# Lecture 11: 句法的算术化与对角线引理

熊　明

## 1　学习目标

(1) 了解句法算术化的基本思想

(2) 了解哥德尔编号的基本过程

(3) 了解 (能行) 可判定、(能行) 可计算的直观定义

(4) 了解可判定关系的 $\Sigma_1$ 可表示性

(5) 了解可计算关系的 $\Sigma_1$ 可表示性

(6) 了解对角线引理及其证明思想

## 2　引导问题

(1) 从原始符号到公式到公式序列，如何一步步进行编码?

(2) 什么叫程序?

(3) 什么是能行可判定关系?

(4) 什么是能行可计算函数?

(5) 句法是如何算术化的?

(6) 对角线引理的对角线在哪里?

## 3 教学纲要

Table 1 初始符号的编码

| $\theta$ | ( | ) | $\neg$ | $\vee$ | $\wedge$ | $\rightarrow$ | $\forall$ | $\exists$ | $\equiv$ | $\boldsymbol{S}$ | $+$ | $\cdot$ | $\mathbf{0}$ | $v_i$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\#(\theta)$ | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 | 19 | 21 | 23 | 25 | 1 | $2i+27$ |

项和公式的编码:

a sequence of symbols, for instance, $)\mathbf{0} \rightarrow \boldsymbol{S}+$, is coded by the number

$$2^5 3^1 5^{13} 7^{21} 11^{23} = ****.$$

$5 \times 1 \times 13 \times 21 \times 23 = ****$

The term $\overline{2}$, i.e., $\boldsymbol{SS}\mathbf{0}$, is coded by the number

$$2^{21} 3^{21} 5^1$$

In particular, a sequence of symbols, $\theta_1\theta_2\ldots\theta_n$, is coded by the number

$$p_1^{\#(\theta_1)} p_2^{\#(\theta_2)} \ldots p_n^{\#(\theta_n)},$$

where for any $i \leq n$, $\theta_i$ is a primitive symbol of $\mathcal{L}_{\mathbb{N}}$, $p_i$ is the $i$-th prime number, that is, $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, ldots. The above number is also called 'the Gödel number (code)' of the sequence $\theta_1\theta_2\ldots\theta_n$.

Quiz 1 The Gödel number of the formula, $\neg\mathbf{0} \equiv \boldsymbol{S}\mathbf{0}$, is ???

公式序列的编码:

Given a sequence of $n$ formulas, $A_1$, $A_2$, ..., $A_n$, we can code this sequence by the number

$$p_1^{\ulcorner A_1 \urcorner} p_2^{\ulcorner A_2 \urcorner} \ldots p_n^{\ulcorner A_n \urcorner}$$

where for any $i \leq n$, $\ulcorner A_i \urcorner$ denotes the Gödel number of $A_i$.

Consider the following sequence:

(1) $\forall v_0(v_0 + \mathbf{0} \equiv v_0)$ $\qquad 2^{15}3^{27}5^{3}7^{27}11^{23}13^{1}17^{19}19^{27}23^{5}$

(2) $\forall v_0(v_0 + \mathbf{0} \equiv v_0) \rightarrow \overline{1} + \mathbf{0} \equiv \overline{1}$

$$* = 2^{15}3^{27}5^{3}7^{27}11^{23}13^{1}17^{19}19^{27}23^{5}29^{13}31^{21}37^{1}41^{23}43^{1}47^{19}53^{21}57^{1}$$

(3) $S\mathbf{0} + \mathbf{0} \equiv S\mathbf{0}$ $\qquad 2^{21}3^{1}5^{23}7^{1}11^{19}13^{21}17^{1}$

The Gödel number of the above sequence is

$$2^{2^{15}3^{27}5^{3}7^{27}11^{23}13^{1}17^{19}19^{27}23^{5}}3^{*}5^{2^{21}3^{1}5^{23}7^{1}11^{19}13^{21}17^{1}}$$

Trick of Gödel's numbering (coding) lies in that not only can we calculate the Gödel number of an expression or a sequence of expressions, but also we can conversely figure out what the expression or the sequence of expressions is once we know the corresponding Gödel number.

The predicate '... is a primitive symbol in $\mathcal{L}_{\mathbb{N}}$'.

The statement: **0** is a primitive symbol.

**1** is not a primitive symbol.

Motivation: we want to arithmetize the statement '**0** is a primitive symbol.' That is, we want to find an arithmetic statement whose meaning is precisely what '**0** is a primitive symbol' expresses.

For this purpose, we would like to find a formal predicate $ps(x)$, which denotes the predicate '$x$ is a primitive symbol'.

Consider the set $\{1, 3, 5, \ldots, \}$, the set of all odd numbers. This set is exactly the set including all Gödel numbers of the primitive symbols. The statement '**0** is a primitive symbol' can be represent as '1 is an odd number', which can even formalized in $\mathcal{L}_{\mathbb{N}}$ as $\exists v(\overline{1} \equiv v \cdot v + \overline{1})$.

