The Axiom of Choice

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Outline

1. The Axiom of Choice
2. Some Applications of AC
3. More Cardinal Arithmetic

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- Definition: Let A and B be two sets. Define

$$^{A}B =_{\text{df}} \{f | f : A \to B\}$$

- Note: In the text (p. 26), the corresponding notation is B^A, which may be confused with the exponentiation operation.

- Question: Is there any question if we define

$$\kappa^\lambda =_{\mathrm{df}} |{}^\lambda\kappa|$$

- Recall: we define

$$\kappa \times \lambda =_{\mathrm{df}} |\kappa \times \lambda|$$

This is well-defined: the lexicographic ordering is an apparent well-ordering for the set $\kappa \times \lambda$, and so there is at least an ordinal equipotent to $\kappa \times \lambda$.

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- Question: Is there any well-ordering for the set $\mathbb{R}$?

- The Well-ordering Principle (WO): Every set can be well-ordered.

  An attempt of proof.

- **The Axiom of Choice (AC)**: For every set A, there exists a function f on A such that for every nonempty $x \in A$, $f(x) \in x$. The function f is called a **choice function** of A.
- ZF $\vdash (AC) \leftrightarrow (WO)$.

The Axiom of Choice is obviously true, the Well−Ordering Principle is obviously false; and who can tell about Zorn's Lemma.

J. L. Bona

- **Zorn's Lemma (ZL):** If every chain in a partially ordered set has an upper bound, then the partially ordered set has a maximal element.
- ZF $\vdash (AC) \leftrightarrow (ZL)$.

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- Recall: A relation R on A is **well-founded**, if for every nonempty subset B of A, there exists a R-minimal element of B, i.e., an element $x \in B$ such that yRx fails for any $y \in B$.
- Fact: A relation R on A is well-founded, iff no elements x_0, x_1, …, of A can form a decreasing chain, that is, $x_1 R x_0$, $x_2 R x_1$, ….
  Proof. The $\Leftarrow$ side needs to employ the Axiom of Choice (AC).

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- **The Principle of Dependent Choices (DC):** If R is a binary relation on a non-empty set A such that for every $x \in A$ there exists $y \in A$, then there exists a sequence of elements in A, x_0, x_1, ..., such that $x_n R x_{n+1}$ for all $n \in \mathbb{N}$.
- ZF $\vdash (AC) \rightarrow (DC)$.

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- **The Axiom of Countable Choice) (CC):** Every countable set has a choice function.
- ZF $\vdash (DC) \rightarrow (CC)$.

- Theorem (AC): For every set A, there exists a (unique) cardinal κ such that $|A| = |\kappa|$.
- Usually, we use $|A|$ to denote the cardinal such that $|A| = |\kappa|$. Thus, $|A| = |\kappa| = \kappa$.
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- Fact:
  - $\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$.
  - $(\kappa^\lambda)^\mu = \kappa^{\lambda\cdot\mu}$.
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  - $\kappa^{\lambda+\mu} = \kappa^{\lambda} \cdot \kappa^{\mu}$.
  - $(\kappa^{\lambda})^{\mu} = \kappa^{\lambda \cdot \mu}$.
  - $(\kappa \cdot \lambda)^{\mu} = \kappa^{\mu} \cdot \lambda^{\mu}$.

- Hausdorff's Formula:

$$\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1}$$

Homework

- Prove: for any sets A and B, $|A| \leq |B|$, iff there exists a function f from A **onto** B.

Thanks for your attention!

Q & A