Simple properties of predicate logic

Ming Hsiung

School of Philosophy and Social Development
South China Normal University

Contents

Contents

Schema and its instances

- $\forall x \varphi \to \exists x \varphi$ is a **schema** (of formulas).
- $\forall x_0 \exists x_1 (x_0 \doteq S x_1) \to \exists x_0 \exists x_1 (x_0 \doteq S x_1)$ is an **instance** of the above schema (in $\mathcal{L}_A$).
- $\forall x_5 Q(a, x_5) \to \exists x_5 Q(a, x_5)$ is an instance of the above schema (in the toy language $\mathcal{L}$).
- $\forall x_5 Q(a, x_5) \to \exists x_6 Q(a, x_6)$ is **not** an instance of the above schema.
- $\forall x_5 Q(a, x_5) \to \exists x_5 Q(b, x_5)$ is **not** an instance of the above schema.

Schema and its instances

- $\forall x\varphi \to \exists x\varphi$ is a **schema** (of formulas).
- $\forall x_0 \exists x_1 (x_0 \doteq \boldsymbol{S} x_1) \to \exists x_0 \exists x_1 (x_0 \doteq \boldsymbol{S} x_1)$ is an **instance** of the above schema (in $\mathcal{L}_A$).
- $\forall x_5 Q(a, x_5) \to \exists x_5 Q(a, x_5)$ is an instance of the above schema (in the toy language $\mathcal{L}$).
- $\forall x_5 Q(a, x_5) \to \exists x_6 Q(a, x_6)$ is **not** an instance of the above schema.
- $\forall x_5 Q(a, x_5) \to \exists x_5 Q(b, x_5)$ is **not** an instance of the above schema.

Schema and its instances

- $\forall x \varphi \to \exists x \varphi$ is a **schema** (of formulas).
- $\forall x_0 \exists x_1 (x_0 \doteq \boldsymbol{S} x_1) \to \exists x_0 \exists x_1 (x_0 \doteq \boldsymbol{S} x_1)$ is an **instance** of the above schema (in $\mathcal{L}_A$).
- $\forall x_5 Q(a, x_5) \to \exists x_5 Q(a, x_5)$ is an instance of the above schema (in the toy language $\mathcal{L}$).
- $\forall x_5 Q(a, x_5) \to \exists x_6 Q(a, x_6)$ is **not** an instance of the above schema.
- $\forall x_5 Q(a, x_5) \to \exists x_5 Q(b, x_5)$ is **not** an instance of the above schema.

Schema and its instances

- $\forall x\varphi \to \exists x\varphi$ is a **schema** (of formulas).
- $\forall x_0 \exists x_1 (x_0 \doteq \boldsymbol{S} x_1) \to \exists x_0 \exists x_1 (x_0 \doteq \boldsymbol{S} x_1)$ is an **instance** of the above schema (in $\mathcal{L}_A$).
- $\forall x_5 Q(a, x_5) \to \exists x_5 Q(a, x_5)$ is an instance of the above schema (in the toy language $\mathcal{L}$).
- $\forall x_5 Q(a, x_5) \to \exists x_6 Q(a, x_6)$ is **not** an instance of the above schema.
- $\forall x_5 Q(a, x_5) \to \exists x_5 Q(b, x_5)$ is **not** an instance of the above schema.

Schema and its instances

- $\forall x\varphi \to \exists x\varphi$ is a **schema** (of formulas).
- $\forall x_0 \exists x_1 (x_0 \doteq \boldsymbol{S} x_1) \to \exists x_0 \exists x_1 (x_0 \doteq \boldsymbol{S} x_1)$ is an **instance** of the above schema (in $\mathcal{L}_A$).
- $\forall x_5 Q(a, x_5) \to \exists x_5 Q(a, x_5)$ is an instance of the above schema (in the toy language $\mathcal{L}$).
- $\forall x_5 Q(a, x_5) \to \exists x_6 Q(a, x_6)$ is **not** an instance of the above schema.
- $\forall x_5 Q(a, x_5) \to \exists x_5 Q(b, x_5)$ is **not** an instance of the above schema.

