Natural Deduction II

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Rules for ∀

∀I

$$\frac{\varphi(x)}{\forall x \varphi(x)}\ \forall\text{I}$$

where the variable x **does not** occur free in any uncanceled hypothesis in the derivation of $\varphi(x)$.

∀E

$$\frac{\forall x \varphi(x)}{\varphi(t)}\ \forall\text{E}$$

where t is free for x in φ.

Rules for $\forall$

$\forall$I

$$\frac{\varphi(x)}{\forall x \varphi(x)} \; \forall\text{I}$$

where the variable x **does not** occur free in any uncanceled hypothesis in the derivation of $\varphi(x)$.

$\forall$E

$$\frac{\forall x \varphi(x)}{\varphi(t)} \; \forall\text{E}$$

where t is free for x in φ.

Why restriction for ∀I

$$\dfrac{\dfrac{\dfrac{\dfrac{[x \doteq \mathbf{0}]}{\forall x (x \doteq \mathbf{0})}\ \forall\mathrm{I?}}{x \doteq \mathbf{0} \to \forall x(x \doteq \mathbf{0})}\ \to \mathrm{I}}{\forall x(x \doteq \mathbf{0} \to \forall x(x \doteq \mathbf{0}))}\ \forall\mathrm{I}}{\mathbf{0} \doteq \mathbf{0} \to \forall x(x \doteq \mathbf{0})}\ \forall\mathrm{E}$$

Why restriction for $\forall$E

$$\dfrac{\dfrac{[\forall x \neg \forall y (x \doteq y)]}{\neg \forall y (y \doteq y)}\ \forall\text{E?}}{\forall x \neg \forall y (x \doteq y) \to \neg \forall y (y \doteq y)} \to \text{I}$$

Derivation

Definition

- A one-element tree φ is a derivation, whose hypothesis and conclusion are both φ.
- If both $\begin{matrix}\mathcal{D}\\ \varphi\end{matrix}$ and $\begin{matrix}\mathcal{D}'\\ \varphi'\end{matrix}$ are derivations, so is

$$\frac{\begin{matrix}\mathcal{D}\\ \varphi\end{matrix} \qquad \begin{matrix}\mathcal{D}'\\ \varphi'\end{matrix}}{\varphi \wedge \varphi'}$$

  whose conclusion is $\varphi \wedge \varphi'$, and whose hypotheses are the union of those in $\begin{matrix}\mathcal{D}\\ \varphi\end{matrix}$ and those in $\begin{matrix}\mathcal{D}'\\ \varphi'\end{matrix}$.

Derivation

Definition

- A one-element tree φ is a derivation, whose hypothesis and conclusion are both φ.
- If both $\begin{array}{c}\mathcal{D}\\ \varphi\end{array}$ and $\begin{array}{c}\mathcal{D}'\\ \varphi'\end{array}$ are derivations, so is

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  whose conclusion is $\varphi \wedge \varphi'$, and whose hypotheses are the union of those in $\begin{array}{c}\mathcal{D}\\ \varphi\end{array}$ and those in $\begin{array}{c}\mathcal{D}'\\ \varphi'\end{array}$.

Derivation (continued)

Definition

- If $\begin{array}{c}\mathcal{D}\\ \varphi \wedge \psi\end{array}$ is a derivation, then the following two items are also derivations

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \varphi \wedge \psi\end{array}}{\varphi}, \qquad \dfrac{\begin{array}{c}\mathcal{D}\\ \varphi \wedge \psi\end{array}}{\psi}$$

whose conclusions are respective φ and ψ, and whose hypotheses are exactly the same as those in $\begin{array}{c}\mathcal{D}\\ \varphi \wedge \psi\end{array}$

Derivation (continued)

Definition

- If both $\begin{matrix}\varphi \\ \mathcal{D} \\ \psi\end{matrix}$ is a derivation, so is

$$\dfrac{\begin{matrix}[\varphi] \\ \mathcal{D} \\ \psi\end{matrix}}{\varphi \to \psi}$$

whose conclusion is $\varphi \to \psi$, and whose hypotheses are those in $\begin{matrix}\varphi \\ \mathcal{D} \\ \psi\end{matrix}$ minus φ.

