Transfinite Recursion on Ordinals

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Outline

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- Theorem: Every well-ordered set is isomorphic to a unique ordinal number.

- The Axiom Schema of Replacement: Let $P(x, y)$ be a property such that for every x, there is a unique y for which $P(x, y)$ holds. For every set A, there is a set B such that, for every $x \in A$, there is $y \in B$ for which $P(x, y)$ holds.
- If A is a set, then $F[A] = \{y | y = F(x), x \in A\}$ is also a set for

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where B is the set that the Axiom Schema of Replacement admits (and $P(x, y)$ is the relation $F(x) = y$).

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- The Transfinite Induction Principle: Let $P(x)$ be a property (possibly with parameters). Assume that, for all ordinal numbers α:

$$\text{If } P(\beta) \text{ holds for all } \beta < \alpha, \text{ then } P(\alpha).$$

Then $P(\alpha)$ holds for all ordinals α.

- The Transfinite Induction Principle, the Second Version: Let $P(x)$ be a property. Assume that
  (a) $P(0)$ holds.
  (b) $P(\alpha)$ implies $P(\alpha+1)$ for all ordinals α.
  (c) For all non-zero limit α, if $P(\beta)$ holds for all $\beta < \alpha$, then $P(\alpha)$ holds.

  Then $P(\alpha)$ holds for all ordinals α.

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- Definition: Let A be a class. If $P(x, y)$ is a property such that for each $x \in A$, there exists a unique y such that $P(x, y)$ (formally, ZFC $\vdash \forall x \in A \exists! y P(x, y)$), then we say that $P(x, y)$ defines a (unary) **operation** (on A).
- As usual, we use $G(x)$ to denote an operation. That is, $G(x) = y =_{\text{df}} P(x, y)$.
- Similarly, we can define a binary operation.

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Example

- The successor operation S is an operation (on V): $S(x) = y =_{\text{df}} y = x \cup \{x\}$.
- The successor operation S is also an operation on the class of all ordinals (**On**): $S(\alpha) = \beta =_{\text{df}} \beta = \alpha \cup \{\alpha\}$.
- The powerset operation $\mathscr{P}$ is an operation: $\mathscr{P}(x) = y =_{\text{df}} y = \{z | z \subseteq x\}$.
- The intersection operation $\cap$ is an operation: $x_1 \cap x_2 = y =_{\text{df}} y = \{z | z \in x_1 \wedge z \in x_2\}$.

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- **The Transfinite Recursion Theorem:** Let G be an operation; then there exists a unique operation F on the class of all the ordinals such that $F(\alpha) = G(F \restriction \alpha)$ for all ordinals α.

- **The Transfinite Recursion Theorem, Parametric Version:** Let G be an operation; then there exists a unique operation F on $V \times \mathrm{On}$ such that $F(z, \alpha) = G(z, F \upharpoonright \alpha)$ for all ordinals α.

- Application 1: For every ordinal β, there exists a unique function $F_\beta : \mathrm{On} \to \mathrm{On}$ such that
  (a) $F_\beta(0) = \beta$;
  (b) $F_\beta(S(\alpha)) = S(F_\beta(\alpha))$ for all α.
  (c) $F_\beta(\alpha) = \bigcup\{F_\beta(\gamma) | \gamma < \alpha\}$ for all non-zero limit α.

- Definition.

$$\beta + \alpha =_{\mathrm{df}} F_\beta(\alpha).$$

- Thus: for all $\alpha, \beta \in \mathrm{On}$,
  (a) $\beta + 0 = \beta$;
  (b) $\beta + S(\alpha) = S(\beta + \alpha)$.
  (c) $\beta + \alpha = \bigcup\{\beta + \gamma | \gamma < \alpha\}$ if α is a non-zero limit.

- Theorem: Let $(A, <_1)$ and $(B, <_2)$ be two disjoint well-ordered sets, isomorphic to ordinals α and β, respectively, and let $<$ be a relation on $A \cup B$ defined by

$$\begin{aligned} a < b \quad =_{\text{df}} \quad & (a, b \in A \wedge a <_1 b) \\ & \vee (a, b \in B \wedge a <_2 b) \\ & \vee (a \in A \wedge b \in B) \end{aligned}$$

  Then $(A \cup B, <)$ is isomorphic to the ordinal $\alpha + \beta$.

- Thus: $1 + \omega = \omega$, but $\omega + 1 \neq \omega$.

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- Definition: for all $\alpha, \beta \in \mathrm{On}$,
  (a) $\beta \cdot 0 = 0$;
  (b) $\beta \cdot S(\alpha) = \beta \cdot \alpha + \beta$.
  (c) $\beta \cdot \alpha = \bigcup \{\beta \cdot \gamma | \gamma < \alpha\}$ if α is a non-zero limit.

- Theorem: Let $(A, <_1)$ and $(B, <_2)$ be two well-ordered sets, isomorphic to ordinals α and β, respectively, and let $<$ be a relation on $B \times A$ defined by

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\begin{aligned}
(b_1, a_1) < (b_2, a_2) \quad =_{\text{df}} \quad & (b_1 <_2 b_2) \\
& \vee (b_1 = b_2 \wedge a_1 <_1 a_2)
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  Then $(B \times A, <)$ is isomorphic to the ordinal $\alpha \cdot \beta$.

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  (a) $\beta^0 = 1$;
  (b) $\beta^{S(\alpha)} = \beta^\alpha \cdot \beta$.
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Homework

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Thanks for your attention!

Q & A