Validity

- A schema is valid, if any of its instances is valid.
- A formula is valid, if the universal closure of it is valid.
- A sentence is valid, if it is true for all structures (of the appropriate similarity type).

Validity

- A schema is valid, if any of its instances is valid.
- A formula is valid, if the universal closure of it is valid.
- A sentence is valid, if it is true for all structures (of the appropriate similarity type).

Validity

- A schema is valid, if any of its instances is valid.
- A formula is valid, if the universal closure of it is valid.
- A sentence is valid, if it is true for all structures (of the appropriate similarity type).

Fact

- $\models \forall x \varphi \leftrightarrow \neg \exists x \neg \varphi$
- $\models \exists x \varphi \leftrightarrow \neg \forall x \neg \varphi$
- $\models \forall x \varphi \rightarrow \exists x \varphi$
- $\models \forall x \varphi \leftrightarrow \varphi$, if x is not free in φ
- $\models \exists x \varphi \leftrightarrow \varphi$, if x is not free in φ

Fact

- $\models \forall x \forall y \varphi \leftrightarrow \forall y \forall x \varphi$
- $\models \exists x \exists y \varphi \leftrightarrow \exists y \exists x \varphi$
- $\models \exists x \forall y \varphi \rightarrow \forall y \exists x \varphi$

Fact

- $\models \forall x(\varphi \land \psi) \leftrightarrow \forall x\varphi \land \forall x\psi$
- $\models \exists x(\varphi \lor \psi) \leftrightarrow \exists x\varphi \lor \exists x\psi$
- $\models \forall x\varphi \lor \forall x\psi \to \forall x(\varphi \lor \psi)$
- $\models \exists x(\varphi \land \psi) \to \exists x\varphi \land \exists x\psi$
- $\models \forall x(\varphi \lor \psi) \to \exists x\varphi \lor \forall x\psi$

Fact

- $\models \forall x(\varphi \to \psi) \to \forall x\varphi \to \forall x\psi$
- $\models \forall x(\varphi \to \psi) \to \exists x\varphi \to \exists x\psi$

Fact

If x is not free in ψ, then

- $\models \forall x(\varphi \wedge \psi) \leftrightarrow \forall x \varphi \wedge \psi$
- $\models \forall x(\varphi \vee \psi) \leftrightarrow \forall x \varphi \vee \psi$
- $\models \exists x(\varphi \wedge \psi) \leftrightarrow \exists x \varphi \wedge \psi$
- $\models \exists x(\varphi \vee \psi) \leftrightarrow \exists x \varphi \vee \psi$

Fact

If x is not free in ψ, then

- $\models \forall x(\psi \to \varphi) \leftrightarrow (\psi \to \forall x\varphi)$
- $\models \exists x(\psi \to \varphi) \leftrightarrow (\psi \to \exists x\varphi)$
- $\models \forall x(\varphi \to \psi) \leftrightarrow (\exists x\varphi \to \psi)$
- $\models \exists x(\varphi \to \psi) \leftrightarrow (\forall x\varphi \to \psi)$

Contents

Fact

- Let x and y be distinct variables such that x does not occur in r then

$$t[s/x][r/y] = t[r/y][s[r/y]/x].$$

- Let
  - $t = x_0 + Sx_1$
  - $s = x_0 \times Sx_1 \times Sx_2$
  - $r = SSx_3$
- Then
  - $t[s/x_0][r/x_1] \quad = (x_0 \times Sx_1 \times Sx_2 + Sx_1)[r/x_1] = x_0 \times SSSx_3 \times Sx_2 + SSSx_3$
  - $t[r/x_1][s[r/x_1]/x_0]$
    $= (x_0 + SSSx_3)[(x_0 \times SSSx_3 \times Sx_2)/x_0]$
    $= x_0 \times SSSx_3 \times Sx_2 + SSSx_3$

