Lecture 13: 无参数的不动点定理

熊　明

1　学习目标

(1) 了解不动点定理的背景

(2) 了解不动点定理中不动点的定义

(3) 了解无参数不动点定理的证明

2　引导问题

(1) 哥德尔语句是哪个公式的不动点?

(2) 一致性语句是哪个公式的不动点?

(3) 亨金语句是哪个公式的不动点?

(4) 给定一个 p 在其中不自由的 p 公式，如何求其不动点?

3　教学纲要

动机:

PA 中可证的事实:

$G \leftrightarrow \neg \mathrm{Bew}\left(\overline{\ulcorner G \urcorner}\right)$	$G \leftrightarrow \neg \mathrm{Bew}\left(\overline{\ulcorner \bot \urcorner}\right)$
$H \leftrightarrow \mathrm{Bew}\left(\overline{\ulcorner H \urcorner}\right)$	$H \leftrightarrow \top$
$\delta \leftrightarrow \mathrm{Bew}\left(\overline{\ulcorner \neg\delta \urcorner}\right)$	$\delta \leftrightarrow \mathrm{Bew}\left(\overline{\ulcorner \bot \urcorner}\right)$
$\delta \leftrightarrow \neg \mathrm{Bew}\left(\overline{\ulcorner \neg\delta \urcorner}\right)$	$\delta \leftrightarrow \bot$
$\delta \leftrightarrow (\mathrm{Bew}\left(\overline{\ulcorner \delta \urcorner}\right) \rightarrow \mathrm{Bew}\left(\overline{\ulcorner \neg\delta \urcorner}\right))$	$\delta \leftrightarrow \left(\mathrm{Bew}\left(\overline{\ulcorner \mathrm{Bew}\left(\overline{\ulcorner \bot \urcorner}\right) \urcorner}\right) \rightarrow \mathrm{Bew}\left(\overline{\ulcorner \bot \urcorner}\right)\right)$

反映到 GL：

方程	无变元的解
$p \leftrightarrow \neg\Box p$	$\neg\Box\bot$
$p \leftrightarrow \Box p$	$\top$
$p \leftrightarrow \Box\neg p$	$\Box\bot$
$p \leftrightarrow \neg\Box\neg p$	$\bot$
$p \leftrightarrow (\Box p \rightarrow \Box\neg p)$	$\Box\Box\bot \rightarrow \Box\bot$

Recall $\boxdot A = \Box A \wedge A$.

Observation: if $A(p)$ is a formula of modal language, in which only p is its variable and any occurrence of p is in the scope of some $\Box$, then there exists a variable-free (=letterless) formula F such that

$$\mathrm{GL} \vdash \boxdot(p \leftrightarrow A(p)) \leftrightarrow \boxdot(p \leftrightarrow F).$$

Equivalently,

$$\mathrm{GL} \vdash \boxdot(p \leftrightarrow A(p)) \to (p \leftrightarrow F).$$

It follows

$$\mathrm{GL} \vdash F \leftrightarrow A(F).$$

Let $A(p) = \Box p \leftrightarrow \Box\neg p$. We come to find a variable-free fixed point of $A(p)$, i.e., a variable-free formula F such that

$$\mathrm{GL} \vdash \boxdot(p \leftrightarrow A(p)) \to (p \leftrightarrow F).$$

Recall: $\mathrm{GL} \vdash C$, iff C is valid in any finite, transitive and irreflexive model.

Thus, we only need to find F such that for any finite, transitive and irreflexive model $\mathcal{M}$, if $\mathcal{M} \models \boxdot(p \leftrightarrow A(p))$, then $\mathcal{M} \models p \leftrightarrow F$. Equivalently, it is sufficient to make sure that if $\mathcal{M} \models p \leftrightarrow A(p)$, then $\mathcal{M} \models p \leftrightarrow F$.

Suppose $\mathcal{M}$ is a finite, transitive and irreflexive model such that $\mathcal{M} \models p \leftrightarrow A(p)$, we investigate the semantical features of the variable p in the model $\mathcal{M}$.

