Lipschitz property of bistable and combustion fronts

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Reaction-diffusion equation

$$\partial_t u - \Delta u = f(u), \quad 0 < u < 1 \quad \text{in } \mathbb{R}^n \times \mathbb{R}. \tag{1}$$

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Travelling wave $u(x, t) = v(x + \kappa t e_n)$, speed κ in direction $-e_n$.

$$-\Delta v + \kappa \partial_n v = f(v) \quad \text{in } \mathbb{R}^n. \tag{2}$$

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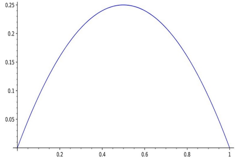
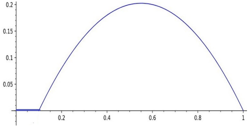
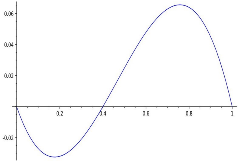


(a) Monostable



(b) Combustion

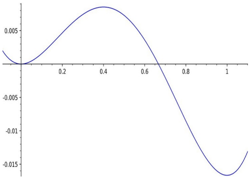


(c) Bistable

Double well structure

$$\partial_t u = \Delta u - W'(u)$$

with W an unbalanced double well potential:



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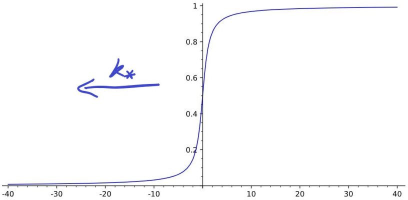

(b) Travelling wave

Travelling waves in $1D$:

$$-g'' + \kappa_* g' = f(g).$$

Propagation phenomena

Theorem (Aronson-Weinberger, Adv. Math. 1978)

Suppose u is a solution to the Cauchy problem for (1) with a compactly supported initial value $u_0 \geq 0$.
If f is monostable, then for any $\delta > 0$, as $t \to \infty$,

- *$u \to 1$ uniformly in $\{|x| < (\kappa_* - \delta)t\}$;*
- *$u \to 0$ uniformly in $\{|x| > (\kappa_* + \delta)t\}$.*

Along the travelling front $\{|x| = \kappa_* t\}$, locally u looks like the $1D$ travelling wave with minimal speed κ_* (with a log shift, [Bramson, Mem. A.M.S. 1983]).

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Fact: 1 represents a more stable state than 0 $\Longrightarrow$ Invading, spreading or propagation.

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- The bistable case is more complicated, because 0 is locally stable.
- Assume the initial value is still compactly supported, then as $t \to \infty$,
  - **Spreading:** $u \to 1$ locally uniformly;
  - **Vanishing:** $u \to 0$ locally uniformly;
  - **Transition:** u converges to an equilibrium state

$$-\Delta w = f(w) \quad \text{in } \mathbb{R}^n.$$

- Zlatos, Matano-Du, Polacik, Du-Lou ...

Front motion

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$$\partial_t u_\varepsilon - \varepsilon \Delta u_\varepsilon = \frac{1}{\varepsilon} f(u_\varepsilon). \tag{3}$$

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[Fife, CBMS Note 1988], [Evans-Souganidis, Indiana 1989], [Barles-Evans-Souganidis, Duke 1990], [Barles-Bronsard-Souganidis, Poincare N.A. 1992], [Barles-Soner-Souganidis, SIAM. C.O. 1993]...

Viscosity approach to front motion

Following Barles-Bronsard-Souganidis, define Φ_ε by $u_\varepsilon := g\left(\frac{\Phi_\varepsilon}{\varepsilon}\right)$.

$$\partial_t \Phi_\varepsilon - \varepsilon \Delta \Phi_\varepsilon = \kappa_* + \frac{g''(\varepsilon^{-1}\Phi_\varepsilon)}{g'(\varepsilon^{-1}\Phi_\varepsilon)} \left(|\nabla \Phi_\varepsilon|^2 - 1\right). \tag{4}$$

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- $\Phi_\varepsilon \to \Phi_\infty$ locally uniformly.

$$\begin{cases} \partial_t \Phi_\infty - \kappa_* - \beta_+ \left(1 - |\nabla \Phi_\infty|^2\right) = 0 & \text{in } \{\Phi_\infty > 0\}, \\ \partial_t \Phi_\infty - \kappa_* + \beta_- \left(1 - |\nabla \Phi_\infty|^2\right) = 0 & \text{in } \{\Phi_\infty > 0\} \end{cases}$$

- Assume $\Phi_\varepsilon(x, 0)$ converges to the signed distance $d(x)$ to a smooth hypersurface $\partial\Omega_0 \Longrightarrow$ $\{\Phi_\infty(x,t) > 0\} = \{d(x) > -\kappa_* t\}$.

Entire solutions and curved TWs

(1) [Hamel-Nadirashvili, CPAM 1999 and ARMA 2001] studied entire solutions: qualitative properties, classification ...