Derivation (continued)

Definition

- If both $\begin{matrix}\mathcal{D}\\ \varphi\end{matrix}$ and $\begin{matrix}\mathcal{D}\\ \varphi \to \psi\end{matrix}$ are derivations, so is

$$\frac{\begin{matrix}\mathcal{D}\\ \varphi\end{matrix} \qquad \begin{matrix}\mathcal{D}'\\ \varphi \to \psi\end{matrix}}{\psi}$$

whose conclusion is ψ, and whose hypotheses are union of those in $\begin{matrix}\mathcal{D}\\ \varphi\end{matrix}$ and those in $\begin{matrix}\mathcal{D}\\ \varphi \to \psi\end{matrix}$

Derivation (continued)

Definition

- If $\begin{array}{c}\mathcal{D}\\ \bot\end{array}$ is a derivation, so is

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \bot\end{array}}{\varphi}$$

whose conclusion is φ, and whose hypotheses are exactly the same as those in $\begin{array}{c}\mathcal{D}\\ \bot\end{array}$

Derivation (continued)

Definition

- If $\begin{matrix}\neg\varphi \\ \mathcal{D} \\ \bot\end{matrix}$ is a derivation, so is

$$\dfrac{\begin{matrix}[\neg\varphi] \\ \mathcal{D} \\ \bot\end{matrix}}{\varphi}$$

whose conclusion is φ, and whose hypotheses are those in $\begin{matrix}\neg\varphi \\ \mathcal{D} \\ \bot\end{matrix}$ minus $\neg\varphi$.

Derivation (continued)

Definition

- If $\begin{array}{c}\mathcal{D}\\ \varphi(x)\end{array}$ is a derivation and x does not occur free in any (uncanceled) hypothesis of $\mathcal{D}$, so is

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \varphi(x)\end{array}}{\forall x\varphi(x)}$$

whose conclusion is $\forall x\varphi$, and whose hypotheses are exactly the same as those in $\begin{array}{c}\mathcal{D}\\ \varphi(x)\end{array}$

Derivation (continued)

Definition

- If $\begin{array}{c}\mathcal{D}\\ \forall x \varphi(x)\end{array}$ is a derivation and t is free for x in $\varphi(x)$, so is

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \forall x \varphi(x)\end{array}}{\varphi(t)}$$

whose conclusion is $\varphi(t)$, and whose hypotheses are exactly the same as those in $\begin{array}{c}\mathcal{D}\\ \forall x \varphi(x)\end{array}$

Derivation (continued)

Definition

- Only the trees obtained by the above rules are the derivations.

Notation

- $\Gamma \vdash \varphi$: there is a derivation with conclusion φ and with all (unconceled) hypotheses in Γ.
  In this case, we also say: φ is derivable from Γ.
- We use $\vdash \varphi$ instead of $\varnothing \vdash \varphi$.
  In this case, we say, φ is a theorem (provable).

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- We use $\vdash \varphi$ instead of $\varnothing \vdash \varphi$.
  In this case, we say, φ is a theorem (provable).

Example

- $\vdash \forall x \forall y \varphi(x,y) \to \forall y \forall x \varphi(x,y)$

Example

- $\vdash \forall x(\varphi \wedge \psi) \leftrightarrow (\forall x\varphi \wedge \forall x\psi)$

Example

- $\vdash \forall x(\varphi \to \psi) \leftrightarrow (\varphi \to \forall x\psi)$, where x does not occur free in φ.

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Definition

In (Van Dalen's) derivations, we use the following definitions:

$$\begin{aligned} \neg\varphi &=_{\text{df}} \varphi \to \bot \\ \varphi \vee \psi &=_{\text{df}} \neg(\neg\varphi \wedge \neg\psi) \\ \varphi \leftrightarrow \psi &=_{\text{df}} (\varphi \to \psi) \wedge (\psi \to \varphi) \\ \exists x\varphi &=_{\text{df}} \neg\forall x\neg\varphi \end{aligned}$$

Example

- If t is free for x in $\varphi(x)$, then $\varphi(t) \vdash \exists x \varphi(x)$

  Note: This can be used as a derived rule.

$\exists$I

$$\frac{\varphi(t)}{\exists x \varphi(x)} \exists\mathrm{I}$$

Example

- If x is not free in ψ or any formula of Γ, and $\Gamma, \varphi \vdash \psi$, then $\Gamma, \exists x \varphi \vdash \psi$

  Note: This can be used as a derived rule.

