First Order Language

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Contents

A toy language

2 The arithmetic language

Structures and Languages

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A toy language

2 The arithmetic language

Structures and Languages

Example

- Socrates is a philosopher.
- Socrates influences Plato.
- Socrates influences all philosophers.
- Some philosopher influences all philosophers.

Symbols	Interpretations
P(x)	x is a philosopher
$Q(x_1, x_2)$	x_1 influences x_2
a	Socrates
b	Plato

- Socrates is a philosopher.
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- A (non-empty) domain A (the set of human beings, or any other set including all philosophers).
- ullet the property R_1 (being a philosophy), as the interpretation of P
- ullet the relation R_2 (influencing), as the interpretation of Q
- the philosopher Socrates, as the interpretation of a
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$$\mathfrak{A} = \langle A, R_1, R_2, \mathsf{Socrates}, \mathsf{Plato} \rangle$$

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- (individual) variables: x_0, x_1, \dots
- connectives: \neg , \wedge , \vee , \rightarrow , \leftrightarrow , \bot
- quantifiers: ∀, ∃
- constant symbols: a, b
- ullet predicate symbols: P, Q
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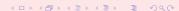
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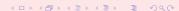
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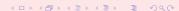
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- ⊥ is a formula.
- If t, t_1 , and t_2 are terms, then expressions of form P(t) or $Q(t_1, t_2)$ are formulas.
- If φ is a formula, so is $(\neg \varphi)$.
- If φ and ψ are formulas, so is $(\varphi \star \psi)$, where \star is \land , \lor , \rightarrow , or \leftrightarrow .
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Terms and formulas for the toy language

Terms are variables or constant symbols.

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Contents

A toy language

The arithmetic language

Structures and Languages

Example

- 0 is not the successor of any number.
- 0 is less than 1.
- 0 is an even number.
- Any number multiplied by 0 is still equal to 0.

Symbols	Interpretations
$x_1 \dot{=} x_2$	x_1 is equal to x_2
$\boldsymbol{S}x$	the successor of x
$x_1 + x_2$	the sum of x_1 and x_2
$x_1 \times x_2$	the product of x_1 and x_2
0	zero

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- ullet the equality relation =, as the interpretation of $\dot{=}$
- the successor operation ', as the interpretation of S
- the addition operation +, as the interpretation of +
- ullet the multiplication operation imes, as the interpretation of imes
- the number 0, as the interpretation of 0

$$\mathfrak{N} = \langle \mathbb{N}, ', +, \times, 0 \rangle$$

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- quantifiers: ∀, ∃
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Terms of the arithmetic language

- Variables are terms (of \mathcal{L}_A).
- 0 is a term.
- If t is a term, so is St.
- If t_1 and t_2 are terms, so are $t_1 + t_2$ and $t_1 \times t_2$.
- Only expressions obtained by the above rules are terms.

- \perp is a formula (of \mathcal{L}_A).
- If t_1 and t_2 are terms, then $(t_1 = t_2)$ is a formula.
- If φ is a formula, so is $(\neg \varphi)$.
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Structure

A structure a is an ordered sequence

$$\langle A, R_1, \dots, R_n, F_1, \dots, F_m, \{c_k | i \in K\} \rangle$$
,

where

- A is a non-empty set.
- $R_1, ..., R_n$ are relations on A.
- *F*₁, ..., *F*_m are functions on *A*.
- for all $k \in K$, c_k is an element of A, and K is an index set.

E.g

- $\mathfrak{A} = \langle A, R_1, R_2, \text{Socrates}, \text{Plato} \rangle$

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- $\mathfrak{A} = \langle A, R_1, R_2, \text{Socrates}, \text{Plato} \rangle$
- $\mathfrak{A} = \langle \mathbb{N}, ', +, \times, 0 \rangle$



Similarity type

The similarity type of a structure $\mathfrak A$

$$\langle A, R_1, \dots, R_n, F_1, \dots, F_m, \{c_i | i \in I\} \rangle$$

is a sequence

$$\langle r_1,\ldots,r_n;a_1,\ldots,a_m;\kappa\rangle$$

where $R_i \subseteq A^{r_i}$, $F_j : A^{a_j} \to A$, the size of $\{c_k | k \in K\}$ is κ .

Structure	Similarity type
$\langle A, R_1, R_2, Socrates, Plato \rangle$	$\langle 1, 2; -; 2 \rangle$
$\langle \mathbb{N},',+,\times,0 \rangle$	$\langle -; 1, 2, 2; 1 \rangle$
$\langle G,R \rangle$	$\langle 1; -; - \rangle$

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About identity relation

Stipulate:

- All structures possess an identity (equality) relation, unless otherwise claimed.
- We always omit the identity relation in a structure.
- Accordingly, we do not mention the arity of the identity relation in the similarity type.

- variables: x_0 , x_1 ,
- connectives: \neg , \wedge , \vee , \rightarrow , \leftrightarrow , \bot
- quantifiers: ∀, ∃
- constant symbols: $\overline{c_k}$ for all $k \in K$
- predicate symbols: $P_1, ..., P_n, (\doteq)$
- function symbols: $f_1, ..., f_m$
- auxiliary symbols:) , (.

$$\mathcal{L} = \langle P_1, \dots, P_n, f_1, \dots, f_m, \{\overline{c_k} | k \in K \} \rangle$$

- variables: x_0, x_1, \ldots
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Terms of \mathcal{L}

- Variables and constant symbols are terms (of \mathcal{L}).
- For any $1 \le i \le m$, if $t_1, ..., t_{a_i}$ are terms, so is $f_i(t_1, ..., t_{a_i})$.
- Only the expressions obtained by the above rules are the terms.

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- \bullet \perp is a formula (of \mathcal{L}).
- For any terms t_1 and t_2 , $t_1 = t_2$ is a formula.
- For any $1 \le i \le n$, if $t_1, ..., t_{r_i}$ are terms, then $P_i(t_1, ..., t_{r_i})$ is a formula.
- If φ is a formula, so is $(\neg \varphi)$.
- If φ and ψ are formulas, so is $(\varphi \star \psi)$, where \star is \land , \lor , \rightarrow , or \leftrightarrow .
- If φ is a formula, then for any variable x, $((\forall x)\varphi)$ and $((\exists x)\varphi)$ are also formulas.
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Other notions

- free occurrence of a variable
- free variable in a formula
- sentence
- substitution s[t/x]
- substitution $\varphi[t/x]$
- ullet t is free for x in φ

Thanks for your attention! Q & A