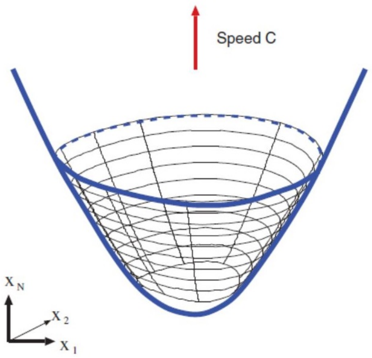

Figure: Conical shaped front, from [Taniguchi, SIAM. M.A. 2015]

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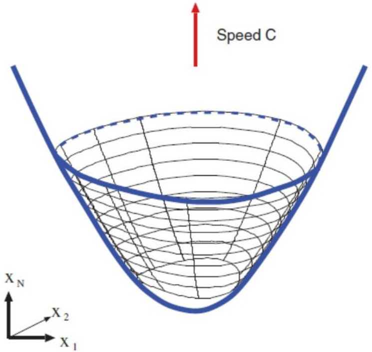
In particular, the notion of "transition fronts" from [Berestycki-Hamel, Contemp. Math. 2007 and CPAM 2012].



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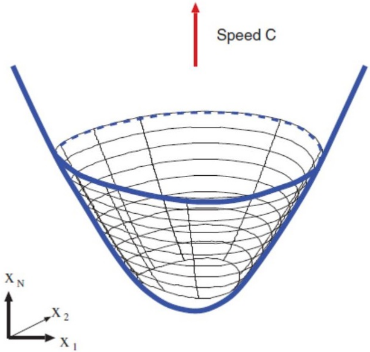
Propagation in heterogenous media, nonlocal diffusions, ...



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- Solutions of (6) $\leftrightarrows$ nonnegative Borel measures on $\mathbb{S}^{n-1}$.

Our goal

Geometric property and its large scale motion law for $\{u(t) = 1/2\}$?

Half Lipschitz property

Theorem

Travelling waves: v *is a solution of* (2), *satisfying* $\sup_{\mathbb{R}^n} v = 1$.
$\exists\, b_0 \in (0,1)$, $\forall\, \lambda \in [1-b_0, 1)$, $\{v = \lambda\}$ *is a globally Lipschitz graph in the* e_n *direction.*

Entire solutions: u *is an entire solution such that* $\forall\, t \in \mathbb{R}$, $\sup_{x\in\mathbb{R}^n} u(x,t) = 1$. *Then for any* $\lambda \in [1-b_0, 1)$, $\{u = \lambda\}$ *is a globally Lipschitz graph in the time direction.*

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There exist "localized" entire solutions: $u(x,t) \to 0$ as $|x| \to +\infty$ (say, for $t \le 0$), whose qualitative properties are very different, see [Hamel-Ninomiya, arXiv:2005.07420].

An example

[Hamel-Roquejoffre. Heteroclinic connections for multidimensional bistable reaction-diffusion equations. DCDS. Series S., 2011].

Theorem

There exists a travelling wave v in $\mathbb{R}^2$, monotone in x_1,

$$\begin{cases} v(x_1, x_2) \to 1 & \textit{uniformly as } x_1 \to +\infty. \\ v(x_1, x_2) \to \varphi(x_2) & \textit{locally uniformly as } \ x_1 \to -\infty, \end{cases}$$

where φ is an L-periodic solution of

$$-\varphi'' = f(\varphi) \quad \textit{in} \ \ \mathbb{R}.$$

For λ close to 0, $\{v = \lambda\}$ is the graph of an L-periodic function h_λ, $h_\lambda(kL) = -\infty$ for any $k \in \mathbb{Z}$. Not globally Lipschitz.

Full Lipschitz property

Theorem

Travelling waves: *Assume that as* $\mathrm{dist}(x, \{v = 1 - b_0\}) \to +\infty$, $v(x) \to 0$ *uniformly.* $\forall \lambda \in (0,1)$, $\{v = \lambda\}$ *is a globally Lipschitz graph in the* e_n *direction.*

Entire solutions: *Assume as* $\mathrm{dist}((x,t), \{u = 1 - b_0\}) \to +\infty$, $u(x,t) \to 0$ *uniformly. If* f *is bistable or combustion type, then* $\forall \lambda \in (0,1)$, $\{u = \lambda\}$ *is a globally Lipschitz graph in the time direction.*

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Question 1: Does a suitable "stability condition" for travelling waves or entire solutions imply the full Lipschitz property?
Question 2: Is an entire solution always monotone in t? (Cf. [Guo-Hamel, J. Elliptic Parabol. Equ. 2016].)