$\exists$E

$$\dfrac{\exists x \varphi \qquad \begin{matrix} [\varphi] \\ \vdots \\ \psi \end{matrix}}{\psi} \; \exists\mathrm{E}$$

Example

- $\vdash \forall x\varphi \to \exists x\varphi$

Example

- $\vdash \exists x \forall y \varphi \to \forall y \exists x \varphi$

Example

- $\vdash \exists x(\varphi \vee \psi) \to \exists x\varphi \vee \exists x\psi$

Contents

Rules for $\doteq$

RI_1

$$\frac{}{x \doteq x}\ \mathrm{RI}_1$$

RI_2

$$\frac{x \doteq y}{y \doteq x}\ \mathrm{RI}_2$$

RI_3

$$\frac{x \doteq y \qquad y \doteq z}{x \doteq z}\ \mathrm{RI}_3$$

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$$\frac{}{x \doteq x}\ \mathsf{RI}_1$$

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RI_3

$$\frac{x \doteq y \qquad y \doteq z}{x \doteq z}\ \mathsf{RI}_3$$

Rules for $\doteq$ (continued)

RI_4

$$\frac{x_1 \doteq y_1, \ldots, x_n \doteq y_n}{t(x_1, \ldots, x_n) \doteq t[y_1/x_1, \ldots, y_n/x_n]}\ \mathsf{RI}_4$$

RI_5

$$\frac{x_1 \doteq y_1, \ldots, x_n \doteq y_n}{\varphi(x_1, \ldots, x_n) \leftrightarrow \varphi[y_1/x_1, \ldots, y_n/x_n]}\ \mathsf{RI}_5$$

where y_1, …, y_n are free for x_1, …, x_n in φ.

Rules for $\doteq$ (continued)

RI_4

$$\frac{x_1 \doteq y_1, \ldots, x_n \doteq y_n}{t(x_1, \ldots, x_n) \doteq t[y_1/x_1, \ldots, y_n/x_n]} \ \mathsf{RI}_4$$

RI_5

$$\frac{x_1 \doteq y_1, \ldots, x_n \doteq y_n}{\varphi(x_1, \ldots, x_n) \leftrightarrow \varphi[y_1/x_1, \ldots, y_n/x_n]} \ \mathsf{RI}_5$$

where y_1, …, y_n are free for x_1, …, x_n in φ.

Example

- If x does not occur in t, then $\vdash \exists x(x \doteq t)$.

Example

- If x does not occur in t, then $\vdash \exists x(x \doteq t)$.

  Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{x \doteq x}\ \mathrm{RI}_1}{\forall x(x \doteq x)}\ \forall\mathrm{I}}{t \doteq t}\ \forall\mathrm{E}}{\exists x(x \doteq t)}\ \exists\mathrm{I}$$

  The trick is in the last step: $t \doteq t$ is taken as $(x \doteq t)[t/x]$.

Example

- If t is free for x in φ, then $x \doteq t \vdash \varphi(x) \leftrightarrow \varphi(t)$.

  Note: This can be used as a derived rule.

$$\frac{x \doteq t}{\varphi(x) \leftrightarrow \varphi(t)}$$

In general, if t_1, ..., t_n are free for x_1, ..., x_n in φ, then

$$x_1 \doteq t_1, \ldots, x_n \doteq t_n \vdash \varphi(x_1, \ldots, x_n) \leftrightarrow \varphi[t_1/x_1, \ldots, t_n/x_n].$$

It is a strengthening of Rule RI_5. We call it "RI_5^+".

Example

- If t is free for x in φ, then $x \doteq t \vdash \varphi(x) \leftrightarrow \varphi(t)$.

  Note: This can be used as a derived rule.

$$\frac{x \doteq t}{\varphi(x) \leftrightarrow \varphi(t)}$$

In general, if t_1, ..., t_n are free for x_1, ..., x_n in φ, then

$$x_1 \doteq t_1, \ldots, x_n \doteq t_n \vdash \varphi(x_1, \ldots, x_n) \leftrightarrow \varphi[t_1/x_1, \ldots, t_n/x_n].$$

It is a strengthening of Rule RI$_5$. We call it "RI$_5^+$".

Example

- If t is free for x in φ, then $x \doteq t \vdash \varphi(x) \leftrightarrow \varphi(t)$.

