Completeness for First-order Logic II

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A equivalence relation

Let T_m be a maximally consistent Henkin set in a language $\mathcal{L}_m$. We define:

- A is the set of all closed terms in $\mathcal{L}_m$.
- $\sim$ is a binary relation on A such that $t \sim s$, iff $T_m \vdash t \doteq s$.

Claim: $\sim$ is an equivalence relation on A.

Proof of Claim.

Since $T_m \vdash t \doteq t$, $\sim$ is reflexive: for any $t \in A$, $t \sim t$.

And, since $T_m \vdash t \doteq s$ implies $T_m \vdash s \doteq t$, $\sim$ is symmetric: for any $s, t \in A$, $t \sim s$ implies $s \sim t$.

Third, since $T_m \vdash t \doteq s$ and $T_m \vdash s \doteq u$ implies $T_m \vdash t \doteq u$, $\sim$ is transitive: for any $s, t, u \in A$, $t \sim s$ and $s \sim u$ implies $t \sim u$.

Claim:

- If $t_i \sim s_i$ $(1 \leq i \leq p)$, then $T_m \vdash P(t_1, \ldots, t_p)$ iff $T_m \vdash P(s_1, \ldots, s_p)$.
- If $t_i \sim s_i$ $(1 \leq i \leq q)$, then $f(t_1, \ldots, t_q) \sim f(s_1, \ldots, s_q)$.

Proof of Claim. If $t_i \sim s_i$ $(1 \leq i \leq p)$, then $T_m \vdash t_i \doteq s_i$ for all $1 \leq i \leq p$. By a lemma of the text, we can deduce $T_m \vdash P(t_1, \ldots, t_p)$ from $T_m \vdash P(s_1, \ldots, s_p)$, and the converse.

Canonical Structure (典范结构)

- For any $t \in A$, let $[t]$, or precisely $[t]_\sim$, be the set $\{s \in A | s \sim t\}$. $[t]$ is called the **equivalence class of t under $\sim$**.
- Let $A/\sim$ be the set $\{[t] | t \in A\}$. $A/\sim$ is called the **quotient set of A by $\sim$**.

We now build the **quotient structure** $\mathfrak{A}$ as follows:

$$\mathfrak{A} = \langle A/\sim, P_1^{\mathfrak{A}}, \ldots, P_n^{\mathfrak{A}}, f_1^{\mathfrak{A}}, \ldots, f_m^{\mathfrak{A}}, \{c_k^{\mathfrak{A}} | k \in K\} \rangle$$

where

- $P_i^{\mathfrak{A}}([t_1], \ldots, [t_{r_i}])$ holds, iff $T_m \vdash P_i(t_1, \ldots, t_{r_i})$.
- $f_j^{\mathfrak{A}}([t_1], \ldots, [t_{a_j}]) = [f_j(t_1, \ldots, t_{a_j})]$.
- $c^{\mathfrak{A}} = [c]$.

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- $P_i^{\mathfrak{A}}([t_1], \ldots, [t_{r_i}])$ holds, iff $T_m \vdash P_i(t_1, \ldots, t_{r_i})$.
- $f_j^{\mathfrak{A}}\left([t_1], \ldots, [t_{a_j}]\right) = [f_j(t_1, \ldots, t_{a_j})]$.
- $c^{\mathfrak{A}} = [c]$.

Claim: $P_i^{\mathfrak{A}}$ and $f_j^{\mathfrak{A}}$ are well-defined, that is, their definitions are independent of the particular elements (reprensentive elements) chosen from the equivalence classes.

$P_i^{\mathfrak{A}}([t_1], \ldots, [t_{r_i}])$ holds, iff $T_m \vdash P_i(t_1, \ldots, t_{r_i})$

$P_i^{\mathfrak{A}}([t'_1], \ldots, [t'_{r_i}])$ holds, iff $T_m \vdash P_i(t'_1, \ldots, t'_{r_i})$, where $[t'_1] = [t_1]$, $\ldots$, $[t'_{r_i}] = [t_{r_i}]$.

When $[t'_1] = [t_1]$ (i.e., $t'_1 \sim t_1$), $\ldots$, $[t'_{r_i}] = [t_{r_i}]$ (i.e., $t'_{r_i} \sim t_{r_i}$), we have:

$T_m \vdash P_i(t'_1, \ldots, t'_{r_i})$, iff $T_m \vdash P_i(t_1, \ldots, t_{r_i})$.