Since $\mathcal{M}$ is conversely well-founded, there exists a dead end, i.e., a point which is accessible to any point. We all know that any formula of form $\Box C$ is true at dead end, more formally, $\mathcal{M}, 0 \models \Box C$, which 0 is a dead end in the (domain of) model $\mathcal{M}$. Thus, $\mathcal{M}, 0 \models A(p)$. But $\mathcal{M} \models p \leftrightarrow A(p)$, and so $\mathcal{M}, 0 \models p$.

Table 1　秩决定 $\Box p \leftrightarrow \Box\neg p$ 的真假

	$\Box p$	$\Box\neg p$	$\Box p \leftrightarrow \Box\neg p$	p	$\neg p$	$\bot$	$\Box\bot$	$\Box\Box\bot$	$\neg\Box\bot$	$\Box\Box\bot \wedge \neg\Box\bot$
0	T	T	T	T	F	F	T	T	F	F
1	T	F	F	F	T	F	F	T	T	T
2	F	F	T	T	F	F	F	F	T	F
3	F	F	T	T	F	F	F	F	T	F

$\neg(\neg\Box\bot \wedge \Box\Box\bot)$, i.e. $\Box\bot \vee \neg\Box\Box\bot$ is the very formula F that we want to find.

Define the notion of rank:

A point of rank 0 is a dead end.

a point of rank 1 is a point which accessible to at least a point of rank 0, and all accessible points from which are of rank 0.

a point of rank 2 is a point which accessible to at least a point of rank 1, and all accessible points from which are of rank ≤ 1.

a point of rank 3 is a point which accessible to at least a point of rank 2, and all accessible points from which are of rank ≤ 2.

We find: the truth value of p at a point is determined by the rank of that point.

We come to verify that a variable-free fixed point of $A(p) = \Box\neg p$ is $\Box\bot$. Suppose $\mathcal{M} \models p \leftrightarrow \Box\neg p$.

Table 2 验证已知无变元不动点的例子

	$\Box\neg p$	p	$\neg p$	$\bot$	$\Box\bot$	$\neg\Box\bot$
0	T	T	F	F	T	F
1	F	F	T	F	F	T
2	F	F	T	F	F	T

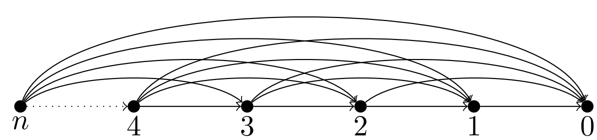

Figure 1 框架 $\langle\{0, 1, \ldots, n\}, >\rangle$

令 $A(p) = \neg\Box p$，求这个公式的无变元不动点，即满足 $\mathrm{GL} \vdash \boxdot(p \leftrightarrow A(p)) \leftrightarrow \boxdot(p \leftrightarrow H)$ 的无变元公式。为此，只需给出表格 4。从这个表格，直接看出带求不动点为 $\neg\Box\bot$。

求 $A(p) = \Box(\neg p \leftrightarrow \Box\bot)$ 的无变元不动点。表列如下。

由这个表可知，待求不动点为 $\Box\bot$。

Table 3 验证已知无变元不动点的例子

	$\Box p$	$\neg\Box p$	p	$\bot$	$\Box\bot$	$\neg\Box\bot$
0	T	F	F	F	T	F
1	F	T	T	F	F	T
2	F	T	T	F	F	T

Table 4 验证已知无变元不动点的例子

	$\Box\bot$	$\Box(\neg p \leftrightarrow \Box\bot)$	p	$\neg p$	$\neg p \leftrightarrow \Box\bot$	$\bot$
0	T	T	T	F	F	F
1	F	F	F	T	F	F
2	F	F	F	T	F	F

4 课后任务

问题 4.1 阅读我的讲义 3.2 节.

问题 4.2 阅读 Boolos (1993), pp. 104-111.