Fact

- Let x and y be distinct variables such that x does not occur in r then
$$t[s/x][r/y] = t[r/y][s[r/y]/x].$$
- Let
  - $t = x_0 + \boldsymbol{S}x_1$
  - $s = x_0 \times \boldsymbol{S}x_1 \times \boldsymbol{S}x_2$
  - $r = \boldsymbol{SS}x_3$
- Then
  - $t[s/x_0][r/x_1] \quad = \quad (x_0 \times \boldsymbol{S}x_1 \times \boldsymbol{S}x_2 + \boldsymbol{S}x_1)[r/x_1]$
    $\quad = \quad x_0 \times \boldsymbol{SSS}x_3 \times \boldsymbol{S}x_2 + \boldsymbol{SSS}x_3$
  - $t[r/x_1][s[r/x_1]/x_0]$
    $= (x_0 + \boldsymbol{SSS}x_3)[(x_0 \times \boldsymbol{SSS}x_3 \times \boldsymbol{S}x_2)/x_0]$
    $= x_0 \times \boldsymbol{SSS}x_3 \times \boldsymbol{S}x_2 + \boldsymbol{SSS}x_3$

Counter-example

- Let x and y be distinct variables such that x **does not occur in** r then

$$t[s/x][r/y] = t[r/y][s[r/y]/x].$$

- Let
  - $t = x_0 + Sx_1$
  - $s = x_0 \times Sx_1 \times Sx_2$
  - $r' = SSx_0$
- Then
  - $t[s/x_0][r'/x_1]$
    $$\begin{aligned} &= (x_0 \times Sx_1 \times Sx_2 + Sx_1)[r'/x_1] \\ &= x_0 \times SSSx_0 \times Sx_2 + SSSx_0 \end{aligned}$$
  - $t[r'/x_1][s[r'/x_1]/x_0]$
    $$\begin{aligned} &= (x_0 + SSSx_0)[(x_0 \times SSSx_0 \times Sx_2)/x_0] \\ &= x_0 \times SSSx_0 \times Sx_2 + SSS(x_0 \times SSSx_0 \times Sx_2) \end{aligned}$$

Counter-example

- Let x and y be distinct variables such that x **does not occur in** r then
$$t[s/x][r/y] = t[r/y][s[r/y]/x].$$
- Let
  - $t = x_0 + \boldsymbol{S}x_1$
  - $s = x_0 \times \boldsymbol{S}x_1 \times \boldsymbol{S}x_2$
  - $r' = \boldsymbol{SS}x_0$
- Then
  - $t[s/x_0][r'/x_1]$
  $$= (x_0 \times \boldsymbol{S}x_1 \times \boldsymbol{S}x_2 + \boldsymbol{S}x_1)[r'/x_1]$$
  $$= x_0 \times \boldsymbol{SSS}x_0 \times \boldsymbol{S}x_2 + \boldsymbol{SSS}x_0$$
  - $t[r'/x_1][s[r'/x_1]/x_0]$
  $$= (x_0 + \boldsymbol{SSS}x_0)[(x_0 \times \boldsymbol{SSS}x_0 \times \boldsymbol{S}x_2)/x_0]$$
  $$= x_0 \times \boldsymbol{SSS}x_0 \times \boldsymbol{S}x_2 + \boldsymbol{SSS}(x_0 \times \boldsymbol{SSS}x_0 \times \boldsymbol{S}x_2)$$

Fact

- Let x and y be distinct variables such that x does not occur in r, and let t and s be free for x and y in φ, then

$$\varphi[s/x][r/y] = \varphi[r/y][s[r/y]/x].$$

Corollary

Let c be a constant symbol.

- If z does not occur in t then

$$t[c/x] = t[z/x][c/z].$$

- If z is free for x in φ and it is not free in φ either, then

$$\varphi[c/x] = \varphi[z/x][c/z].$$

Theorem

- (Change of Bound variables) If x and y are free for z in φ and neither x nor y is free in φ, then
  - $\models \exists x \varphi[x/z] \leftrightarrow \exists y \varphi[y/z]$
  - $\models \forall x \varphi[x/z] \leftrightarrow \forall y \varphi[y/z]$

(counter-)examples:

