\|\nabla u\|_6^{\frac45}\left(\|u\|_6^{\frac34}\|\nabla u\|_6^{\frac14}\right)^{\frac45} \\ &\lesssim \bar\rho\|\nabla u\|_2\|\nabla u\|_6 \lesssim \bar\rho\|\nabla u\|_2(\|\nabla G\|_2 + \|\nabla\omega\|_2 + \bar\rho\|\nabla\theta\|_2),\end{aligned}$$

where, in the last step, (2.20) has been used. Thanks to this and recalling (2.11), the conclusion follows from (2.23). $\square$

2.2. A Priori Estimates

Proposition 2.7. *Assume that $2\mu > \lambda$. Denote*

$$\mathscr{N}_T = \bar{\rho} \sup_{0 \leqq t \leqq T} (\|\rho\|_3 + \bar{\rho}^2 \|\sqrt{\rho} u\|_2^2)(t) \sup_{0 \leqq t \leqq T} (\|\nabla u\|_2^2 + \bar{\rho} \|\sqrt{\rho} E\|_2^2)(t).$$

Then, there is a positive constant η_0 depending only on R, γ, μ, λ and κ, such that if

$$\eta \leqq \eta_0, \quad \sup_{0 \leqq t \leqq T} \|\rho\|_\infty \leqq 4\bar{\rho}, \quad \text{and} \quad \mathscr{N}_T \leqq \sqrt{\eta},$$

then the following estimates hold:

$$\sup_{0 \leqq t \leqq T} \|\sqrt{\rho} E\|_2^2 + \int_0^T \|(\nabla \theta, |u| \nabla u)\|_2^2 \, \mathrm{d}t \leqq C \|\sqrt{\rho_0} E_0\|_2^2,$$

$$\sup_{0 \leqq t \leqq T} \|\rho\|_3 + \left(\int_0^T \int \rho^3 p \, \mathrm{d}x \, \mathrm{d}t \right)^{\frac{1}{3}} \leqq C(\|\rho_0\|_3 + \bar{\rho}^2 \|\sqrt{\rho_0} u_0\|_2^2),$$

$$\bar{\rho}^2 \left(\sup_{0 \leqq t \leqq T} \|\sqrt{\rho} u\|_2^2 + \int_0^T \|\nabla u\|_2^2 \, \mathrm{d}t \right) \leqq C(\|\rho_0\|_3 + \bar{\rho}^2 \|\sqrt{\rho_0} u_0\|_2^2),$$

$$\sup_{0 \leqq t \leqq T} \|\nabla u\|_2^2 + \int_0^T \left\| \left(\sqrt{\rho} u_t, \frac{\nabla G}{\sqrt{\bar{\rho}}}, \frac{\nabla \omega}{\sqrt{\bar{\rho}}} \right) \right\|_2^2 \mathrm{d}t \leqq C(\|\nabla u_0\|_2^2 + \bar{\rho} \|\sqrt{\rho_0} E_0\|_2^2),$$

$$\sup_{0 \leqq t \leqq T} \|\rho\|_\infty \leqq \bar{\rho} e^{C \mathscr{N}_0^{\frac{1}{6}} + C \mathscr{N}_0^{\frac{1}{2}}},$$

for a positive constant C depending only on R, γ, μ, λ and κ, where

$$\mathscr{N}_0 = \bar{\rho} (\|\rho_0\|_3 + \bar{\rho}^2 \|\sqrt{\rho_0} u_0\|_2^2)(\|\nabla u_0\|_2^2 + \bar{\rho} \|\sqrt{\rho_0} E_0\|_2^2).$$

Proof. By assumption, it follows from Proposition 2.3 that

$$\begin{aligned} &\sup_{0 \leqq t \leqq T} \|\sqrt{\rho} E\|_2^2 + \int_0^T (\|\nabla \theta\|_2^2 + \||u| \nabla u\|_2^2) \, \mathrm{d}t \\ &\leqq C \|\sqrt{\rho_0} E_0\|_2^2 + C \eta_0^{\frac{1}{4}} \int_0^T (\|\nabla \theta\|_2^2 + \||u| \nabla u\|_2^2) \, \mathrm{d}t, \end{aligned}$$

which, by choosing η_0 suitably small, implies that

$$\sup_{0 \leqq t \leqq T} \|\sqrt{\rho} E\|_2^2 + \int_0^T (\|\nabla \theta\|_2^2 + \||u| \nabla u\|_2^2) \, \mathrm{d}t \leqq C \|\sqrt{\rho_0} E_0\|_2^2. \tag{2.24}$$

