Models and Relative Consistency

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Outline

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1. The Cumulative Hierarchy of Sets
2. Models of Set Theory
3. A Typical Relative Consistency Theorem

- Definition: for all ordinals α,

$$\begin{aligned} V_0 &= 0; \\ V_{\alpha+1} &= \mathscr{P}(V_\alpha); \\ V_\alpha &= \bigcup_{\beta<\alpha} V_\beta, \text{ if } \alpha \text{ is a non-zero limit.} \end{aligned}$$

- Fact: If $x \in V_\alpha$ and $y \in x$ then $y \in V_\beta$ for some $\beta < \alpha$.
- Fact: For all α, V_α is a transitive set.
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- Recall: A is well-founded by $\in$, if for any non-empty subset B of A, there exists an element x of B such that $x \cap B = \varnothing$.
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- **The Axiom of Foundation**: Every set is well-founded by $\in$.
- Equivalent, every set occurs in some level of the cumulative hierarchy V_α ($\alpha \in \mathrm{On}$).
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- Definition: For any formula φ and any class M we define φ^M, **the relativization of φ to M**, by:
  - $(x = y)^M$ is $x = y$.
  - $(x \in y)^M$ is $x \in y$.
  - $(\neg\varphi)^M$ is $\neg\varphi^M$.
  - $(\varphi * \psi)^M$ is $\varphi^M * \psi^M$, where $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$
  - $(\forall x\varphi)^M$ is $\forall x(x \in M \rightarrow \varphi^M)$, that is, $\forall x(P(x) \rightarrow \varphi^M)$.
  - $(\exists x\varphi)^M$ is $\exists x(x \in M \wedge \varphi^M)$.

- Definition: Let T be a theory such as ZF, ZFC and so on. Let φ be a sentence, and M be a class.
  - If $T \vdash \varphi^M$, we say that relative to T, M is a (standard) **model** of φ, or φ is **true in M**.
  - We say that relative to T, M is a **model** of a set Σ of sentences, whenever M is a model of each sentence in Σ.
- Sometimes, we can use more suggestive notation $T \vdash$ '$M \models \varphi$' instead of $T \vdash \varphi^M$.

- Notation: ZF^- denotes the theory ZF minus the Axiom of Foundation.

- Recall: M is transitive, if

$$\forall x \in M(x \subseteq M).$$

- Lemma: Relative to ZF^{-}, if M is transitive, then the Axiom of Extensionality is true in M.

Two equivalent statements:

- Lemma (ZF^{-}): If M is transitive, then the Axiom of Extensionality is true in M.
- Lemma: Working in ZF^{-}, we have: if M is transitive, then the Axiom of Extensionality is true in M.

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- Definition: M is **super-transitive**, if

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- Lemma (ZF^-): If M satisfies

$$\forall x, y \in M \exists z \in M (x \in z \wedge y \in z),$$

then the Axiom of Pair is true in M.

- Lemma (ZF^-): If M satisfies

$$\forall x \in M \exists y \in M (\bigcup x \subseteq y),$$

then the Axiom of Union is true in M.

- Lemma (ZF^-): If M satisfies

$$\forall x \in M \exists y \in M (\mathscr{P}(x) \cap M \subseteq y),$$

then the Axiom of Power Set is true in M.

- Lemma (ZF^-): If M satisfies the following condition: for any formula $\varphi(x, y)$,

$$\forall x \in A \exists! y \in M \varphi^M(x, y) \rightarrow \exists x \in M \left(\{y | \exists x \in A\, \varphi^M(x, y)\} \subseteq y \right),$$

then the Axiom Schema of Replacement is true in M.

- Lemma (ZF^-): If $\omega \in M$, then the Axiom of Infinity is true in M.

- Lemma (ZF^-): If $M \subseteq WF$, then the Axiom of Foundation is true in M.

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- Lemma: For any sentence ψ (of $\mathcal{L}_S$) and any class M, if $\vdash \psi$, then

$$\vdash \exists x(x \in M) \to \psi^M.$$

Note: About the axiom schema $\forall x \varphi(x) \to \varphi(y/x)$, we should consider its universal closure

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- Lemma: If φ_1, φ_2, ..., φ_n, ψ are sentences, and $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$, then for any class M,

$$\vdash \left(\exists x(x \in M) \wedge \varphi_1^M \wedge \varphi_2^M \ldots \wedge \varphi_n^M\right) \rightarrow \psi^M.$$

- Theorem: Let T_0 and T_1 be two theories in the language $\mathcal{L}_S$, and suppose the we can find a class M such that relative to T_0, M is a model of T_1. Then the consistency of T_0 implies that of T_1.

- Theorem: If ZF^- is consistent, so is ZF.
  Proof. If suffices to verify that relative to ZF^-, the class WF is a model of ZF.
  - The Axiom of Extensionality
  - The Axiom Schema of Comprehension
  - The Axiom of Pair
  - The Axiom of Union
  - The Axiom of Power Set
  - The Axiom of Infinity
  - The Axiom Schema of Replacement
  - The Axiom of Foundation

Homework

- Working in ZF, for each of the axioms (axiom schemata) in ZF, determine whether it is true in the set V_ω.
- Which consistency result can you get from your answer to the above question?

Thanks for your attention!

Q & A