- $\exists x \neg(w \doteq z)[x/z] \leftrightarrow \exists y \neg(w \doteq z)[y/z]$
- $\exists x \neg(x \doteq z)[x/z] \leftrightarrow \exists y \neg(x \doteq z)[y/z]$
- $\forall x \exists w \neg(w \doteq z)[x/z] \leftrightarrow \forall y \exists w \neg(w \doteq z)[y/z]$
- $\forall x \exists x \neg(x \doteq z)[x/z] \leftrightarrow \forall y \exists x \neg(x \doteq z)[y/z]$

Theorem

- If x and y are free for z in φ and neither x nor y is free in φ, then
  - $\models \exists x\varphi[x/z] \leftrightarrow \exists y\varphi[y/z]$

Proof.

For any structure $\mathfrak{A}$, we have

$$\begin{aligned}\mathfrak{A} \models \exists x\varphi[x/z] \quad &\text{iff} \quad \mathfrak{A} \models \varphi[x/z][\overline{a}/x] \text{ for some } a \in A \\ &\text{iff} \quad \mathfrak{A} \models \varphi[\overline{a}/z] \text{ for some } a \in A \\ &\text{iff} \quad \mathfrak{A} \models \varphi[y/z][\overline{a}/y] \text{ for some } a \in A \\ &\text{iff} \quad \mathfrak{A} \models \exists y\varphi[y/z]\end{aligned}$$

Hence, for any structure $\mathfrak{A}$, $\mathfrak{A} \models \exists x\varphi[x/z]$, iff $\mathfrak{A} \models \exists y\varphi[y/z]$. We thus can conclude that $\models \exists x\varphi[x/z] \leftrightarrow \exists y\varphi[y/z]$. □

Corollary

- Every formula is equivalent to one in which no variable occurs both free and bound.

Substitution theorem for terms

- $$\models t_1 \doteq t_2 \to s\,[t_1/x] \doteq s\,[t_2/x]\,.$$

- If t is a closed term and $s(x)$ is a term, then

$$(s[t/x])^{\mathfrak{A}} = \left(s\left[\overline{t^{\mathfrak{A}}}/x\right]\right)^{\mathfrak{A}}.$$

- If $t(y_1, \ldots, y_n)$ and $s(x)$ are terms, then

$$\left(s\left[t\left[\overline{a_1}/y_1, \ldots, \overline{a_n}/y_n\right]/x\right]\right)^{\mathfrak{A}} = \left(s\left[\overline{\left(t\left[\overline{a_1}/y_1, \ldots, \overline{a_n}/y_n\right]^{\mathfrak{A}}\right)}/x\right]\right)^{\mathfrak{A}}.$$

Substitution theorem for terms

- $$\models t_1 \doteq t_2 \to s\,[t_1/x] \doteq s\,[t_2/x]\,.$$

- If t is a closed term and $s(x)$ is a term, then

$$(s[t/x])^{\mathfrak{A}} = \left(s\left[\overline{t^{\mathfrak{A}}}/x\right]\right)^{\mathfrak{A}}.$$

- If $t(y_1, \ldots, y_n)$ and $s(x)$ are terms, then

$$\left(s\left[t\left[\overline{a_1}/y_1, \ldots, \overline{a_n}/y_n\right]/x\right]\right)^{\mathfrak{A}} = \left(s\left[\overline{\left(t\left[\overline{a_1}/y_1, \ldots, \overline{a_n}/y_n\right]^{\mathfrak{A}}\right)}/x\right]\right)^{\mathfrak{A}}.$$

Substitution theorem for terms

- $$\models t_1 \doteq t_2 \to s\,[t_1/x] \doteq s\,[t_2/x]\,.$$

- If t is a closed term and $s(x)$ is a term, then
$$(s[t/x])^{\mathfrak{A}} = \left(s\left[\overline{t^{\mathfrak{A}}}/x\right]\right)^{\mathfrak{A}}.$$

- If $t(y_1, \ldots, y_n)$ and $s(x)$ are terms, then
$$(s\,[t\,[\overline{a_1}/y_1, \ldots, \overline{a_n}/y_n]\,/x])^{\mathfrak{A}} = \left(s\left[\overline{\left(t\,[\overline{a_1}/y_1, \ldots, \overline{a_n}/y_n]^{\mathfrak{A}}\right)}/x\right]\right)^{\mathfrak{A}}.$$