Claim: For any closed term t in $\mathcal{L}_m$, $t^{\mathfrak{A}} = [t]$.

Proof of Claim. (1) if $t = c$, then by $t^{\mathfrak{A}} = [c]$.

(2) Suppose $t = f_j(t_1, \ldots, t_{a_j})$, then

$$\begin{aligned} t^{\mathfrak{A}} &= (f_j(t_1, \ldots, t_{a_j}))^{\mathfrak{A}} \\ &= f_j^{\mathfrak{A}}\left(t_1^{\mathfrak{A}}, \ldots, t_{a_j}^{\mathfrak{A}}\right) \\ &= f_j^{\mathfrak{A}}\left([t_1], \ldots, [t_{a_j}]\right) \qquad \text{(by IH } t_1^{\mathfrak{A}} = [t_1] \text{ and so on)} \\ &= [f_j(t_1, \ldots, t_{a_j})] \\ &= [t]. \end{aligned}$$

Claim: For any sentence φ in $\mathcal{L}_m$, $\mathfrak{A} \models \varphi$, iff $T_m \vdash \varphi$.

We prove this claim by induction on φ.

Case (i): φ is atomic.

Case (i)-a: φ is $P_i(t_1, \ldots, t_{r_i})$. We want $\mathfrak{A} \models P_i(t_1, \ldots, t_{r_i})$, iff $T_m \vdash P_i(t_1, \ldots, t_{r_i})$. That is, $P_i^{\mathfrak{A}}\left(t_1^{\mathfrak{A}}, \ldots, t_{r_i}^{\mathfrak{A}}\right)$, iff $T_m \vdash P_i(t_1, \ldots, t_{r_i})$. Equivalently, $P_i^{\mathfrak{A}}([t_1], \ldots, [t_{r_i}])$, iff $T_m \vdash P_i(t_1, \ldots, t_{r_i})$. We are done.

Case (i)-b: φ is $t_1 \doteq t_2$. We want $\mathfrak{A} \models t_1 \doteq t_2$, iff $T_m \vdash t_1 \doteq t_2$. That is, $t_1^{\mathfrak{A}} = t_2^{\mathfrak{A}}$, iff $T_m \vdash t_1 \doteq t_2$. Equivalently, $[t_1] = [t_2]$, iff $T_m \vdash t_1 \doteq t_2$. In other words, $t_1 \sim t_2$, iff $T_m \vdash t_1 \doteq t_2$. We are done.

Claim: For any sentence φ in $\mathcal{L}_m$, $\mathfrak{A} \models \varphi$, iff $T_m \vdash \varphi$.

We prove this claim by induction on φ.

Case (ii): $\varphi = \neg\psi$.

We want $\mathfrak{A} \models \neg\psi$, iff $T_m \vdash \neg\psi$.

In other words, $\mathfrak{A} \not\models \psi$, iff $T_m \vdash \neg\psi$.

By IH, $\mathfrak{A} \not\models \psi$, iff $T_m \nvdash \psi$.

Now, we need to prove $T_m \nvdash \psi$, iff $T_m \vdash \neg\psi$.

Claim: For any sentence φ in $\mathcal{L}_m$, $\mathfrak{A} \models \varphi$, iff $T_m \vdash \varphi$.

We prove this claim by induction on φ.

Case (iii): $\varphi = \sigma \to \tau$.

Claim: For any sentence φ in $\mathcal{L}_m$, $\mathfrak{A} \models \varphi$, iff $T_m \vdash \varphi$.

We prove this claim by induction on φ.

Case (iv): $\varphi = \forall x \psi(x)$.

We want $\mathfrak{A} \models \forall x \psi(x)$, iff $T_m \vdash \forall x \psi(x)$.

$$
\begin{aligned}
\mathfrak{A} \models \forall x \psi(x), \quad & \text{iff} \quad \mathfrak{A} \models \psi(\overline{a}) \text{ for all } a \in A/\sim \\
& \text{iff} \quad T_m \vdash \psi(\overline{a}) \text{ for all } a \in A/\sim \\
& \text{iff} \quad T_m \vdash \psi(\overline{a}) \text{ for all } a = [t] \in A/\sim \\
& \text{iff} \quad T_m \vdash \psi\left(\overline{[t]}\right) \text{ for all closed term } t
\end{aligned}
$$

To prove the original result, we only need to prove $T_m \vdash \forall x \psi(x)$, iff $T_m \vdash \psi(\overline{a})$ holds for all $a \in A/\sim$.

