From Orderings to Ordinals

Ming Hsiung
mingshone@163.com

School of Philosophy and Social Development
South China Normal University

Set Theory, Spring 2022

Outline

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Definition:

- A binary relation R in A is **asymmetric** if $\forall x, y \in A(xRy \to \neg yRx)$.
- A binary relation R in A is **transitive** if $\forall x, y, z \in A(xRy \wedge yRz \to \neg xRz)$.
- A binary relation R in A is a **strict (partial) ordering** if it is asymmetric and transitive.
- A strict ordering R in A is **linear** or **total** if any two elements of A are **comparable** by R, i.e., $\forall x, y \in A(xRy \vee yRx \vee x = y)$.

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A stipulation

- When R is a (partial) ordering in A, we can equivalent say:
- A is **(partially) ordered** by A;
- (A, R) is **(partially) ordered** set.

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Definition: Let $(P, <)$ and $(P', <')$ be two (strict) ordered sets.

- An **isomorphism** from $(P, <)$ to $(P', <')$ is a one-to-one correspondence h from P to P' such that for all $p_1, p_2 \in P$,

$$p_1 < p_2 \text{ iff } h(p_1) <' h(p_2).$$

If an isomorphism exists between $(P, <)$ and $(P, <')$, then $(P, <)$ and $(P', <')$ are **isomorphic**.

Quiz

In the following, $<$ always denotes the less-than relation on the relevant sets.

- Are $(\mathbb{N}, <)$ and $(\{n^2 | n \in \mathbb{N}\}, <)$ isomorphic.
- How about $(\mathbb{N}, <)$ and $(\mathbb{Z}, <)$?
- How about $(\mathbb{Z}, <)$ and $(\mathbb{Q}, <)$?
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- A relation R on A is **well-founded**, if for every nonempty subset B of A, there exists a R-minimal element of B, i.e., an element $x \in B$ such that yRx fails for any $y \in B$.
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- Definition: If a be an element of a well-ordered set $(W, <)$, we call the set
$$W[a] = \{x \in W | x < a\}$$
the **initial segment of W given by a**.

- Theorem: If $(W_1, <_1)$ and $(W_2, <_2)$ are well-ordered sets, then exactly one of the following holds:
  (a) either W_1 and W_2 are isomorphic, or
  (b) W_1 is isomorphic to $W_2[a_2]$ for some $a_2 \in W_2$, or
  (c) W_2 is isomorphic to $W_1[a_1]$ for some $a_1 \in W_1$.

  In each case, the isomorphism is unique.

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- Definition: A set A is **transitive** if every element of A is a subset of A.
- Definition: A set is an **ordinal (number)** if it is a transitive and well-ordered by $\in$.
- Fact: Every natural number is an ordinal.
- Fact: The set $\mathbb{N}$ is an ordinal.
- Notation: $\omega =_{\text{df}} \mathbb{N}$.

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Homework

Let $(W_1, <_1)$ and $(W_2, <_2)$ be two disjoint well-ordered sets.

- Define $<$ on $W_1 \cup W_2$ by: $a < b$ iff $(a < b \wedge a \in W_1 \wedge b \in W_1) \vee (a < b \wedge a \in W_2 \wedge b \in W_2) \vee (a \in W_1 \wedge b \in W_2)$.
  Question: Is $(W_1 \cup W_2, <)$ a well-ordered set?

Homework

Let $(W_1, <_1)$ and $(W_2, <_2)$ be two well-ordered sets.

- Define $<$ on $W_1 \times W_2$ by: $(a_1, a_2) < (b_1, b_2)$ iff $a_1 <_1 b_1 \vee (a_1 = b_1 \wedge a_2 <_2 b_2)$.
  Question: Is $(W_1 \times W_2, <)$ a well-ordered set?
- Define $<$ on $W_1 \times W_2$ by: $(a_1, a_2) < (b_1, b_2)$ iff $a_1 <_1 b_1 \wedge a_2 <_2 b_2$.
  Question: Is $(W_1 \times W_2, <)$ a well-ordered set?

Homework

- Prove: any two countable dense linearly ordered sets without endpoints are isomorphic. (See Theorem 4.9)

Thanks for your attention!

Q & A