Interpreting closed terms and sentences

Ming Hsiung

School of Philosophy and Social Development
South China Normal University

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Recall

The structure of natural numbers:

$$\mathfrak{N} = \langle \mathbb{N}, ', +, \times, 0 \rangle$$

A toy structure

$$\mathfrak{R} = \left\langle \mathbb{R}, \sqrt[3]{\cdot}, +, \times, 0 \right\rangle$$

The similarity type of $\mathfrak{N}$ ($\mathfrak{R}$):

$$\langle -; 1, 2, 2; 1 \rangle$$

The arithmetic language

$$\mathcal{L}_A = \langle \boldsymbol{S}, +, \times, \mathbf{0} \rangle$$

true or false

- $\phi = \forall x(x \times \mathbf{0} \doteq \mathbf{0})$
- $\psi = \forall x_1 \exists x_2 (x_1 \doteq \boldsymbol{S} x_2)$

$$\mathfrak{N} = \langle \mathbb{N}, ', +, \times, 0 \rangle$$

$$\mathfrak{R} = \left\langle \mathbb{R}, \sqrt[3]{\cdot}, +, \times, 0 \right\rangle$$

Formal semantics

It is Tarski who first made it clear in what sense a sentence is true in a structure.

- $\phi = \forall x(x \times \mathbf{0} \doteq \mathbf{0})$
- $\psi = \forall x_1 \exists x_2 (x_1 \doteq \boldsymbol{S} x_2)$

$$\mathfrak{N} = \langle \mathbb{N}, ', +, \times, 0 \rangle$$

$$\mathfrak{R} = \left\langle \mathbb{R}, \sqrt[3]{\cdot}, +, \times, 0 \right\rangle$$

- $\mathfrak{N} \models \phi$ ($[\phi]_{\mathfrak{N}} = 1$, $\mathcal{V}_{\mathfrak{N}}(\phi) = 1$), $\mathfrak{R} \models \phi$
- $\mathfrak{N} \not\models \psi$ ($[\psi]_{\mathfrak{N}} = 0$, $\mathcal{V}_{\mathfrak{N}}(\psi) = 0$), $\mathfrak{R} \models \psi$

Expanding languages by adding constants

$$\mathcal{L}_A = \langle \boldsymbol{S}, +, \times, \mathbf{0} \rangle$$

$$\mathfrak{N} = \langle \mathbb{N}, ', +, \times, 0 \rangle$$

$$\mathcal{L}_A(\mathfrak{N}) = \langle \boldsymbol{S}, +, \times, \mathbf{0}, \overline{0}, \overline{1}, \overline{2}, \ldots \rangle$$

or more precisely,

$$\mathcal{L}_A(\mathfrak{N}) = \langle \boldsymbol{S}, +, \times, \mathbf{0}, \{\overline{n} | n \in \mathbb{N}\} \rangle$$

$$\mathfrak{R} = \left\langle \mathbb{R}, \sqrt[3]{\cdot}, +, \times, 0 \right\rangle$$

$$\mathcal{L}_A(\mathfrak{R}) = \langle \boldsymbol{S}, +, \times, \mathbf{0}, \{\overline{r} | r \in \mathbb{R}\} \rangle$$

Terms of $\mathcal{L}_A(\mathfrak{N})$

- Variables are terms (of $\mathcal{L}_A(\mathfrak{N})$).
- $\mathbf{0}$ is a term, and for any $n \in \mathbb{N}$, $\overline{n}$ is a term.
- If t is a term, so is $\boldsymbol{S}t$.
- If t_1 and t_2 are terms, so are $t_1 + t_2$ and $t_1 \times t_2$.
- Only expressions obtained by the above rules are terms.

Interpreting closed terms of $\mathcal{L}_A(\mathfrak{N})$

For any closed term t of $\mathcal{L}_A(\mathfrak{N})$, we define recursively $t^{\mathfrak{N}}$ as follows:

- $\mathbf{0}^{\mathfrak{N}} = 0$.
- For any $n \in \mathbb{N}$, $\overline{n}^{\mathfrak{N}} = n$.
- $(\boldsymbol{S}t)^{\mathfrak{N}} = (t^{\mathfrak{N}})'$.
- $(t_1 + t_2)^{\mathfrak{N}} = t_1^{\mathfrak{N}} + t_2^{\mathfrak{N}}$.
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$(\cdot)^{\mathfrak{N}}$ is a mapping from the set of closed terms of $\mathcal{L}_A(\mathfrak{N})$ to $\mathbb{N}$, which is usually **an interpretation of closed terms of $\mathcal{L}_A(\mathfrak{N})$ in $\mathfrak{N}$**.

