Boolean Functions and Formulas

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Recall

<table>
<tr><th>φ</th><th>ψ</th><th>$\neg\varphi$</th><th>$\varphi \vee \psi$</th><th>$\varphi \wedge \psi$</th><th>$\varphi \rightarrow \psi$</th><th>$\varphi \leftrightarrow \psi$</th></tr>
<tr><td>1</td><td>1</td><td rowspan="2">0</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
<tr><td>1</td><td>0</td><td>1</td><td>0</td><td>0</td><td>0</td></tr>
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<tr><td>0</td><td>0</td><td>0</td><td>0</td><td>1</td><td>1</td></tr>
</table>

Example

p_1	p_2	$(p_1 \vee \neg p_2) \to (p_1 \wedge \neg p_2)$
1	1	
1	0	
0	1	
0	0	

Example

p_1	p_2	$\neg p_2$	$p_1 \vee \neg p_2$	$p_1 \wedge \neg p_2$	$(p_1 \vee \neg p_2) \rightarrow (p_1 \wedge \neg p_2)$
1	1	0	1	0	0
1	0	1	1	1	1
0	1	0	0	0	1
0	0	1	1	0	0

Example

p_1	p_2	$\neg p_2$	$p_1 \vee \neg p_2$	$p_1 \wedge \neg p_2$	$(p_1 \vee \neg p_2) \rightarrow (p_1 \wedge \neg p_2)$
1	1	0	1	0	0
1	0	1	1	1	1
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0	0	1	1	0	0

Example

p_1	p_2	$\neg p_2$	$p_1 \vee \neg p_2$	$p_1 \wedge \neg p_2$	$(p_1 \vee \neg p_2) \to (p_1 \wedge \neg p_2)$
1	1	0	1	0	0
1	0	1	1	1	1
0	1	0	0	0	1
0	0	1	1	0	0

Example

p_1	p_2	$\neg p_2$	$p_1 \vee \neg p_2$	$p_1 \wedge \neg p_2$	$(p_1 \vee \neg p_2) \rightarrow (p_1 \wedge \neg p_2)$
1	1	0	1	0	0
1	0	1	1	1	1
0	1	0	0	0	1
0	0	1	1	0	0

Example

p_1	p_2	$\neg p_2$	$p_1 \lor \neg p_2$	$p_1 \land \neg p_2$	$(p_1 \lor \neg p_2) \to (p_1 \land \neg p_2)$
1	1	0	1	0	0
1	0	1	1	1	1
0	1	0	0	0	1
0	0	1	1	0	0

Contents

Example

- Find a formula φ, such that its corresponding truth table is

p	q	φ
1	1	0
1	0	1
0	1	1
0	0	0

Solution

p	q	φ
1	1	0
1	0	**1**
0	1	**1**
0	0	0

p	q	φ_1
1	1	0
1	0	**1**
0	1	0
0	0	0

p	q	φ_2
1	1	0
1	0	0
0	1	**1**
0	0	0

p	q	φ
1	1	0
1	0	**1**
0	1	**1**
0	0	0

$$\varphi \iff \varphi_1 \vee \varphi_2$$

p	q	φ_1
1	1	0
1	0	**1**
0	1	0
0	0	0

p	q	φ_2
1	1	0
1	0	0
0	1	**1**
0	0	0

p	q	φ
1	1	0
1	0	**1**
0	1	**1**
0	0	0

$$\varphi \iff \varphi_1 \vee \varphi_2$$

p	q	φ_1
1	1	0
1	0	**1**
0	1	0
0	0	0

$$\varphi_1 \iff p \wedge \neg q$$

p	q	φ_2
1	1	0
1	0	0
0	1	**1**
0	0	0

$$\varphi_2 \iff \ldots\ldots$$

p	q	φ
1	1	0
1	0	**1**
0	1	**1**
0	0	0

$$\varphi \iff \varphi_1 \vee \varphi_2$$

p	q	φ_1
1	1	0
1	0	**1**
0	1	0
0	0	0

$$\varphi_1 \iff p \wedge \neg q$$

p	q	φ_2
1	1	0
1	0	0
0	1	**1**
0	0	0

$$\varphi_2 \iff \neg p \wedge q$$

Quiz

- Find a formula φ, such that its corresponding truth table is

p	q	r	φ
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	1
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	1

A procedure from a truth table to a formula

- Click: **A procedure from a truth table to a formula**

Contents

A truth table is essentially a Boolean function!

Boolean function

An n-ary Boolean function is a function from the set of $\{0,1\}^n$ to $\{0,1\}$.

- Every Boolean function is a finite function in the sense that its domain is finite.
- As usual, every Boolean function can be fixed by a table (called truth table).

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unitary Boolean functions

There are altogether four unitary Boolean functions. We give them by the corresponding tables:

p	$f_1(p)$
1	1
0	1

p	$f_2(p)$
1	1
0	0

p	$f_3(p)$
1	0
0	1

p	$f_4(p)$
1	0
0	0

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1	0
0	0

An example of binary Boolean functions

p_1	p_2	$f(p_1, p_2)$
1	1	0
1	0	1
0	1	1
0	0	0

Fact

For any $n \geq 1$, there are exactly 2^{2^n} n-ary Boolean functions.

Contents

Any Boolean function can be represented by a formula in the following sense.

Theorem

For any n-ary Boolean function f, there exists a formula φ containing only variables p_1, …, p_n, and connectives $\neg$, $\wedge$, and $\vee$, such that for any assignment σ,

$$[\varphi]_\sigma = f(\sigma(p_1), \ldots, \sigma(p_n)).$$

Some notions

- A **literal** is a variable or the negation of a variable.
- A formula is **in conjunctive normal form** (CNT) if it is a conjunction consisting of one or more conjuncts, each of which is a disjunction of one or more literals.
- A formula is **in disjunctive normal form** (DNT) if it is a disjunction consisting of one or more disjuncts, each of which is a conjunction of one or more literals.

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Example

- $p_1 \vee \neg p_2 \vee p_3$.
- $p_1 \wedge \neg p_2 \wedge p_3$.
- $(p_1 \wedge \neg p_2) \vee p_3$.
- $(p_1 \vee \neg p_2 \vee p_3) \wedge (\neg p_4 \vee p_5) \wedge \neg p_3$.
- $(p_1 \wedge \neg p_2 \wedge p_3) \vee (\neg p_4 \wedge p_5) \vee \neg p_3$

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Corollary

For any formula φ, we can find a formula $\varphi^\wedge$ in CNF and a formula $\varphi^\vee$ in DNF such that

$$\varphi \iff \varphi^\wedge,$$

and

$$\varphi \iff \varphi^\vee.$$

Thanks for your attention!

Q & A