Natural Deduction I

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Rules for $\wedge$

$\wedge$I

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi} \wedge\mathrm{I}$$

$\wedge$E

$$\frac{\varphi \wedge \psi}{\varphi} \wedge\mathrm{E}$$

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Rules for $\rightarrow$

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$$\dfrac{\begin{array}{c}[\varphi] \\ \vdots \\ \psi\end{array}}{\varphi \rightarrow \psi} \rightarrow \mathrm{I}$$

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Rules for $\bot$

$\bot$

$$\frac{\bot}{\varphi}\ \bot$$

RAA

$$\frac{\begin{array}{c}[\neg\varphi]\\ \vdots\\ \bot\end{array}}{\varphi}\ \mathrm{RAA}$$

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$$\begin{array}{c}[\neg\varphi]\\ \vdots\\ \dfrac{\bot}{\varphi}\ \mathrm{RAA}\end{array}$$

Example

$$\dfrac{\dfrac{\varphi \qquad \psi}{\varphi \wedge \psi}\wedge\mathsf{I} \qquad \varphi \wedge \psi \to \sigma}{\sigma}\to\mathsf{E}$$

- This is a formal deduction (**derivation**), which shows that from φ, ψ, and $\varphi \wedge \psi \to \sigma$, we can deduce σ.
- Notation: $\{\varphi, \psi, \varphi \wedge \psi \to \sigma\} \vdash \sigma$,
  or briefly, $\varphi, \psi, \varphi \wedge \psi \to \sigma \vdash \sigma$.

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$$\dfrac{\dfrac{\varphi \qquad\qquad \psi}{\varphi \land \psi}\land\mathrm{I} \qquad\qquad \varphi \land \psi \to \sigma}{\sigma}\to\mathrm{E}$$

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- This is actually a derivation **schema**. One of its instances in propositional logic is as follows:

$$\dfrac{\dfrac{p \qquad\qquad q}{p \wedge q}\wedge\mathsf{I} \qquad\qquad p \wedge q \to r}{r}\to\mathsf{E}$$

- We will always say the top one is a derivation, even we know it is a schema.

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Example

- A derivation for $\varphi \wedge \psi \vdash \psi \wedge \varphi$

$$\dfrac{\dfrac{\varphi \wedge \psi}{\psi}\ \wedge\mathrm{E} \qquad \dfrac{\varphi \wedge \psi}{\varphi}\ \wedge\mathrm{E}}{\psi \wedge \varphi}\ \wedge\mathrm{I}$$

Example

- A derivation for $\varnothing \vdash \varphi \wedge \psi \rightarrow \psi \wedge \varphi$, of briefly, $\vdash \varphi \wedge \psi \rightarrow \psi \wedge \varphi$

$$\dfrac{\dfrac{\dfrac{[\varphi \wedge \psi]}{\psi}\,\wedge\mathrm{E} \qquad \dfrac{[\varphi \wedge \psi]}{\varphi}\,\wedge\mathrm{E}}{\psi \wedge \varphi}\,\wedge\mathrm{I}}{\varphi \wedge \psi \rightarrow \psi \wedge \varphi}\,\rightarrow\mathrm{I}$$

Example

- A derivation for $\vdash \varphi \to ((\varphi \to \bot) \to \bot)$

$$\dfrac{\dfrac{\dfrac{[\varphi]^2 \qquad [\varphi \to \bot]^1}{\bot} \to \mathrm{E}}{(\varphi \to \bot) \to \bot} \to \mathrm{I}_1}{\varphi \to ((\varphi \to \bot) \to \bot)} \to \mathrm{I}_2$$

Example

- A derivation for $\vdash (\varphi \to \psi \to \sigma) \to \varphi \wedge \psi \to \sigma$

$$
\dfrac{
  \dfrac{
    \dfrac{[\varphi \wedge \psi]^1}{\psi}\,\wedge\mathrm{E}
    \qquad
    \dfrac{
      \dfrac{[\varphi \wedge \psi]^1}{\varphi}\,\wedge\mathrm{E}
      \qquad
      [\varphi \to \psi \to \sigma]^2
    }{\psi \to \sigma}\,\to\mathrm{E}
  }{
    \dfrac{\sigma}{\varphi \wedge \psi \to \sigma}\,\to\mathrm{I}_1
  }\,\to\mathrm{E}
}{(\varphi \to \psi \to \sigma) \to \varphi \wedge \psi \to \sigma}\,\to\mathrm{I}_2
$$

Note

- In the textbook, $\neg\varphi$ is a defined expression for $\varphi \to \bot$.
- Thus, the above derivation also shows $\vdash \varphi \to \neg\neg\varphi$.