Thanks to (2.24), using the assumptions, and applying Proposition 2.2, one obtains

$$\begin{aligned}\sup_{0\leqq t\leqq T}\|\sqrt{\rho}u\|_2^2+\int_0^T\|\nabla u\|_2^2\,\mathrm{d}t &\leqq C\|\sqrt{\rho_0}u_0\|_2^2+C\|\sqrt{\rho_0}E_0\|_2^2\sup_{0\leqq t\leqq T}\|\rho\|_3^2\\ &\leqq C\|\sqrt{\rho_0}u_0\|_2^2+C\sup_{0\leqq t\leqq T}\|\sqrt{\rho}E\|_2^2\sup_{0\leqq t\leqq T}\|\rho\|_3^2\\ &\leqq C\|\sqrt{\rho_0}u_0\|_2^2+\frac{C\sqrt{\eta_0}}{\bar\rho^2}\sup_{0\leqq t\leqq T}\|\rho\|_3. \qquad (2.25)\end{aligned}$$

Using the assumptions and (2.25), it follows from Proposition 2.4 and the Young inequality that

$$\begin{aligned}&\sup_{0\leqq t\leqq T}\|\rho\|_3^3+\int_0^T\int\rho^3p\,\mathrm{d}x\,\mathrm{d}t\\ &\leqq C\|\rho_0\|_3^3+C\eta_0^{\frac{1}{12}}\sup_{0\leqq t\leqq T}\|\rho_0\|_3^3+C\bar\rho^2\left(\|\sqrt{\rho_0}u_0\|_2^2+\frac{\sqrt{\eta_0}}{\bar\rho^2}\sup_{0\leqq t\leqq T}\|\rho\|_3\right)\sup_{0\leqq t\leqq T}\|\rho\|_3^2\\ &\leqq C\|\rho_0\|_3^3+\left(C\eta_0^{\frac{1}{12}}+\frac14+C\sqrt{\eta_0}\right)\sup_{0\leqq t\leqq T}\|\rho\|_3^3+C\bar\rho^6\|\sqrt{\rho_0}u_0\|_2^6,\end{aligned}$$

from which, by choosing η_0 sufficiently small, one obtains

$$\sup_{0\leqq t\leqq T}\|\rho\|_3+\left(\int_0^T\int\rho^3p\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac13}\leqq C(\|\rho_0\|_3+\bar\rho^2\|\sqrt{\rho_0}u_0\|_2^2). \qquad (2.26)$$

Combing (2.25) with (2.26) yields

$$\bar\rho^2\left(\sup_{0\leqq t\leqq T}\|\sqrt{\rho}u\|_2^2+\int_0^T\|\nabla u\|_2^2\,\mathrm{d}t\right)\leqq C(\|\rho_0\|_3+\bar\rho^2\|\sqrt{\rho_0}u_0\|_2^2). \qquad (2.27)$$

Using (2.24) and (2.27), it follows from Proposition 2.5 that

$$\begin{aligned}&\sup_{0\leqq t\leqq T}\|\nabla u\|_2^2+\int_0^T\left\|\left(\sqrt{\rho}u_t,\frac{\nabla G}{\sqrt{\bar\rho}},\frac{\nabla\omega}{\sqrt{\bar\rho}}\right)\right\|_2^2\mathrm{d}t\\ &\lesssim\|\nabla u_0\|_2^2+\bar\rho\|\sqrt{\rho_0}E_0\|_2^2+\bar\rho^3\int_0^T\|\nabla u\|_2^2\,\mathrm{d}t\sup_{0\leqq t\leqq T}\left(\|\nabla u\|_2^2+\bar\rho\|\sqrt{\rho}\theta\|_2^2\right)\\ &\quad\times\sup_{0\leqq t\leqq T}\|\nabla u\|_2^2+\left(\bar\rho+\bar\rho^2\sup_{0\leqq t\leqq T}\|\rho\|_3^{\frac12}\|\sqrt{\rho}\theta\|_2\right)\int_0^T\|(\nabla\theta,|u|\nabla u)\|_2^2\,\mathrm{d}t\\ &\lesssim\bar\rho(\|\rho_0\|_3+\bar\rho^2\|\sqrt{\rho_0}u_0\|_2^2)\sup_{0\leqq t\leqq T}\left(\|\nabla u\|_2^2+\bar\rho\|\sqrt{\rho}E\|_2^2\right)\sup_{0\leqq t\leqq T}\|\nabla u\|_2^2\\ &\quad+\bar\rho^2\sup_{0\leqq t\leqq T}\|\rho\|_3^{\frac12}\|\sqrt{\rho}\theta\|_2\|\sqrt{\rho_0}E_0\|_2^2+\|\nabla u_0\|_2^2+\bar\rho\|\sqrt{\rho_0}E_0\|_2^2. \qquad (2.28)\end{aligned}$$

