

Vanishing Viscosity Limit of the Navier-Stokes Equations to the Euler Equations for Compressible Fluid Flow with Vacuum

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Introduction

3-D isentropic compressible Navier-Stokes equations:

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ (\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla p = \nabla \cdot T. \end{cases} \quad (0.1)$$

Space and time variables:

$$x = (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t \geq 0,$$

density and velocity of fluid:

$$\rho, \quad u = (u^{(1)}, u^{(2)}, u^{(3)}) \in \mathbb{R}^3.$$

viscous stress tensor:

$$T = \mu(\rho)(\nabla u + (\nabla u)^\top) + \lambda(\rho)\operatorname{div} u \mathbb{I}_3,$$

where \mathbb{I}_3 is 3×3 matrix.

Introduction

Based on the law of Boyle and Gay-Lussac, it holds

$$\mu(\rho) = \epsilon \alpha \rho^\delta, \quad \lambda(\rho) = \epsilon \beta \rho^\delta,$$

$\epsilon \in (0, 1]$ is a constant, $\mu(\rho), \lambda(\rho)$ are shear and bulk viscosity coefficient, respectively. α, β are constants, satisfying $\alpha > 0, \quad 2\alpha + 3\beta \geq 0$.

3-D isentropic compressible Euler equations ($\epsilon = 0$):

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ (\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla p = 0. \end{cases} \quad (0.2)$$

Given the same initial data to (0.1), (0.2)

$$\begin{cases} (\rho, u)|_{t=0} = (\rho_0(x) \geq 0, u_0(x)), \\ (\rho, u) \rightarrow (0, 0), \text{ as } |x| \rightarrow +\infty. \end{cases} \quad (0.3)$$

Introduction

We consider the equation of state

$$p = A\rho^\gamma, \quad A > 0, \gamma > 1.$$

Thus the momentum equation becomes

$$\underbrace{\rho(u_t + u \cdot \nabla u) + A\gamma\rho^{\gamma-1}\nabla\rho}_{\text{Hyperbolic}} = \underbrace{-\epsilon\rho^\delta Lu}_{\text{Elliptic}} + \underbrace{\epsilon\nabla\rho^\delta \cdot S(u)}_{\text{Source}},$$

where

$$Lu = -\alpha\Delta u - (\alpha + \beta)\nabla\operatorname{div}u, \quad S(u) = \alpha(\nabla u + (\nabla u)^\top) + \beta\operatorname{div}u\mathbb{I}_3.$$

Two kinds of degeneracy caused by vacuum or the decay in the far field:

- Degeneracy of time involution;
- Degeneracy of viscosities.

Introduction

For smooth solution (ρ, u) away from the vacuum, the momentum equation could be written as

$$\underbrace{u_t + u \cdot \nabla u + \frac{A\gamma}{\gamma-1} \nabla \rho^{\gamma-1} - \frac{\delta\epsilon}{\delta-1} \nabla \rho^{\delta-1} \cdot S(u)}_{\text{Lower order}} \\ = \underbrace{-\epsilon \rho^{\delta-1} Lu}_{\text{Higher order}}.$$

As $\rho \rightarrow 0$, the above equality formally becomes

$$u_t + u \cdot \nabla u = 0, \quad \delta > 1, \gamma > 1,$$

and which implies that the velocity u can be governed by a nonlinear parabolic system if density contains vacuum.

- **Existence with vacuum state**

$\delta = 0$ (the viscosity coefficients are constants) :

- Choe-Kim (2004-2006) Local strong solution:
Initial compatibility condition.
- Duan-Luo-Zheng (2012 SIAM J. Math. Anal.) 2D.

$\delta = 1$ (Local regular solution):

$$\begin{cases} \phi_t + u \cdot \nabla \phi + \frac{\gamma-1}{2} \phi \nabla \cdot u = 0, \\ u_t + u \cdot \nabla u + \frac{2A\gamma}{\gamma-1} \phi \nabla \phi + Lu = \psi \cdot S(u), \\ \phi = \rho^{\frac{\gamma-1}{2}}, \psi = \frac{\nabla \rho}{\rho} = \frac{2}{\gamma-2} \nabla \phi / \phi. \end{cases} \quad (0.4)$$

- Li-Pan-Zhu (2017, J. Math. Fluid Mech.) 2D, shallow water:
The estimates of $\nabla \rho / \rho \in L^6(\mathbb{R}^3) \cap D^1(\mathbb{R}^3)$ is major.

Known Works

$1 < \delta \leq \min\{3, \frac{\gamma+1}{2}\}$ (Local regular solution):

$$\begin{cases} \phi_t + u \cdot \nabla \phi + \frac{\delta-1}{2} \phi \nabla \cdot u = 0, \\ u_t + u \cdot \nabla u + \frac{2A\gamma}{\delta-1} \phi^{\frac{2\gamma-\delta-1}{\delta-1}} \nabla \phi + \phi^2 Lu = \nabla \phi^2 \cdot \mathbb{S}(u), \\ \phi = \rho^{\frac{\delta-1}{2}}. \end{cases} \quad (0.5)$$

- Li-Pan-Zhu (2019, Arch. Ration. Mech. Anal.) 3D case.

Global regular solution with small data

- Huang-Li-Xin (2012 Commun. Pure Appl. Math,) $\delta = 0$, global existence with small data but large oscillations.
- Xin-Zhu (2018, arXiv:1806.02383v2,) $\delta > 1, \gamma > 1$.

$$\text{Dist}(sp(\nabla u_0(x)), R_-) \geq \kappa, \quad \|\rho_0\|_{H^3} \leq C.$$

Weak solution with vacuum state

- Bresch-Desjardins (2003, CMP, CPDE; 2007, JMPA, AMFM.)