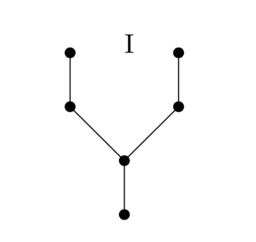
$$\dfrac{\dfrac{\dfrac{[\varphi]^2 \qquad [\varphi \to \bot]^1}{\bot}}{(\varphi \to \bot) \to \bot}}{\varphi \to ((\varphi \to \bot) \to \bot)}$$

$$\dfrac{\dfrac{\dfrac{[\varphi]^2 \qquad [\neg\varphi]^1}{\bot}}{\neg\neg\varphi}}{\varphi \to \neg\neg\varphi}$$

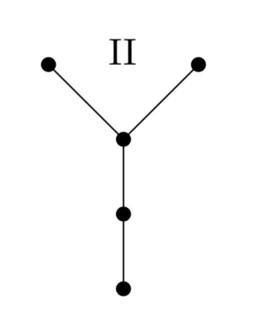
Contents

All derivations that we have given have the form of **trees**.

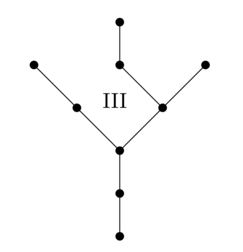
$$\dfrac{\dfrac{\dfrac{[\varphi \wedge \psi]}{\psi}\,\wedge\mathrm{E} \qquad \dfrac{[\varphi \wedge \psi]}{\varphi}\,\wedge\mathrm{E}}{\psi \wedge \varphi}\,\wedge\mathrm{I}}{\varphi \wedge \psi \to \psi \wedge \varphi}\,\to\mathrm{I}$$



$$\cfrac{\cfrac{\cfrac{[\varphi]^2 \qquad\qquad [\varphi \to \bot]^1}{\bot}\ \to \mathrm{E}}{(\varphi \to \bot) \to \bot}\ \to \mathrm{I}_1}{\varphi \to ((\varphi \to \bot) \to \bot)}\ \to \mathrm{I}_2$$



$$\dfrac{\dfrac{\dfrac{\dfrac{[\varphi \wedge \psi]^1}{\psi}\,\wedge\mathrm{E} \qquad \dfrac{\dfrac{[\varphi \wedge \psi]^1}{\varphi}\,\wedge\mathrm{E} \qquad [\varphi \to \psi \to \sigma]^2}{\psi \to \sigma}\,\to\mathrm{E}}{\sigma}\,\to\mathrm{E}}{\varphi \wedge \psi \to \sigma}\,\to\mathrm{I}_1}{(\varphi \to \psi \to \sigma) \to \varphi \wedge \psi \to \sigma}\,\to\mathrm{I}_2$$



Derivation

Definition

- A one-element tree φ is a derivation, whose hypothesis and conclusion are both φ.
- If both $\begin{matrix}\mathcal{D}\\ \varphi\end{matrix}$ and $\begin{matrix}\mathcal{D}'\\ \varphi'\end{matrix}$ are derivations, so is

$$\frac{\begin{matrix}\mathcal{D}\\ \varphi\end{matrix} \qquad \begin{matrix}\mathcal{D}'\\ \varphi'\end{matrix}}{\varphi \wedge \varphi'}$$

  whose conclusion is $\varphi \wedge \varphi'$, and whose hypotheses are the union of those in $\begin{matrix}\mathcal{D}\\ \varphi\end{matrix}$ and those in $\begin{matrix}\mathcal{D}'\\ \varphi'\end{matrix}$.

