Cardinals and Cardinal Arithmetic

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Outline

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- Definition: An ordinal number α is called an **initial ordinal** (or a **cardinal**) if it is not equipotent to any $\beta < \alpha$.

- $|\omega| = |\omega + 1| = |\omega + \omega| = |\omega \cdot \omega|$.
- ω is a cardinal, but none of $\omega + 1, \omega + \omega, \omega \cdot \omega$ is so.

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- Definition: For any set A, let $h(A)$ be the least ordinal number which is not equipotent to any subset of A. $h(A)$ is called the **Hartog's number** of A.

- Fact: For any A, $h(A)$, if exists, is a cardinal.

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- Lemma: For any A, $h(A)$ exists.

  Proof. Let $B = \{\beta | |\beta| \leq |A|\}$.

  Claim 1: B is a set.

  Claim 2: $h(A) = B$.

- Lemma: For any cardinal κ, $h(\kappa)$ is the least cardinal that is greater than κ.

- Lemma: If A is a set of cardinals, $\bigcup A$ is also a cardinal, and it is the least upper bound of A.

- Definition: Define ω_α by transfinite recursion on α by

$$\begin{aligned} \omega_0 &= \omega \\ \omega_{\alpha+1} &= h(\omega_\alpha) \\ \omega_\alpha &= \bigcup\{\omega_\beta | \beta < \alpha\}, \text{ if } \alpha \text{ is a non-zero limit} \end{aligned}$$

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- Fact: Every ω_α is an infinite cardinal.

- Theorem: Every infinite cardinal is some ω_α.

- Notation:

$$\aleph_\alpha =_{\text{df}} \omega_\alpha$$

To avoid confusion, we employ the convention of using the ω-symbolism when the ordinal operations are involved, and the $\aleph$-symbolism for the cardinal operations.

Hrbacek & Jech, 1999, p. 132.

- $\omega_0 + \omega_0 = \omega_0 \cdot 2 \neq \omega_0$; $\aleph_0 + \aleph_0 = \aleph_0 \cdot 2 = \aleph_0$
- $2^{\omega_0} = \omega_0$; $2^{\aleph_0} > \aleph_0$.

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1. Cardinals

2. Cardinal Arithmetic

- In the following, we always use κ, λ, μ, and so on to denote the cardinals.
- Definition:

$$\kappa + \lambda =_{\mathrm{df}} |(\kappa \times \{0\}) \cup (\lambda \times \{1\})|$$

- Precisely, $\kappa + \lambda$ is the cardinal μ which is equipotent to the set $(\kappa \times \{0\}) \cup (\lambda \times \{1\})$.

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Fact:

- $\kappa + \lambda = \lambda + \kappa$
- $\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$
- $\aleph_\alpha + n = \aleph_\alpha$.
- $\aleph_\alpha + \aleph_\beta = \aleph_{\max(\alpha,\beta)}$.

- Definition:

$$\kappa \cdot \lambda =_{\text{df}} |\kappa \times \lambda|$$

Fact:

- $\kappa \cdot \lambda = \lambda \cdot \kappa$
- $\kappa \cdot (\lambda \cdot \mu) = (\kappa \cdot \lambda) \cdot \mu$
- $\kappa \cdot (\lambda + \mu) = \kappa \cdot \lambda + \kappa \cdot \mu$
- $\aleph_\alpha \cdot n = \aleph_\alpha$.

- Theorem: $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$
- Corollary: $\aleph_\alpha \cdot \aleph_\beta = \aleph_{\max(\alpha,\beta)}$.

Homework

- Proof: If α and β are at most countable ordinals then $\alpha + \beta$ is also at most countable.
- Show that the ordinal and cardinal additions $n + m$ are equal.

Thanks for your attention!

Q & A