Countable and Uncountable Sets

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Outline

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1. Cardinality of Sets
2. Countable Sets
3. Uncountable Sets

- Definition: Sets A and B are **equipotent** (have **the same cardinality**) if there is a one-to-one function f from A onto B. We denote this by $|A| = |B|$.
- Fact: The equipotency relation is an equivalence relation.

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- Example: $|\mathbb{N}| = |\{0, 1, 4, \ldots, n^2, \ldots\}|$

- Example: $|\mathbb{N}| = |\mathbb{Z}|$

- Example: $|(3, 2022)| = |(0, 1)|$, where $(0, 1)$ and $(3, 2022)$ are both intervals of real numbers.

- Example: $|\mathbb{R}| = |(0,1)|$, where $(0,1)$ is an interval of real numbers.

- Definition: **The cardinality of A is less than or equal to the cardinality of B** (notation: $|A| \leq |B|$) if there is a one-to-one mapping of A into B. $|A| = |B|$.
- Fact: The above relation $\leq$ is reflexive and transitive.

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Theorem (Cantor-Bernstein Theorem)

If $|X| \leq |Y|$ *and* $|Y| \leq |X|$, *then* $|X| = |Y|$.

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- Definition: A set S is **countable** if $|S| = |\mathbb{N}|$. A set S is **at most countable** if $|S| \leq |\mathbb{N}|$.
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- Example: $\mathbb{Z}$ is countable (again).
- Generally, If A and B are countable, then $A \cup B$ is countable.

- Example: $\mathbb{N} \times \mathbb{N}$ is countable.
- Generally, If A and B are countable, then $A \times B$ is countable.

- Example: $\mathbb{Q}$ is countable.

- Definition: $\mathrm{Seq}(\mathbb{N}) =_{\mathrm{df}} \bigcup\{\mathbb{N}^n | n \in \mathbb{N}\}$, i.e., the set of all finite sequence of natural numbers.
- Example: $\mathrm{Seq}(\mathbb{N})$ is countable.
- Generally, if A is countable, then $\mathrm{Seq}(A)$ is countable.

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- Example: The interval $(0, 1)$ is uncountable.

- Example: The set $\mathscr{P}(\mathbb{N})$ is uncountable.
- Generally, $|A| < |\mathscr{P}(A)|$.

- Definition: $2^{\mathbb{N}} =_{\text{df}} \{f | f : \mathbb{N} \to 2\}$.
- Example: The set $2^{\mathbb{N}}$ is uncountable.

The Continuum Hypothesis

A summary:

- $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|$.
- $|\mathbb{R}| = |(0,1)| = |(a,b)|$, where $a < b$.
- $|\mathbb{N}| < |\mathbb{R}|$.

The Continuum Hypothesis: There is no set A such that $|\mathbb{N}| < |A| < |\mathbb{R}|$.

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Homework

Prove:

- $|2^A| = |\mathscr{P}(A)|$.
- $|2^{\mathbb{N}}| = |\mathbb{R}|$.
- $|2^{\mathbb{N}}| = |\mathbb{N}^{\mathbb{N}}|$.
- $|\mathbb{N}| < |\mathrm{Sym}(\mathbb{N})|$, where $\mathrm{Sym}(\mathbb{N})$ is the set of all permutations on $\mathbb{N}$. A permutation on $\mathbb{N}$ is a bijection from $\mathbb{N}$ to $\mathbb{N}$.

Thanks for your attention!

Q & A