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Derivation (continued)

Definition

- If $\begin{array}{c}\mathcal{D}\\ \varphi \wedge \psi\end{array}$ is a derivation, then the following two items are also derivations

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \varphi \wedge \psi\end{array}}{\varphi}, \qquad \dfrac{\begin{array}{c}\mathcal{D}\\ \varphi \wedge \psi\end{array}}{\psi}$$

whose conclusions are respectively φ and ψ, and whose hypotheses are exactly the same as those in $\begin{array}{c}\mathcal{D}\\ \varphi \wedge \psi\end{array}$

Derivation (continued)

Definition

- If both $\begin{matrix}\varphi \\ \mathcal{D} \\ \psi\end{matrix}$ is a derivation, so is

$$\frac{\begin{matrix}[\varphi] \\ \mathcal{D} \\ \psi\end{matrix}}{\varphi \to \psi}$$

whose conclusion is $\varphi \to \psi$, and whose hypotheses are those in $\begin{matrix}\varphi \\ \mathcal{D} \\ \psi\end{matrix}$ minus φ.

Derivation (continued)

Definition

- If both $\begin{array}{c}\mathcal{D}\\ \varphi\end{array}$ and $\begin{array}{c}\mathcal{D}\\ \varphi \to \psi\end{array}$ are derivations, so is

$$\frac{\begin{array}{c}\mathcal{D}\\ \varphi\end{array} \qquad \begin{array}{c}\mathcal{D}'\\ \varphi \to \psi\end{array}}{\psi}$$

whose conclusion is ψ, and whose hypotheses are union of those in $\begin{array}{c}\mathcal{D}\\ \varphi\end{array}$ and those in $\begin{array}{c}\mathcal{D}\\ \varphi \to \psi\end{array}$

Derivation (continued)

Definition

- If $\begin{array}{c}\mathcal{D}\\ \bot\end{array}$ is a derivation, so is

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \bot\end{array}}{\varphi}$$

whose conclusion is φ, and whose hypotheses are exactly the same as those in $\begin{array}{c}\mathcal{D}\\ \bot\end{array}$

Derivation (continued)

Definition

- If $\begin{array}{c}\neg\varphi\\ \mathcal{D}\\ \bot\end{array}$ is a derivation, so is

$$\dfrac{\begin{array}{c}[\neg\varphi]\\ \mathcal{D}\\ \bot\end{array}}{\varphi}$$

  whose conclusion is φ, and whose hypotheses are those in $\begin{array}{c}\neg\varphi\\ \mathcal{D}\\ \bot\end{array}$ minus $\neg\varphi$.

Derivation (continued)

Definition

- Only the trees obtained by the above rules are the derivations.

Notation

- $\Gamma \vdash \varphi$: there is a derivation with conclusion φ and with all (unconceled) hypotheses in Γ.
  In this case, we also say: φ is derivable from Γ.
- We use $\vdash \varphi$ instead of $\varnothing \vdash \varphi$.
  In this case, we say, φ is a theorem.

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Example

- $\vdash (\varphi \to \psi) \to (\psi \to \sigma) \to (\varphi \to \sigma)$

Example

- $\vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$

Definition

In (Van Dalen's) derivations, we use the following definitions:

$$
\begin{aligned}
\neg\varphi &=_{\text{df}} \varphi \to \bot \\
\varphi \vee \psi &=_{\text{df}} \neg(\neg\varphi \wedge \neg\psi) \\
\varphi \leftrightarrow \psi &=_{\text{df}} (\varphi \to \psi) \wedge (\psi \to \varphi)
\end{aligned}
$$

Example

- $\varphi \vdash \varphi \vee \psi$
- $\psi \vdash \varphi \vee \psi$

  Note: This can be used as derived rules.

$\vee$I

$$\frac{\varphi}{\varphi \vee \psi} \vee\mathrm{I} \qquad \frac{\psi}{\varphi \vee \psi} \vee\mathrm{I}$$

Example

- If $\Gamma, \varphi \vdash \sigma$ and $\Gamma, \psi \vdash \sigma$, then $\Gamma, \varphi \vee \psi \vdash \sigma$
  Note: This can be used as a derived rule.

$\vee$E

$$\cfrac{\varphi \vee \psi \qquad \begin{array}{c}[\varphi]\\ \vdots\\ \sigma\end{array} \qquad \begin{array}{c}[\psi]\\ \vdots\\ \sigma\end{array}}{\sigma}\ \vee\text{E}$$

See pp. 47-48 of the textbook for more derived rules.

Example

- $\vdash \varphi \vee \neg\varphi$

  Hint: use derived rules!

Thanks for your attention!

Q & A