Lecture 12: 哥德尔定理与勒布定理

熊　明

1　学习目标

(1) 了解哥德尔第一不完全性定理及其证明思想

(2) 了解哥德尔第二不完全性定理及其证明思想

(3) 了解 Henkin 问题及勒布的解决（勒布定理）

2　引导问题

(1) 哥德尔语句是如何构造出来的?

(2) 哥德尔语句的直观意思是什么?

(3) 什么是 ω-一致?

(4) 在什么意义上，哥德尔对其定理的证明是构造性的?

(5) 哥德尔第二不完全性定理中的一致性语句什么是什么?

(6) Henkin 问了什么问题，勒布是如何解决的?

3 教学纲要

Recall (arithmatization of the syntactics):

'x is a formula' is a syntactical predicate, and we want to find an arithmetical predicate (i.e., a formula in $\mathcal{L}_{\mathbb{N}}$, whose's free variables have at most one), say $\overline{formula}(x)$, which 'represents' the previous predicate.

Let $form$ be the set of the Gödel numbers of all formulas (in $\mathcal{L}_{\mathbb{N}}$), i.e.,

$$form = \{\ulcorner A \urcorner \mid A \,\text{is a formula}\}.$$

Instead of saying 'A is a formula', we can say 'the Gödel number of A belongs to the set $form$', i.e., $\ulcorner A \urcorner \in form$. Or, more intuitively, we can use $form\,(\ulcorner A \urcorner)$.

We can ACTUALLY find an arithmetical predicate $\overline{formula}(x)$, such that

- if $form\,(\ulcorner A \urcorner)$ holds, then $\text{PA} \vdash \overline{formula}(\overline{\ulcorner A \urcorner})$.
- if $form\,(\ulcorner A \urcorner)$ fails, then $\text{PA} \vdash \neg\overline{formula}(\overline{\ulcorner A \urcorner})$.

More generally, for any number m,

- if $form\,(m)$ holds, then $\text{PA} \vdash \overline{formula}(\overline{m})$.
- if $form\,(m)$ fails, then $\text{PA} \vdash \neg\overline{formula}(\overline{m})$.

In this sense, we say $\overline{formula}(x)$ represents $form\,(\ulcorner A \urcorner)$.

Remark. We can also utilize the following condition:

- $form\,(\ulcorner A \urcorner)$ holds, iff $\mathrm{PA} \vdash \overline{formula}(\overline{\ulcorner A \urcorner})$.

- if $form\,(\ulcorner A \urcorner)$ holds, then $\mathrm{PA} \vdash \overline{formula}(\overline{\ulcorner A \urcorner})$.

- if $form\,(\ulcorner A \urcorner)$ fails, then $\mathrm{PA} \nvdash \overline{formula}(\overline{\ulcorner A \urcorner})$.

In this case, we say that $\overline{formula}(x)$ defines $form\,(\ulcorner A \urcorner)$.

定义 3.1 对 $\mathcal{L}_{\mathbb{N}}$ 中的一个语句集 Σ，如果存在公式 $A(x)$ 使得 $\Sigma \vdash \exists x A(x)$，同时还使得对任意自然数 n，$\Sigma \vdash \neg A\,(\overline{n})$，那么称 Σ 是 ω-**不一致的**。否则，称 Σ 是 ω-**一致的**。(Remark: $\omega = \mathbb{N}$)

定理 3.2 (**哥德尔第一不完全性定理**) 在 $\mathcal{L}_{\mathbb{N}}$ 中存在一个语句 G，使得

(1) 若 PA 是一致的，则 $\mathrm{PA} \nvdash G$。

(2) 若 PA 是 ω-一致的，则 $\mathrm{PA} \nvdash \neg G$。

证 明: Consider the binary syntactical relation $\mathrm{prf}(x, y)$ whose intended meaning is that $\mathrm{prf}(m, n)$, iff m is the Gödel number of a proof in PA, and the last formula in this proof is the one whose Gödel number is precisely n.

It is EVIDENT that $\mathrm{prf}(x, y)$ is effectively decidable. Then, it can be represented by a formula, say $\mathrm{Prf}(x, y)$.

Consider the formula $A(y) = \neg\exists x \text{Prf}(x, y)$, by the Gödel diagonal lemma, we can find a sentence G, such that

$$\text{PA} \vdash G \leftrightarrow A(\overline{\ulcorner G \urcorner}).$$

$$\text{PA} \vdash G \leftrightarrow \neg\exists x \text{Prf}(x, \overline{\ulcorner G \urcorner}).$$

Assume that $\text{PA} \vdash G$, then there exists a number m, such that m is the Gödel number of a proof in PA and G is the last formula of this proof. It means $\text{prf}(m, \ulcorner G \urcorner)$. By the definition of representation, we know

$$\text{PA} \vdash \text{Prf}\left(\overline{m}, \overline{\ulcorner G \urcorner}\right).$$

By logic,

$$\text{PA} \vdash \exists x \text{Prf}\left(x, \overline{\ulcorner G \urcorner}\right).$$

i.e.,

$$\text{PA} \vdash \neg G.$$

Thus, PA is inconsistent, a contradiction!