Recalling the definition of $\mathcal{N}_T$ and the assumption that $\mathcal{N}_T \leqq \sqrt{\eta_0}$, it is clear that

$$\begin{aligned}
&\bar\rho(\|\rho_0\|_3 + \bar\rho^2\|\sqrt{\rho_0}u_0\|_2^2) \sup_{0\leqq t\leqq T}\left(\|\nabla u\|_2^2 + \bar\rho\|\sqrt{\rho}E\|_2^2\right)\\
&\quad\leqq \bar\rho \sup_{0\leqq t\leqq T}(\|\rho\|_3 + \bar\rho^2\|\sqrt{\rho}u\|_2^2) \sup_{0\leqq t\leqq T}\left(\|\nabla u\|_2^2 + \bar\rho\|\sqrt{\rho}E\|_2^2\right) \leqq \mathcal{N}_T \leqq \sqrt{\eta_0}
\end{aligned}$$

and

$$\bar\rho \sup_{0\leqq t\leqq T}\|\rho\|_3^{\frac12}\|\sqrt{\rho}\theta\|_2 \leqq \left(\bar\rho^2 \sup_{0\leqq t\leqq T}\|\rho\|_3 \sup_{0\leqq t\leqq T}\|\sqrt{\rho}E\|_2^2\right)^{\frac12} \leqq \mathcal{N}_T^{\frac12} \leqq \eta_0^{\frac14}.$$

Thanks to the above two estimates, by choosing η_0 sufficiently small, one can easily derive from (2.28) that

$$\sup_{0\leqq t\leqq T}\|\nabla u\|_2^2 + \int_0^T \left\|\left(\sqrt{\rho}u_t, \frac{\nabla G}{\sqrt{\bar\rho}}, \frac{\nabla\omega}{\sqrt{\bar\rho}}\right)\right\|_2^2 \mathrm{d}t \leqq C(\|\nabla u_0\|_2^2 + \bar\rho\|\sqrt{\rho_0}E_0\|_2^2). \tag{2.29}$$

The estimate for $\|\rho\|_\infty$ follows from Proposition 2.6 by (2.24), (2.27), and (2.29). $\square$

Proposition 2.8. *Assume that $2\mu > \lambda$. Let η_0, $\mathcal{N}_T$, and $\mathcal{N}_0$ be as in Proposition 2.7. Then, the following two things hold:*

(i) *There is a number $\varepsilon_0 \in (0, \eta_0)$ depending only on R, γ, μ, λ and κ, such that if*

$$\sup_{0\leqq t\leqq T}\|\rho\|_\infty \leqq 4\bar\rho, \quad \mathcal{N}_T \leqq \sqrt{\varepsilon_0}, \quad \textit{and} \quad \mathcal{N}_0 \leqq \varepsilon_0,$$

then

$$\sup_{0\leqq t\leqq T}\|\rho\|_\infty \leqq 2\bar\rho \quad \textit{and} \quad \mathcal{N}_T \leqq \frac{\sqrt{\varepsilon_0}}{2}.$$

(ii) *As a consequence of (i), the following estimates hold:*

$$\mathcal{N}_T \leqq \frac{\sqrt{\varepsilon_0}}{2} \quad \textit{and} \quad \sup_{0\leqq t\leqq T}\|\rho\|_\infty \leqq 2\bar\rho,$$

as long as $\mathcal{N}_0 \leqq \varepsilon_0$.

Proof. (i) Let $\varepsilon_0 \leqq \eta_0$ be sufficiently small. By assumption, all the conditions in Proposition 2.7 hold, and thus

$$\begin{aligned}
\mathcal{N}_T &\leqq C\bar\rho(\|\rho_0\|_3 + \bar\rho^2\|\sqrt{\rho_0}u_0\|_2^2)(\|\nabla u_0\|_2^2 + \bar\rho\|\sqrt{\rho_0}E_0\|_2^2)\\
&= C\mathcal{N}_0 \leqq C\varepsilon_0 \leqq \frac{\sqrt{\varepsilon_0}}{2}
\end{aligned}$$

and

$$\sup_{0\leqq t\leqq T}\|\rho\|_\infty \leqq \bar\rho e^{C\mathcal{N}_0^{\frac16}+C\mathcal{N}_0^{\frac12}} \leqq \bar\rho e^{C\varepsilon_0^{\frac16}+C\varepsilon_0^{\frac12}} \leqq 2\bar\rho,$$

as long as ε_0 is sufficiently small. The first conclusion follows.