Proof. Let y be a variable that does not occur in φ. We have the following derivation:

$$
\dfrac{x \doteq t \qquad \dfrac{\dfrac{\dfrac{\dfrac{[x \doteq y]}{\varphi(x) \leftrightarrow \varphi[y/x]}\,\mathrm{RI}_5}{x \doteq y \to (\varphi(x) \leftrightarrow \varphi[y/x])}\to \mathrm{I}}{\forall y(x \doteq y \to (\varphi(x) \leftrightarrow \varphi[y/x]))}\,\forall \mathrm{I}}{x \doteq t \to (\varphi(x) \leftrightarrow \varphi[y/x][t/y]))}\,\forall \mathrm{E}}{\varphi(x) \leftrightarrow \varphi[y/x][t/y]}\to \mathrm{E}
$$

Note that $\varphi[y/x][t/y]$ is exactly $\varphi[t/x]$. We thus obtain $x \doteq t \vdash \varphi(x) \leftrightarrow \varphi(t)$.

Example

- If x does not occur in t and t is free for x in φ, then $\vdash \varphi[t/x] \leftrightarrow \exists x(x \doteq t \wedge \varphi)$.

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- If x does not occur in t and t is free for x in φ, then $\vdash \varphi[t/x] \leftrightarrow \exists x(x \doteq t \wedge \varphi)$.

  Proof.

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{x \doteq x}\,\mathrm{RI}_1}{\forall x(x \doteq x)}\,\forall\mathrm{I}}{t \doteq t}\,\forall\mathrm{E} \qquad [\varphi[t/x]]}{\dfrac{\dfrac{t \doteq t \wedge \varphi[t/x]}{\exists x(x \doteq t \wedge \varphi)}\,\exists\mathrm{I}}{\varphi[t/x] \to \exists x(x \doteq t \wedge \varphi)}\,\to\mathrm{I}}\,\wedge\mathrm{I}
$$

  This shows $\vdash \varphi[t/x] \to \exists x(x \doteq t \wedge \varphi)$.

Example

- If x does not occur in t and t is free for x in φ, then $\vdash \varphi[t/x] \leftrightarrow \exists x(x \doteq t \wedge \varphi)$.

  Proof (continued).

$$
\dfrac{[\exists x(x \doteq t \wedge \varphi)]^2 \qquad \dfrac{\dfrac{\dfrac{\dfrac{[x \doteq t \wedge \varphi]^1}{x \doteq t}\wedge\mathrm{E}}{\varphi(x) \leftrightarrow \varphi[t/x]}\mathrm{RI}_5^+}{\varphi(x) \to \varphi[t/x]}\wedge\mathrm{E} \qquad \dfrac{[x \doteq t \wedge \varphi]^1}{\varphi}\wedge\mathrm{E}}{\varphi[t/x]}\to\mathrm{E}}{\dfrac{\varphi[t/x]}{\exists x(x \doteq t \wedge \varphi) \to \varphi[t/x]}\to\mathrm{I}}\exists\mathrm{E}
$$

  This shows $\vdash \exists x(x \doteq t \wedge \varphi) \to \varphi[t/x]$.

  In conclusion, we obtain $\vdash \varphi[t/x] \leftrightarrow \exists x(x \doteq t \wedge \varphi)$.

Example

- If x does not occur in t and t is free for x in φ, then $\vdash \varphi[t/x] \leftrightarrow \exists x(x\doteq t \wedge \varphi)$.

  Proof (continued).

$$
\dfrac{[\exists x(x\doteq t \wedge \varphi)]^2 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[x\doteq t \wedge \varphi]^1}{x\doteq t}\wedge E}{\varphi(x) \leftrightarrow \varphi[t/x]}RI_5^+}{\varphi(x) \to \varphi[t/x]}\wedge E \qquad \dfrac{[x\doteq t \wedge \varphi]^1}{\varphi}\wedge E}{\varphi[t/x]}\to E}{\dfrac{\varphi[t/x]}{\exists x(x\doteq t \wedge \varphi) \to \varphi[t/x]}\to I}\exists E
$$

  This shows $\vdash \exists x(x\doteq t \wedge \varphi) \to \varphi[t/x]$.

  In conclusion, we obtain $\vdash \varphi[t/x] \leftrightarrow \exists x(x\doteq t \wedge \varphi)$.

Example

- If x does not occur in t and t is free for x in φ, then $\vdash \varphi[t/x] \leftrightarrow \forall x(x \doteq t \to \varphi)$.

Thanks for your attention!

Q & A