Suppose PA is ω-consistent, and suppose $\mathrm{PA} \vdash \neg G$, then on the one hand,

$$\mathrm{PA} \vdash \exists x \mathrm{Prf}(x, \overline{\ulcorner G \urcorner}).$$

On the other hand, we have $\mathrm{PA} \nvdash G$, and so for any number n, $\mathrm{prf}(n, \ulcorner G \urcorner)$ fails. By the definition of representation, we obtain,

$$\text{for any number } n, \mathrm{PA} \vdash \neg \mathrm{Prf}(\overline{n}, \overline{\ulcorner G \urcorner})$$

This is not equivalent to

$$\mathrm{PA} \vdash \forall x \neg \mathrm{Prf}(x, \overline{\ulcorner G \urcorner})$$

Then, we know PA is ω-inconsistent, contradiction!

鬼斧神工，化腐朽为神奇！

$$\text{sentence (1) is untrue} \tag{1}$$

$$\text{sentence (2) is unprovable (in PA)} \tag{2}$$

Suppose sentence (2) is provable, then it must be true, and so sentence (2) is unprovable, a contradiction!

Thus, sentence (2) is unprovable!

It follows immediately that it must be true!

Hence, we obtain a true but unprovable sentence! In other words, this sentence and its negation are both unprovable!

Sentence (2) is a typical example of self-referential sentences.

定理 3.3 (**哥德尔-罗塞尔不完全性定理**) 在 $\mathcal{L}_{\mathbb{N}}$ 中存在一个语句 R，使得若 PA 是一致的，则 $\mathrm{PA} \nvdash R$，并且 $\mathrm{PA} \nvdash \neg R$。

Notation: $\mathrm{Bew}(y) = \exists x \mathrm{Prf}(x, y)$. Intuitively, $\mathrm{Bew}(\overline{\ulcorner A \urcorner})$, i.e., $\exists x \mathrm{Prf}(x, \overline{\ulcorner A \urcorner})$ denotes that A is provable (in PA).

$\neg \mathrm{Bew}\left(\overline{\ulcorner \bot \urcorner}\right)$ denotes that PA is consistent. 哥德尔第二不完全性定理 says that if PA is consistent, then PA $\nvdash$ 'PA is consistent'.

定理 3.4 (**哥德尔第二不完全性定理**) 若 PA 是一致的,则 $\mathrm{PA} \nvdash \neg \mathrm{Bew}\left(\overline{\ulcorner \bot \urcorner}\right)$。

证明:

- 若 PA 是一致的，则 $\mathrm{PA} \nvdash G$。

We can formalize the above statement as

$$\neg \mathrm{Bew}\left(\overline{\ulcorner \bot \urcorner}\right) \to \neg \mathrm{Bew}(\overline{\ulcorner G \urcorner}).$$

To prove Gödel second incompleteness theorem, we need to prove

$$\mathrm{PA} \vdash \neg\mathrm{Bew}\left(\overline{\ulcorner \bot \urcorner}\right) \to \neg\mathrm{Bew}(\overline{\ulcorner G \urcorner}),$$

i.e.,

$$\mathrm{PA} \vdash \mathrm{Bew}(\overline{\ulcorner G \urcorner}) \to \mathrm{Bew}\left(\overline{\ulcorner \bot \urcorner}\right),$$

引理 3.5 (**可推演条件**) 对任何语句 A、B，

(1) 如果 $\mathrm{PA} \vdash A$，那么 $\mathrm{PA} \vdash \mathrm{Bew}\left(\overline{\ulcorner A \urcorner}\right)$。

(2) $\mathrm{PA} \vdash \mathrm{Bew}\left(\overline{\ulcorner A \to B \urcorner}\right) \to \mathrm{Bew}\left(\overline{\ulcorner A \urcorner}\right) \to \mathrm{Bew}\left(\overline{\ulcorner B \urcorner}\right)$。

(3) $\mathrm{PA} \vdash \mathrm{Bew}\left(\overline{\ulcorner A \urcorner}\right) \to \mathrm{Bew}\left(\overline{\ulcorner \mathrm{Bew}\left(\overline{\ulcorner A \urcorner}\right) \urcorner}\right)$。

Henkin 问题：We know $\mathrm{PA} \vdash G \leftrightarrow \neg\mathrm{Bew}\left(\overline{\ulcorner G \urcorner}\right)$.

Consider the sentence $\mathrm{PA} \vdash H \leftrightarrow \mathrm{Bew}\left(\overline{\ulcorner H \urcorner}\right)$. Question: is the sentence H provable or not?

It should be provable!

定理 3.6 (**勒布定理**) 对任何闭式 A，若 $\mathrm{PA} \vdash \mathrm{Bew}\left(\overline{\ulcorner A \urcorner}\right) \to A$，则 $\mathrm{PA} \vdash A$。特别地，亨金语句 H 在一阶算术中是可证的。

Curry's paradox:

$$\text{if sentence (3) is true, then Santa Claus exists.} \tag{3}$$

Suppose sentence (3) is true, then if sentence (3) is true, then Santa Claus exists. Then, Santa Claus exists.

Thus, if sentence (3) is true, then Santa Claus exists.

Hence, sentence (3) is true.

Therefore, Santa Claus exists!

4 课后任务

Quiz 1 阅读我的讲义 2.6、2.7、2.8 节.