Language for Propositional Logic

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The language of propositional logic

The language of propositional logic:

- the alphabet
- the formation rules

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The alphabet

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- proposition symbols: p_0, p_1, p_2,...
- connectives: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\perp$
- auxiliary symbols:), (.

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Formation rules

- proposition symbols are formulas, and $\bot$ is a formula. (atomic formulas)
- if φ is a formula, so is $(\neg\varphi)$.
- if φ and ψ are formulas, so are $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \to \psi)$, $(\varphi \leftrightarrow \psi)$.
- only the sequence of symbols obtained by the above rules are the formulas.

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Examples

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Induction Principle

If we can prove

- all atomic formulas have a property P;
- whenever a formula φ has the property P, $(\neg\varphi)$ does so;
- whenever formulas φ and ψ have the property P, $(\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \to \psi), (\varphi \leftrightarrow \psi)$ do so,

then we conclude that all formulas have the property P.

Example

- Prove: Each formula has an even number of brackets.

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Recursion principle

Theorem 2.1.6 (p. 10)

Example

The number of brackets of a formula can be defined as follows

Example

The number of connectives of a formula can be defined as follows

Example

The set of subformulas of a formula can be defined as follows

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Definition of valuation

A **valuation** for PL is a function v from the set of formulas to the set $\{0, 1\}$, satisfying the following conditions: for any formulas φ and ψ,

- $v(\bot) = 0$.
- $v(\neg\varphi) = 1$, iff $v(\varphi) = 0$.
- $v(\varphi \wedge \psi) = 1$, iff $v(\varphi) = 1$, $v(\psi) = 1$.
- $v(\varphi \vee \psi) = 0$, iff $v(\varphi) = 0$, $v(\psi) = 0$.
- $v(\varphi \to \psi) = 0$, iff $v(\varphi) = 1$, $v(\psi) = 0$。
- $v(\varphi \leftrightarrow \psi) = 0$, iff $v(\varphi) = v(\psi)$。

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Basic Truth Table

φ	ψ	$\neg\varphi$	$\varphi \vee \psi$	$\varphi \wedge \psi$	$\varphi \rightarrow \psi$	$\varphi \leftrightarrow \psi$
1	1	0	1	1	1	1
1	0		1	0	0	0
0	1	1	1	0	1	0
0	0		0	0	1	1

Example

- Prove: If $v(p_i) = v'(p_i)$ for all p_i occurring in φ, then $v(\varphi) = v'(\varphi)$.

Fact

Any assignment σ (that is, a function from the set of variables to $\{0, 1\}$) can be extended uniquely to be a valuation v such that $v(p_i) = \sigma(p_i)$.

Notation. $[\cdot]_\sigma$: the (unique) valuation induced from the assignment σ.

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Some notions

- $\models \varphi$ (φ is a tautology): for any valuation v, $v(\varphi) = 1$; or equivalently, for any assignment σ, $[\varphi]_\sigma = 1$
- $\varphi \iff \psi$ (φ is logically equivalent to ψ): $\models \varphi \leftrightarrow \psi$. Or equivalently, for any valuation v, $v(\varphi) = v(\psi)$
- Let Σ be a set of formulas. $\Sigma \models \varphi$ (φ is a (semantic) consequence of Σ), if $v(\varphi) = 1$ holds for any valuation v such that for any $\psi \in \Sigma$, $v(\psi) = 1$.

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Facts

- $\neg\neg\varphi \iff \varphi$
- $\neg(\varphi \wedge \psi) \iff \neg\varphi \vee \neg\psi$
- $\neg(\varphi \vee \psi) \iff \neg\varphi \wedge \neg\psi$
- $\neg(\varphi \to \psi) \iff \varphi \wedge \neg\psi$
- $\neg\varphi \iff \varphi \to \bot$
- ...
- $\varphi, \varphi \to \psi \models \psi$ (that is, $\{\varphi, \varphi \to \psi\} \models \psi$)
- $\Sigma \cup \{\varphi\} \models \psi$, iff $\Sigma \models \varphi \to \psi$.
- $\ldots$

  Please refer to D. Van Dalen's textbook for more similar facts.

Thanks for your attention!

Q & A

Supplementary exercise 1

Let X be the smallest set with the properties

- $p_i \in X$, and $\perp \in X$.
- if $\varphi \in X$, then $(\neg\varphi) \in X$.
- if $\varphi, \psi \in X$, then $(\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \to \psi), (\varphi \leftrightarrow \psi) \in X$.

Prove: Every formula is in X.

Supplementary exercise 2

Let X be as above, and let Y be the set of all formulas.

- Try to give a set-theoretical representation of Y.
- Prove: Every member of Y is in X.