Lipschitz property and cone of monotonicity

Given $\lambda > 0$, denote the cone

$$\mathcal{C}_\lambda^+(x,t) := \{(y,s): \ s > t, \quad |y-x| < \lambda(s-t)\}.$$

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Lemma (Characterization of Lipschitz graphs)

Suppose $\Sigma \subset \mathbb{R}^n \times \mathbb{R}$ satisfies the cone condition:

$$\forall (x,t) \in \Sigma, \quad \mathcal{C}_\lambda^+(x,t) \cap \Sigma = \emptyset.$$

Then Σ lies on a Lipschitz graph $\{t = h(x)\}$.

Large scale Lipschitz property

Lemma

$\exists\ D > 0, 0 < b_2 < 1,\ \forall\ (x, t) \in \{u = 1 - b_2\}$,

$$u > 1 - b_2 \quad in \quad \mathcal{C}^{+}_{\kappa_*-\delta}(x, t + D).$$

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- A cone of monotonicity condition with time delay, and only for one level set.
- $u(x,t) \sim 1 \Longrightarrow u(\cdot, t) > 1 - b$ in a large $B_R(x)$.

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- A cone of monotonicity condition with time delay, and only for one level set.
- $u(x,t) \sim 1 \Longrightarrow u(\cdot, t) > 1 - b$ in a large $B_R(x)$.
- **Comparison function which spreads to 1:**

$$\begin{cases} \partial_t w - \Delta w = f(w) & \text{in } \mathbb{R}^n \times (0, +\infty), \\ w(0) = (1 - b)\,\chi_{B_R}, \end{cases}$$

where $b \ll 1$, $R \geq R(b, \delta)$, then

$$w(x,t) > 1 - b \quad \text{in} \quad \mathcal{C}^+_{\kappa_* - \delta}(0, D).$$

Blowing down analysis

Set $\Psi := g^{-1} \circ v$ and $\Psi_\varepsilon(x) := \varepsilon \Psi(\varepsilon^{-1} x)$. They are globally Lipschitz and satisfies

$$-\varepsilon \Delta \Psi_\varepsilon + \kappa \partial_n \Psi_\varepsilon = \kappa_* + \frac{g''(\varepsilon^{-1}\Psi_\varepsilon)}{g'(\varepsilon^{-1}\Psi_\varepsilon)} \left(|\nabla \Psi_\varepsilon|^2 - 1 \right).$$

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Take a subsequence $\Psi_\varepsilon \to \Psi_\infty$, then

Lemma

$\{\Psi_\infty > 0\} = \{x_n > h_\infty(x')\}$, *with* h_∞ *globally Lipschitz.*

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Idea: Scaling *the cone property with time delay* gives a cone of monotonicity for $\{\Psi_\infty > 0\}$:

$$x \in \{\Psi_\infty > 0\} \Longrightarrow \mathcal{C}^+_{\kappa_*/\kappa}(x) \subset \{\Psi_\infty > 0\}.$$

Propagation of cone of monotonicity

Step 1. In the open set $\{\Psi_\infty > 0\}$, Ψ_∞ is a viscosity solution of

$$\kappa \partial_n \Psi_\infty - \kappa_* + \beta_+ \left(|\nabla \Psi_\infty|^2 - 1 \right) = 0.$$

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Step 2. For any $x \in \{\Psi_\infty > 0\}$, $\Psi_\infty(x)$ equals

$$\inf_{y' \in \mathbb{R}^{n-1}} \left[K \sqrt{|x' - y'|^2 + (x_n - h_\infty(y'))^2} - \frac{\kappa}{2\beta_+} \left(x_n - h_\infty(y') \right) \right].$$

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$\Longleftarrow$ Uniform semi-concavity in the vanishing viscosity approximation of H-J equations.

Completion of the proof

- If $v \to 0$ uniformly on the other side, $\Psi_\infty < 0$ below $\{x_n = h_\infty(x')\}$.
  $\Longrightarrow$ In the same way we obtain monotonicity of Ψ in $\{\Psi < -L\}$.

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$$\sup_{\{v>1/2\}} \frac{|\nabla v| + |\partial_t v|}{1-v} < +\infty, \quad \sup_{\{v<1/2\}} \frac{|\nabla v| + |\partial_t v|}{v} < +\infty.$$

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We need the global Lipschitz property of Φ to perform the blowing down analysis.

An elementary approach

- The level set $\{u = 1 - b_2\}$ belongs to the D-neighborhood of a globally Lipschitz graph $\{t = h^*(x)\}$.

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- If $f'(0) < 0$, the maximum principle holds for

$$\partial_t \varphi - \Delta \varphi = f'(u)\varphi \quad \text{in} \quad \{t < h(x) - L\}.$$

Relations between u_t and $|\nabla u|$ in $\{t > h(x)\}$ can be extended to $\{t < h(x)\}$.

Geometric motion

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In the bistable or combustion case, if $u \to 0$ on the negative side, the blowing down limit h_∞ is a viscosity solution of

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Corollary (Minimal speed)

(i) For any travelling wave v, its speed $\kappa \geq \kappa_*$.
(ii) If $\kappa = \kappa_*$, then $v(x) \equiv g(x_n + b)$.

Thanks for your attention!