Eg.

- $(\boldsymbol{SS}\mathbf{0})^{\mathfrak{N}} = 2$
- $(\boldsymbol{SS}\mathbf{0} + \overline{3})^{\mathfrak{N}} = 5$

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- $(\boldsymbol{S}\boldsymbol{S}\mathbf{0})^{\mathfrak{N}} = 2$
- $(\boldsymbol{S}\boldsymbol{S}\mathbf{0} + \overline{3})^{\mathfrak{N}} = 5$

Quiz

Prove: for any $n \in \mathbb{N}$,

$$(\boldsymbol{S}\boldsymbol{S}\ldots\boldsymbol{S}\mathbf{0})^{\mathfrak{N}} = (\overline{n})^{\mathfrak{N}},$$

where there are n occurrences of $\boldsymbol{S}$ in the above equation.

Terms of $\mathcal{L}_A(\mathfrak{R})$

Interpreting closed terms of $\mathcal{L}_A(\mathfrak{R})$

Quiz

Find all $r \in \mathbb{R}$, such that

$$(\boldsymbol{S}\overline{r})^{\mathfrak{R}} = (\overline{r})^{\mathfrak{R}}.$$

Formulas of $\mathcal{L}_A(\mathfrak{N})$

- $\perp$ is a formula (of $\mathcal{L}_A$).
- If t_1 and t_2 are terms, then $(t_1 \doteq t_2)$ is a formula.
- If φ is a formula, so is $(\neg\varphi)$.
- If φ and ψ are formulas, so is $(\varphi \star \psi)$, where $\star$ is $\wedge$, $\vee$, $\rightarrow$, or $\leftrightarrow$.
- If φ is a formula, then for any variable x, $((\forall x)\varphi)$ and $((\exists x)\varphi)$ are also formulas.
- Formulas are exactly those expressions obtained by the above rules.

Interpreting sentences of $\mathcal{L}_A(\mathfrak{N})$ in $\mathfrak{N}$

- $[\perp]_{\mathfrak{N}} = 0$ (another notation: $\mathcal{V}_{\mathfrak{N}}(\perp) = 0$)
- $[t_1 \doteq t_2]_{\mathfrak{N}} = 1$, iff $t_1^{\mathfrak{N}} = t_2^{\mathfrak{N}}$; otherwise, $[t_1 \doteq t_2]_{\mathfrak{N}} = 0$.
- $[\neg\varphi]_{\mathfrak{N}} = 1$, iff $[\varphi]_{\mathfrak{N}} = 0$.
- $[\varphi \to \psi]_{\mathfrak{N}} = 0$, iff $[\varphi]_{\mathfrak{N}} = 1$ and $[\psi]_{\mathfrak{N}} = 0$; ...
- $[(\forall x)\varphi]_{\mathfrak{N}} = 1$, iff for all $n \in \mathbb{N}$, $[\varphi(\overline{n}/x)]_{\mathfrak{N}} = 1$.
- $[(\exists x)\varphi]_{\mathfrak{N}} = 1$, iff

$[\cdot]_{\mathfrak{N}}$ is a mapping from the set of sentences of $\mathcal{L}_A(\mathfrak{N})$ to $\{1, 0\}$, which is usually **an interpretation (evaluation) of sentences of $\mathcal{L}_A(\mathfrak{N})$ in $\mathfrak{N}$**.

Usually, we use $\mathfrak{N} \models \phi$ instead of $[\phi]_{\mathfrak{N}} = 1$.

Interpreting sentences of $\mathcal{L}_A(\mathfrak{N})$ in $\mathfrak{N}$

- $[\bot]_\mathfrak{N} = 0$ (another notation: $\mathcal{V}_\mathfrak{N}(\bot) = 0$)
- $[t_1 \dot{=} t_2]_\mathfrak{N} = 1$, iff $t_1^\mathfrak{N} = t_2^\mathfrak{N}$; otherwise, $[t_1 \dot{=} t_2]_\mathfrak{N} = 0$.
- $[\neg\varphi]_\mathfrak{N} = 1$, iff $[\varphi]_\mathfrak{N} = 0$.
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- $[\mathbf{0} \doteq \overline{0}]_{\mathfrak{N}} =$
- $[\boldsymbol{S}\mathbf{0} \doteq \overline{1}]_{\mathfrak{N}} =$
- $[\forall x (x \times \mathbf{0} \doteq \mathbf{0})]_{\mathfrak{N}} =$
- $[\forall x_1 \exists x_2 (x_1 \doteq \boldsymbol{S} x_2)]_{\mathfrak{N}} =$

Formulas of $\mathcal{L}_A(\mathfrak{R})$

Interpreting sentences of $\mathcal{L}_A(\mathfrak{R})$ in $\mathfrak{R}$

Eg.

