Soundness for First-order Logic

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Definition

Let Γ be a set of **formulas** of $\mathcal{L}$, and let σ be a **formulas** of $\mathcal{L}$. Suppose all formulas in $\Gamma \cup \varphi$ contain only free variables x_1, …, x_n.

- For a structure $\mathfrak{A}$ (for the language $\mathcal{L}(\mathfrak{A})$) and for a_1, …, $a_n \in A$, we use $\mathfrak{A} \models \Gamma(a_1, \ldots, a_n)$ to denote $\mathfrak{A} \models \varphi(a_1, \ldots, a_n)$ holds for all $\varphi \in \Gamma$.
- By $\Gamma \models \sigma$, we mean that for any structure $\mathfrak{A}$, and for any a_1, …, $a_n \in A$, $\mathfrak{A} \models \Gamma(a_1, \ldots, a_n)$ implies $\mathfrak{A} \models \sigma(a_1, \ldots, a_n)$.

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Soundness Theorem and its proof

The soundness theorem

Let Γ and σ be as above. If $\Gamma \vdash \sigma$, then $\Gamma \models \sigma$.

Proof. It suffices to show that for each derivation $\mathcal{D}$ with conclusion σ and hypotheses in Γ, we have $\Gamma \models \sigma$. We prove this result by induction on $\mathcal{D}$.

(basis)

($\wedge$I)

Soundness Theorem and its proof (continued)

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(RAA)

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($\forall$I)

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(∀E)

Soundness Theorem and its proof (continued)

(It suffices to show that for each derivation $\mathcal{D}$ with conclusion σ and hypotheses in Γ, we have $\Gamma \models \sigma$. We prove this result by induction on $\mathcal{D}$.)

(RI$_1$)

(RI$_2$)

Soundness Theorem and its proof (continued)

(It suffices to show that for each derivation $\mathcal{D}$ with conclusion σ and hypotheses in Γ, we have $\Gamma \models \sigma$. We prove this result by induction on $\mathcal{D}$.)

(RI$_3$)

Soundness Theorem and its proof (continued)

(It suffices to show that for each derivation $\mathcal{D}$ with conclusion σ and hypotheses in Γ, we have $\Gamma \models \sigma$. We prove this result by induction on $\mathcal{D}$.)

(RI$_4$)

Soundness Theorem and its proof (continued)

(It suffices to show that for each derivation $\mathcal{D}$ with conclusion σ and hypotheses in Γ, we have $\Gamma \models \sigma$. We prove this result by induction on $\mathcal{D}$.)

(RI$_5$)

Thanks for your attention!

Q & A