(ii) Define

$$T_\# := \max\left\{\mathcal{T}\in(0,T]\,\middle|\,\mathcal{N}_\mathcal{T}\leqq\sqrt{\varepsilon_0},\ \sup_{0\leqq t\leqq\mathcal{T}}\|\rho\|_\infty\leqq 4\bar\rho\right\}.$$

Then, by (i), we have

$$\mathcal{N}_\mathcal{T}\leqq\frac{\sqrt{\varepsilon_0}}{2},\qquad \sup_{0\leqq t\leqq\mathcal{T}}\|\rho\|_\infty\leqq 2\bar\rho,\quad \forall\mathcal{T}\in(0,T_\#). \tag{2.30}$$

If $T_\# < T$, noticing that $\mathcal{N}_\mathcal{T}$ and $\sup_{0\leqq t\leqq\mathcal{T}}\|\rho\|_\infty$ are continuous on $[0,T]$, there is another time $T_{\#\#}\in(T_\#,T]$ such that

$$\mathcal{N}_{T_{\#\#}}\leqq\sqrt{\varepsilon_0}\qquad\text{and}\qquad\sup_{0\leqq t\leqq T_{\#\#}}\|\rho\|_\infty\leqq 4\bar\rho,$$

which is the contradiction to the definition of $T_\#$. Thus, we have $T_\# = T$, and the conclusion follows from (2.30) and the continuity of $\mathcal{N}_\mathcal{T}$ and $\sup_{0\leqq t\leqq\mathcal{T}}\|\rho\|_\infty$ on $[0,T]$. $\square$

The following corollary is a straightforward consequence of Proposition 2.7 and (ii) of Proposition 2.8:

Corollary 2.1. *Assume that $2\mu>\lambda$. Let ε_0 be as in Proposition 2.8 and assume that $\mathcal{N}_0\leqq\varepsilon_0$. Then, there is a positive constant C depending only on R, γ, μ, λ, κ, $\bar\rho$, $\|\rho_0\|_3$, $\|\sqrt{\rho_0}u_0\|_2$, $\|\sqrt{\rho_0}E_0\|_2$ and $\|\nabla u_0\|_2$, such that the following estimates hold:*

$$\sup_{0\leqq t\leqq T}\left(\|(\sqrt\rho E,\sqrt\rho u,\nabla u)\|_2^2+\|\rho\|_3+\|\rho\|_\infty\right)\leqq C,$$
$$\int_0^T\left(\|\left(\nabla\theta,|u|\nabla u,\sqrt\rho u_t,\nabla G,\nabla\omega\right)\|_2^2+\|\nabla u\|_6^2+\int\rho^3p\,dx\right)dt\leqq C.$$

3. Proof of Theorem 1.1

The following blow-up criteria is cited from Huang–Li [17].

Proposition 3.1. *Let $T^*<\infty$ be the maximal time of existence of a solution (ρ,u,θ) to system (1.1)–(1.3), with initial data (ρ_0,u_0,θ_0). Then,*

$$\lim_{T\to T^*}\left(\|\rho\|_{L^\infty(0,T;L^\infty)}+\|u\|_{L^s(0,T;L^r)}\right)=\infty$$

for any (s,r) such that $\frac2s+\frac3r\leqq1$ and $3<r\leqq\infty$.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let ε_0 and $\mathscr{N}_T$ be as in Proposition 2.8 and assume that $\mathscr{N}_0 \leqq \varepsilon_0$. By Proposition 2.1, there is a unique local strong solution (ρ, u, θ) to system (1.1)–(1.3), with initial data (ρ_0, u_0, θ_0). Extend the local solution (ρ, u, θ) to the maximal time of existence $T_{\max}$. If $T_{\max} = \infty$, then (ρ, u, θ) is a global solution and we are down. Assume that $T_{\max} < \infty$. Then, by Proposition 3.1, it holds that

$$\lim_{T\to T_{\max}} (\|\rho\|_{L^\infty(0,T;L^\infty)} + \|u\|_{L^4(0,T;L^6)}) = \infty. \tag{3.1}$$

By Corollary 2.1, it follows that we have $\sup_{0\leqq t\leqq T}(\|\rho\|_\infty + \|\nabla u\|_2^2) \leqq C$ which, by the Sobolev embedding inequality, gives $\|\rho\|_{L^\infty(0,T;L^\infty)} + \|u\|_{L^4(0,T;L^6)} \leqq C$ for any $T \in (0, T_{\max})$ for a positive constant C independent of T. This implies that

$$\lim_{T\to T_{\max}} (\|\rho\|_{L^\infty(0,T;L^\infty)} + \|u\|_{L^4(0,T;L^6)}) \leqq C < \infty,$$

which is in contradiction to (3.1). Therefore, we must have that $T_{\max} = \infty$, proving Theorem 1.1. $\square$

Acknowledgements. The author is grateful to the anonymous referees for the kind suggestions that improved this paper. This work was supported in part by the National Natural Science Foundation of China Grants 11971009, 11871005, and 11771156, by the Natural Science Foundation of Guangdong Province Grant 2019A1515011621, by the South China Normal University start-up Grant 550-8S0315, and by the Hong Kong RGC Grant CUHK 14302917.

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Global Solutions of Compressible Navier–Stokes Equations with Vacuum

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(*Received June 22, 2019 / Accepted March 25, 2020*)