- $[\mathbf{0} \doteq \overline{0}]_{\mathfrak{R}} =$
- $[\boldsymbol{S}\mathbf{0} \doteq \overline{1}]_{\mathfrak{R}} =$
- $\left[\boldsymbol{S}\overline{2} \doteq \overline{\sqrt{2}}\right]_{\mathfrak{R}} =$
- $[\forall x(x \times \mathbf{0} \doteq \mathbf{0})]_{\mathfrak{R}} =$
- $[\forall x_1 \exists x_2 (x_1 \doteq \boldsymbol{S} x_2)]_{\mathfrak{R}} =$

Notations

- Usually, we use $\mathfrak{R} \models \phi$ instead of $[\phi]_{\mathfrak{R}} = 1$.
- We use $\models \phi$ to denote that for all structure $\mathfrak{A}$ of the same type as $\mathfrak{N}$, $\mathfrak{A} \models \phi$ always holds.
- $\Gamma \models \phi$: whenever $\mathfrak{A} \models \Gamma$, we have $\mathfrak{A} \models \phi$.
- If ϕ is a formula (not necessarily a sentence), by $\mathfrak{A} \models \phi$, we always mean $\mathfrak{A} \models Cl(\phi)$, where $Cl(\phi)$ is the universal closure of ϕ.

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- Usually, we use $\mathfrak{R} \models \phi$ instead of $[\phi]_{\mathfrak{R}} = 1$.
- We use $\models \phi$ to denote that for all structure $\mathfrak{A}$ of the same type as $\mathfrak{N}$, $\mathfrak{A} \models \phi$ always holds.
- $\Gamma \models \phi$: whenever $\mathfrak{A} \models \Gamma$, we have $\mathfrak{A} \models \phi$.
- If ϕ is a formula (not necessarily a sentence), by $\mathfrak{A} \models \phi$, we always mean $\mathfrak{A} \models Cl(\phi)$, where $Cl(\phi)$ is the universal closure of ϕ.

Contents

Recall

A structure $\mathfrak{A}$:

$$\langle A, R_1, \ldots, R_n, F_1, \ldots, F_m, \{c_k | i \in K\} \rangle,$$

The similarity type of $\mathfrak{A}$

$$\langle r_1, \ldots, r_n; a_1, \ldots, a_m; \kappa \rangle$$

where $R_i \subseteq A^{r_i}$, $F_j : A^{a_j} \to A$, the size of $\{c_k | k \in K\}$ is κ.

The corresponding language:

$$\mathcal{L} = \langle P_1, \ldots, P_n, f_1, \ldots, f_m, \{\overline{c_k} | k \in K\} \rangle$$

$\mathcal{L}(\mathfrak{A})$

$$\mathcal{L} = \langle P_1, \ldots, P_n, f_1, \ldots, f_m, \{\overline{c_k} | k \in K\} \rangle$$

$$\mathfrak{A} = \langle A, R_1, \ldots, R_n, F_1, \ldots, F_m, \{c_k | i \in K\} \rangle,$$

$$\mathcal{L}(\mathfrak{A}) = \langle P_1, \ldots, P_n, f_1, \ldots, f_m, \{\overline{c_k} | k \in K\}, \{\overline{a} | a \in A\} \rangle$$

Terms of $\mathcal{L}(\mathfrak{A})$

Interpreting closed terms of $\mathcal{L}(\mathfrak{A})$

Formulas of $\mathcal{L}(\mathfrak{A})$

Interpreting sentences of $\mathcal{L}(\mathfrak{A})$

Interpreting sentences of $\mathcal{L}_A$ again

Is it possible to directly interpret sentences of $\mathcal{L}_A$ in $\mathfrak{N}$ without the help of the language $\mathcal{L}_A(\mathfrak{N})$?

Thanks for your attention!

Q & A