**1** is not a primitive symbol.

'**0** is a term'

'$S\mathbf{0}$ is a term'

'$\forall v_0(v_0 + \mathbf{0} \equiv v_0)$ is a formula'

'$\forall v_0(v_0 + \mathbf{0} \equiv v_0)$ is a sentence'

**定义** 3.1 (**非形式定义**) $R$ 是自然数集上的 $k$ 元关系，如果存在一个"程序"，使得不论"输入"什么样的自然数 $n_1$、⋯、$n_k$，都能根据这个程序判定 $R(n_1, \ldots, n_k)$ 是否成立，那么就称 $R$ 是**（能行）可判定的**。

**定义** 3.2 (**非形式定义**) $f$ 是自然数集上的 $k$ 元函数，如果存在一个"程序"，使得不论"输入"什么样的自然数 $n_1$、⋯、$n_k$，都能根据这个程序计算出 $f(n_1, \ldots, n_k)$ 的取值，那么就称 $f$ 是**（能行）可计算的**。

**定理** 3.3 (**非形式定理**) 自然数集上的可判定关系在一阶算术中是**$\Sigma_1$-可表达的**。具体而言，如果 $R$ 是可判定的 $k$ 元关系，那么存在 $\Sigma_1$ 公式 $A(x_1, \ldots, x_k)$，使得对任意自然数 $n_1, \ldots, n_k$，

(1) 若 $R(n_1, \ldots, n_k)$ 成立，则 $\mathrm{PA} \vdash A\left(\overline{n_1}, \ldots, \overline{n_k}\right)$。

(2) 若 $R(n_1, \ldots, n_k)$ 不成立，则 $\mathrm{PA} \vdash \neg A\left(\overline{n_1}, \ldots, \overline{n_k}\right)$。

**定理** 3.4 (**非形式定理**) 自然数集上的可计算函数在一阶算术中是**$\Sigma_1$-可表达的**。具体而言，如果 $f$ 是可计算的 $k$ 元函数，那么存在 $\Sigma_1$ 公式 $A(x_1, \ldots, x_k, y)$，使得对任意自然数 $n_1, \ldots, n_k, m$，若 $f(n_1, \ldots, n_k) = m$，则

$$\mathrm{PA} \vdash \forall v\left(A\left(\overline{n_1}, \ldots, \overline{n_k}, v\right) \rightarrow v \equiv \overline{m}\right).$$

**引理** 3.5 (**对角线引理**) For any formula $A(v_1)$, there exists a sentence $\delta$ such that

$$\mathrm{PA} \vdash \delta \leftrightarrow A\left(\overline{\ulcorner \delta \urcorner}\right).$$

For example, $A(v_1)$ is the formula $v_1 \equiv \mathbf{0}$, then we find $\delta$, such that

$$\mathrm{PA} \vdash \delta \leftrightarrow \overline{\ulcorner \delta \urcorner} \equiv \mathbf{0}$$

**证明：** We define a function $d$ as follows: for any number $n$, if $n$ is the Gödel number of a formula $B(v_0)$, then $d(n)$ is the Gödel number of $B(\overline{n})$; otherwise, $d(n) = n$. Then, it is evident that the function $d$ is effectively computable (we usually show this point by giving an intuitive procedure (program)). Then by Theorem 4, we can find a formula $D(v_0, v_1)$, which represents the function $d$.

We have:

(1) if $d(n) = m$, then $\mathrm{PA} \vdash D(\overline{n}, \overline{m})$.

(2) if $d(n) \neq m$, then $\mathrm{PA} \vdash \neg D(\overline{n}, \overline{m})$.

(3) $\mathrm{PA} \vdash \forall x \left(D(\overline{n}, x) \to x = \overline{d(n)}\right)$.

Consider the formula $\exists v_1 (D(v_0, v_1) \wedge A(v_1))$, and let it be $B(v_0)$. Let $n$ be its Gödel number. We denote $B(\overline{n})$ with $\delta$. We now prove $\delta$ satisfies the desired condition.

$$\mathrm{PA} \ \vdash \ \delta \leftrightarrow \exists v_1 \left(D(\overline{n}, v_1) \wedge A(v_1)\right)$$

Let $d(n) = m$. Then by definition of $d$, $m$ is the Gödel number of $B(\overline{n})$, i.e., Gödel number of $\delta$.

$$\mathrm{PA} \ \vdash \ D\left(\overline{n}, \overline{m}\right) \wedge \forall x \left(D\left(\overline{n}, x\right) \rightarrow x = \overline{m}\right)$$

Then,

$$\mathrm{PA} \ \vdash \ \exists v_1 \left(D(\overline{n}, v_1) \wedge A(v_1)\right) \leftrightarrow D(\overline{n}, \overline{m}) \wedge A(\overline{m})$$

Thus, we get

$$\mathrm{PA} \ \vdash \ \delta \leftrightarrow D(\overline{n}, \overline{m}) \wedge A(\overline{m})$$

$$\mathrm{PA} \ \vdash \ \delta \leftrightarrow \overline{d(n)} \equiv \overline{m} \wedge A(\overline{m})$$

$$\mathrm{PA} \ \vdash \ \delta \leftrightarrow A(\overline{d(n)})$$

$$\text{PA} \vdash \delta \leftrightarrow A\left(\ulcorner \delta \urcorner\right)$$

## 4 课后任务

**问题** 4.1 完成我的讲义 2.4 和 2.5 节.