The side from left to right is obvious owing to the fact $\vdash \forall x \psi(x) \to \psi(\overline{a})$.

Claim: For any sentence φ in $\mathcal{L}_m$, $\mathfrak{A} \models \varphi$, iff $T_m \vdash \varphi$.

We prove this claim by induction on φ.

Case (iv): $\varphi = \forall x \psi(x)$.

We prove $T_m \vdash \forall x \psi(x)$ from $T_m \vdash \psi(\overline{a})$ holds for all $a \in A/\sim$.

We suppose $T_m \nvdash \forall x \psi(x)$, by maximal consistency of T_m, $T_m \vdash \neg \forall x \psi(x)$, then $T_m \vdash \exists x \neg \psi(x)$. Since T_m is a Henkin theory, $T_m \vdash \exists x \neg \psi(x) \rightarrow \neg \psi(c)$ for some constant c in $\mathcal{L}_m$. Thus, $T_m \vdash \neg \psi(c)$ for some constant c in $\mathcal{L}_m$.

$T_m \vdash c \doteq \overline{[c]}$, iff $\mathfrak{A} \models c \doteq \overline{[c]}$, iff $c^{\mathfrak{A}} = \overline{[c]}^{\mathfrak{A}}$. Since $\overline{[c]}^{\mathfrak{A}} = [c]$, $c^{\mathfrak{A}} = \overline{[c]}^{\mathfrak{A}}$. Hence, $T_m \vdash c \doteq \overline{[c]}$.

Therefore, $T_m \vdash \neg \psi\left(\overline{[c]}\right)$ for some constant c in $\mathcal{L}_m$. That is, $T_m \vdash \neg \psi\left(\overline{[c]}\right)$ for some $[c] \in A/\sim$.

We are done.

Proof of the completeness theorem

Let Γ be a set of sentences of $\mathcal{L}$ and σ be a sentence of $\mathcal{L}$. If Γ is consistent, then Γ has a model.

Proof. We extend Γ to a maximally consistent Henkin set T_m. Let T_m is contained in the language $\mathcal{L}_m$.

Then, we have known that the quotient structure $\mathfrak{A}$ is a model of T_m. Let $\mathfrak{A} \restriction_{\mathcal{L}}$ be the structure obtained from $\mathfrak{A}$ by dismissing the interpretations of those constant symbols occurring in $\mathcal{L}_m$ but not in $\mathcal{L}$. Then, clearly, $\mathfrak{A} \restriction_{\mathcal{L}}$ is a model of Γ. QED

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Gödel (1929)

Let Γ be a set of sentences of $\mathcal{L}$. Γ has a model, iff each finite subset of Γ has a model.

Proof.

An application

Let $Th(\mathfrak{N})$ be the set of all sentences φ satisfying $\mathfrak{A} \models \varphi$. It is called the **true arithmetic**.

Let $\mathcal{L}_A^+$ be $\mathcal{L}_A \cup \{c\}$, where c is a new constant. Consider the set of $\mathcal{L}_A^+$:

$$Th(\mathfrak{N}) \cup \{c > \overline{n} | n \in \mathbb{N}\}$$

Claim. The above set has a model.

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Proof.

An application

Let $\mathfrak{A}$ be the model that we just claim. Then, we can pick $\mathfrak{A}$ such that it is also countable. Let $\mathfrak{A} \upharpoonright_{\mathcal{L}_A}$ be the structure obtained from $\mathfrak{A}$ by dismissing the interpretation of c.

Then, $\mathfrak{A} \upharpoonright_{\mathcal{L}_A}$ is a structure of $\mathcal{L}_A$, which have the same true sentences as $\mathfrak{N}$, since $\mathfrak{A} \upharpoonright_{\mathcal{L}_A} \models \varphi$, iff $\mathfrak{N} \models \varphi$. However, $\mathfrak{A} \upharpoonright_{\mathcal{L}_A}$ contains an infinitely large "number". $\mathfrak{A} \upharpoonright_{\mathcal{L}_A}$ is a **non-standard** model for true arithmetic.

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Overspill lemma

If $\varphi(\overline{n})$ holds in a non-standard model for infinitely many (finite) numbers n, then $\varphi(\overline{a})$ holds for at least one infinite number a.

Proof.

Thanks for your attention!

Q & A