Completeness for First-order Logic I

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Contents

A stipulation

In the following, unless otherwise claimed, φ, ψ and so on are used to denote a sentence, and Γ, Σ, T and so on are used to denote a set of sentences.

Gödel's Completeness Theorem

Gödel (1929)

Let Γ be a set of sentences of $\mathcal{L}$ and σ be a sentence of $\mathcal{L}$. If $\Gamma \models \sigma$, then $\Gamma \vdash \sigma$.

Note. This result also holds for any set Γ of formulas and any formula σ. However, we usually state and prove this result for sentences.

An equivalent statement

- A set Γ of formulas is **consistent**, if it is not the case $\Gamma \vdash \bot$, otherwise, it is inconsistent.

Let Γ be a set of sentences of $\mathcal{L}$ and σ be a sentence of $\mathcal{L}$. If Γ is consistent, then Γ has a model.

Proof of equivalence.

The sketch of the proof

Let Γ be a set of sentences of $\mathcal{L}$ and σ be a sentence of $\mathcal{L}$. If Γ is consistent, then Γ has a model.

Sketch of Proof.

Step 1: extend Γ to a maximally consistent and Henkin set T_m.

Step 2: "construct a model of T_m using T_m itself."

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Definition

- A set Γ of sentences is **closed under derivability**, if whenever $\Gamma \vdash \varphi$, we always have $\varphi \in \Gamma$.
- A **theory** is a set of sentences that is closed under derivability.

Let $T = \{\varphi | \Gamma \vdash \varphi\}$. Then, T is a theory. Usually, Γ is called an **axiom set** of T. The elements of Γ are called **axioms**.

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Axioms for Peano arithmetic

- $\forall x(\neg \boldsymbol{S}x \doteq \mathbf{0})$
- $\forall x_1 \forall x_2(\boldsymbol{S}x_1 \doteq \boldsymbol{S}x_2 \to x_1 \doteq x_2)$
- $\forall x(x + \mathbf{0} \doteq x)$
- $\forall x_1 \forall x_2(x_1 + \boldsymbol{S}x_2 \doteq \boldsymbol{S}(x_1 + x_2))$
- $\forall x(x \times \mathbf{0} \doteq \mathbf{0})$
- $\forall x_1 \forall x_2(x_1 \times \boldsymbol{S}x_2 \doteq (x_1 \times x_2) + x_1)$

and all instances from the following schema (**induction schema**):

- $\varphi(\mathbf{0}/x) \wedge \forall x(\varphi(x) \to \varphi(\boldsymbol{S}x/x)) \to \forall x \varphi(x)$

Peano arithmetic

Let Γ be the set of the above-mentioned axioms, and let

$$\mathbf{PA} = \{\varphi | \Gamma \vdash \varphi\}$$

where φ is a sentence in the first-order arithmetic language $\mathcal{L}_A$.

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Example

Prove: $\mathbf{PA} \vdash 0 + S0 \doteq S0$.

Definition

- A set Γ of sentences is **inconsistent** if $\Gamma \vdash \bot$; otherwise, it is **consistent**.
- A set Γ of sentences is **maximally consistent** if
  - Γ is consistent, and
  - provided that $\Gamma \subsetneq \Gamma'$, then Γ' is inconsistent.

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Facts about (in)consistency

The following three conditions are equivalent:

- Γ is inconsistent,
- For some φ, $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$,
- For any φ, $\Gamma \vdash \varphi$.

Facts about (in)consistency

- $\Gamma \cup \{\neg\varphi\}$ is inconsistent, iff $\Gamma \vdash \varphi$
- $\Gamma \cup \{\varphi\}$ is inconsistent, iff $\Gamma \vdash \neg\varphi$.

Example

Prove. $\mathbf{PA}$ is consistent. (Hint: by the soundness theorem)

Facts about maximal consistency

Let Γ be a maximally consistent set of sentences.

- Γ is closed under derivability, and thus is a theory.
- for any φ, $\neg\varphi \in \Gamma$, iff $\varphi \notin \Gamma$.
- for any φ, $\varphi \to \psi \in \Gamma$, iff either $\varphi \notin \Gamma$, or $\psi \in \Gamma$.

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Lemma

Each consistent set can be extended to be a maximally consistent theory.

Proof.

Contents

Definition

- A theory T in a language $\mathcal{L}$ is called a **Henkin theory** if for each sentence $\exists x \varphi(x)$, there is a constant c in $\mathcal{L}$ such that $\exists x \varphi(x) \to \varphi(c) \in T$ (such a c is called a **witness** for $\exists x \varphi(x)$).
- Let T_i be a theory in the language $\mathcal{L}_i$, $i = 0, 1$.
  - T_1 is an **extension** of T_0, if T_0 is a subset of T_1.
  - T_1 is an **conservative extension** of T_0, if the sentences in both T_1 and $\mathcal{L}_0$ are exactly those in T_0.

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Definition

- Let T be a theory in the language $\mathcal{L}$. Let $\mathcal{L}^*$ be the language obtained from $\mathcal{L}$ by adding infinite new constants:

$$\mathcal{L}^* = \mathcal{L} \cup \{c_\varphi | \exists x \varphi \in \mathcal{L}\}.$$

Let T^* be the theory whose axiom set are

$$T \cup \{\exists x \varphi(x) \to \varphi(c_\varphi) | \exists x \varphi \in \mathcal{L}\}.$$

Lemma

If T is consistent, so is T^*.

Proof.

If T is consistent, $T \cup \{\exists x \varphi(x) \to \varphi(c)\}$ is consistent.