Substitution theorem for formulas

- If t_1 and t_2 are free for x in φ, then

$$\models t_1 \doteq t_2 \to \varphi\,[t_1/x] \doteq \varphi\,[t_2/x]\,.$$

- If t is a closed term and $\varphi(x)$ is a formula, then

$$\mathfrak{A} \models \varphi[t/x] \iff \mathfrak{A} \models \varphi\left[\overline{t^{\mathfrak{A}}}/x\right].$$

- If $t(y_1, \ldots, y_n)$ and $\varphi(x)$ are terms, then

$$\mathfrak{A} \models \varphi\left[t\left[\overline{a_1}/y_1, \ldots, \overline{a_n}/y_n\right]/x\right] \iff \mathfrak{A} \models \varphi\left[\overline{\left(t\left[\overline{a_1}/y_1, \ldots, \overline{a_n}/y_n\right]^{\mathfrak{A}}\right)}/x\right]$$

Substitution theorem for formulas

- If t_1 and t_2 are free for x in φ, then

$$\models t_1 \doteq t_2 \rightarrow \varphi[t_1/x] \doteq \varphi[t_2/x] \, .$$

- If t is a closed term and $\varphi(x)$ is a formula, then

$$\mathfrak{A} \models \varphi[t/x] \iff \mathfrak{A} \models \varphi\left[\overline{t^{\mathfrak{A}}}/x\right] .$$

- If $t(y_1, \ldots, y_n)$ and $\varphi(x)$ are terms, then

$$\mathfrak{A} \models \varphi\left[t\left[\overline{a_1}/y_1, \ldots, \overline{a_n}/y_n\right]/x\right]$$
$$\iff \mathfrak{A} \models \varphi\left[\overline{\left(t\left[\overline{a_1}/y_1, \ldots, \overline{a_n}/y_n\right]^{\mathfrak{A}}\right)}/x\right]$$

Substitution theorem for formulas

- If t_1 and t_2 are free for x in φ, then

$$\models t_1 \doteq t_2 \to \varphi\left[t_1/x\right] \doteq \varphi\left[t_2/x\right].$$

- If t is a closed term and $\varphi(x)$ is a formula, then

$$\mathfrak{A} \models \varphi[t/x] \iff \mathfrak{A} \models \varphi\left[\overline{t^{\mathfrak{A}}}/x\right].$$

- If $t(y_1, \ldots, y_n)$ and $\varphi(x)$ are terms, then

$$\begin{aligned} & \mathfrak{A} \models \varphi\left[t\left[\overline{a_1}/y_1, \ldots, \overline{a_n}/y_n\right]/x\right] \\ \iff & \mathfrak{A} \models \varphi\left[\overline{\left(t\left[\overline{a_1}/y_1, \ldots, \overline{a_n}/y_n\right]^{\mathfrak{A}}\right)}/x\right] \end{aligned}$$

Contents

Fact

If x does not occur in t, then

- $\models \exists x(x \doteq t)$.
- $\models \varphi[t/x] \leftrightarrow \forall x(x \doteq t \rightarrow \varphi(x))$
- $\models \varphi[t/x] \leftrightarrow \exists x(x \doteq t \wedge \varphi(x))$

$\forall$-introduction rules

- If t is free for x in φ and $\Sigma, \varphi[t/x] \models \psi$, then $\Sigma, \forall x \varphi(x) \models \psi$.
- If x is not free in any formula of Σ, and $\Sigma \models \psi$, then $\Sigma \models \forall x \psi$.

$\exists$-introduction rules

- If t is free for x in φ and $\Sigma \models \varphi[t/x]$, then $\Sigma \models \exists x \varphi(x)$.
- If x is not free in any formula of $\Sigma \cup \{\varphi\}$, and $\Sigma, \varphi \models \psi$, then $\Sigma, \exists x \varphi \models \psi$.

Thanks for your attention!

Q & A