Suppose $T \cup \{\exists x \varphi(x) \to \varphi(c)\}$ is inconsistent, that is, $T \cup \{\exists x \varphi(x) \to \varphi(c)\} \vdash \bot$. Then, $T \vdash \neg(\exists x \varphi(x) \to \varphi(c))$, and so

$$T \vdash \exists x \varphi(x) \wedge \neg\varphi(c).$$

Since $T \vdash \neg\varphi(c)$, it follows that $T \vdash \forall x \neg\varphi(x)$. On the other hand, $T \vdash \exists x \varphi(x)$, that is, $T \vdash \neg\forall x \neg\varphi(x)$. Thus, T is inconsistent.

$$T^* = \{\varphi | \underline{T \cup \{\exists x \varphi(x) \to \varphi(c_\varphi) | \exists x \varphi \in \mathcal{L}\}} \vdash \varphi\}$$

To prove T^* is consistent, we only need to prove $T \cup \{\exists x \varphi(x) \to \varphi(c_\varphi) | \exists x \varphi \in \mathcal{L}\}$ is consistent.

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Suppose $T \cup \{\exists x\varphi(x) \rightarrow \varphi(c_\varphi) | \exists x\varphi \in \mathcal{L}\}$ is inconsistent, then $T \cup \{\exists x\varphi(x) \rightarrow \varphi(c_\varphi) | \exists x\varphi \in \mathcal{L}\} \vdash \perp$, and so, there exists N such that $T \cup \{\exists x\varphi_i(x) \rightarrow \varphi_i(c_\varphi) | \exists x\varphi_i \in \mathcal{L}, 1 \leq i \leq N\} \vdash \perp$. Therefore, By the result that we just prove, T is inconsistent. A contradiction!

Lemma

Suppose T_0 is a consistent. For any $n \in \mathbb{N}$, let $T_{n+1} = (T_n)^*$. Let $T_\omega = \bigcup_{n\in\mathbb{N}} T_n$. Then T_ω is a consistent Henkin theory.

Proof. Note that T^* is not necessarily a Henkin theory in $\mathcal{L}^*$.

$$T_0 \subseteq (T_0)^* = T_1 \subseteq (T_1)^* = T_2 \subseteq (T_2)^* = T_3 \subseteq \ldots\ldots$$

$$\mathcal{L}_0 \subseteq (\mathcal{L}_0)^* = \mathcal{L}_1 \subseteq (\mathcal{L}_1)^* = \mathcal{L}_2 \subseteq (\mathcal{L}_2)^* = \mathcal{L}_3 \subseteq \ldots\ldots$$

$$T_\omega = T_0 \cup T_1 \cup T_2 \cup T_3 \cup \ldots\ldots$$

By the finiteness of derivation, we can see T_ω is consistent from every T_n is consistent. Suppose T_ω is inconsistent, then $T_\omega \vdash \bot$. Thus, $\varphi_1, \ldots, \varphi_N \vdash \bot$, where $\varphi_1, \ldots, \varphi_N \in T_\omega$. For any $1 \leq i \leq N$, $\varphi_i \in T_\omega$, then there exists m_i such that $\varphi_i \in T_{m_i}$.

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Proof.

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Lemma

Suppose T_0 is a consistent. For any $n \in \mathbb{N}$, let $T_{n+1} = (T_n)^*$. Let $T_\omega = \bigcup_{n\in\mathbb{N}} T_n$. Then T_ω is a consistent Henkin theory.

Proof. We next prove T_ω is a theory. Suppose $T_\omega \vdash \varphi$, we want $\varphi \in T_\omega$. By the finiteness of derivation, from $T_\omega \vdash \varphi$, $\varphi_1, \dots, \varphi_N \vdash \varphi$, where $\varphi_1, \dots, \varphi_N \in T_\omega$. For any $1 \le i \le N$, $\varphi_i \in T_\omega$, then there exists m_i such that $\varphi_i \in T_{m_i}$. Let m be the largest one among m_i for all $1 \le i \le N$. Then, for all $1 \le i \le N$, $\varphi_i \in T_m$. Then, $T_m \vdash \varphi$, and since T_m is a theory, $\varphi \in T_m$. Hence, $\varphi \in T_\omega$.

Lemma

Suppose T_0 is a consistent. For any $n \in \mathbb{N}$, let $T_{n+1} = (T_n)^*$. Let $T_\omega = \bigcup_{n\in\mathbb{N}} T_n$. Then T_ω is a consistent Henkin theory.

Proof. We next prove T_ω is a Henkin theory.

$$\mathcal{L}_\omega = \mathcal{L}_0 \cup \mathcal{L}_1 \cup \mathcal{L}_2 \cup \ldots\ldots,$$

For any $\exists x\varphi(x) \in \mathcal{L}_\omega$, we want to prove that for some $c \in \mathcal{L}_\omega$,

$$\exists x\varphi(x) \to \varphi(c) \in T_\omega.$$

Since $\exists x\varphi(x) \in \mathcal{L}_\omega$, we know $\exists x\varphi(x) \in \mathcal{L}_N$ for some N.

$$T_0 \subseteq \ldots\ldots \subseteq T_N \subseteq (T_N)^* = T_{N+1} \subseteq \ldots\ldots$$

Then, $\exists x\varphi(x) \to \varphi(c_\varphi) \in T_{N+1}$. Thus, $\exists x\varphi(x) \to \varphi(c_\varphi) \in T_\omega$.

Corollary

Each consistent set can be extended to be a maximally consistent theory such that it is also a Henkin theory.

Proof. Suppose Σ is a consistent set, then by the previous lemma, there exists $T \supseteq \Sigma$ such that T is a consistent Henkin Theory. Then, by Lindenbaum's lemma, T can be extended to T' such that T' is a maximally consistent theory.

We claim T' is a Henkin theory. This is easy by Lemma 4.1.10.

Thanks for your attention!

Q & A