- **Vanishing viscosity limit**

1-D :

- Gilbarg (1951 Amer. J. Math.) Shock layer .
- Hoff-Liu (1989 Indiana Univ. Math. J.) Shock data.
- Chen-Perepelitsa (2010 Comm. Pure Appl. Math) Based on the uniform energy estimates and compactness compensated argument, established the convergent result to the finite energy entropy weak solution with spherical symmetry and large initial data.
- Huang-Wang-Yang (2012 Arch. Rational Mech. Anal) the vanishing viscosity limit of Riemann solution.

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M-D without vacuum:

- Klainerman-Majda (1981 Comm. Pure Appl. Math.)

M-D with vacuum:

Even for $\delta = 0$ case, the existence of strong solutions to the viscous (Choe-Kim) and inviscid flow (Makino-Ukai-Kawashima) are usually established in different frame-works. The proofs shown in N-S equations essentially depend on uniform ellipticity of Lemé operator L and the life spans T which strictly depend on the real viscosities. that is

$$\begin{cases} |u|_{D^{k+2}} \leq C(\frac{1}{\epsilon\alpha}, \frac{1}{\epsilon\alpha})(|u_t + u \cdot \nabla u + \nabla p|_{D^k}) \\ T \sim O(\epsilon\alpha, \epsilon\beta). \end{cases}$$

- Ding-Zhu (2017, J. Math. Pure Appl.) $\delta = 1$, 2-D, the vanishing viscosity with vacuum.
- Geng-Li-Zhu (2019, Arch. Rational Mech. Anal.)
 $\gamma > 1, 1 < \delta \leq \min\{3, \gamma\}$, 3-D, the vanishing viscosity with vacuum.

Theorem

(Makino-Ukai-Kawshima, 1986, Existence of Regular solution of Euler equations,) *Let $\gamma > 1$. If the initial data (ρ_0, u_0) satisfy*

$$\rho_0 \geq 0, \quad (\rho_0^{\frac{\gamma-1}{2}}, u_0) \in H^3(\mathbb{R}^3),$$

then there exists a time $T_0 > 0$ and a unique regular solution (ρ, u) to Cauchy problem (0.2) with (0.5) satisfying

$$(\rho^{\frac{\gamma-1}{2}}, u) \in C([0, T_0]; H^3), \quad ((\rho^{\frac{\gamma-1}{2}})_t, u_t) \in C([0, T_0]; H^2),$$

where the regular solution (ρ, u) to (0.2) with (0.5) is defined by

$$(A)) \quad \rho \geq 0, \quad (\rho^{\frac{\gamma-1}{2}}, u) \in C^1([0, T_0]; \mathbb{R}^3),$$

$$(B) \quad u_t + u \cdot \nabla u = 0 \quad \text{as} \quad \rho(t, x) = 0.$$

Main Results

Definition

(Regular solution of N-S equations) Let $T > 0$ be a finite constant. A solution (ρ, u) to the Cauchy problem (0.1) with (0.5) is called a regular solution in $[0, T] \times \mathbb{R}^3$ if (ρ, u) satisfies this problem in the sense of distributions and:

$$(A) \quad \rho \geq 0, \quad \left(\rho^{\frac{\delta-1}{2}}, \rho^{\frac{\gamma-1}{2}} \right) \in C([0, T]; H^3);$$

$$(B) \quad u \in C([0, T]; H^{s'}) \cap L^\infty([0, T]; H^3), \\ \rho^{\frac{\delta-1}{2}} \nabla^4 u \in L^2([0, T]; L^2);$$

$$(C) \quad u_t + u \cdot \nabla u = 0 \quad \text{as} \quad \rho(t, x) = 0,$$

where $s' \in [2, 3)$ is an arbitrary constant.

Main Results

Theorem

(**Geng-Li-Zhu, ARMA 2019, Uniform regularity to N-S equations**) *If the initial data satisfies*

$$\rho_0 \geq 0, \quad \left(\rho_0^{\frac{\delta-1}{2}}, \rho_0^{\frac{\gamma-1}{2}}, u_0 \right) \in H^3, \quad (0.6)$$

then there exists a time $T_ > 0$ independent of ϵ , and a unique regular solution (ρ, u) to Cauchy problem (0.1) with (0.5) satisfying following estimates:*

$$\begin{aligned} & \sup_{0 \leq t \leq T_*} \left(\|\rho^{\frac{\gamma-1}{2}}\|_3^2 + \|\rho^{\frac{\delta-1}{2}}\|_2^2 + \epsilon \|\rho^{\frac{\delta-1}{2}}\|_{D^3}^2 + \|u\|_2^2 \right)(t) \\ & + \operatorname{ess\,sup}_{0 \leq t \leq T_*} |u(t)|_{D^3}^2 + \int_0^{T_*} \epsilon \|\rho^{\frac{\delta-1}{2}} \nabla^4 u\|_2^2 ds \leq C^0, \end{aligned}$$

for positive constant $C^0 = C^0(\alpha, \beta, A, \gamma, \delta, \rho_0, u_0)$.

Main Results

Corollary

Actually, (ρ, u) satisfies Cauchy problem (0.1) with (0.5) classically in positive time $(0, T_]$.*

Moreover, if the following condition holds:

$$1 < \min\{\delta, \gamma\} \leq \frac{5}{3}, \quad \delta = 2, 3, \quad \gamma = 2, 3$$

we still have

$$\rho \in C((0, T_*]; H^3), \quad \rho_t \in C((0, T]; H^2).$$

Main Results

Theorem

(Geng-Li-Zhu, ARMA 2019, Inviscid limit) *Let $(\rho^\epsilon, u^\epsilon)$ and (ρ, u) are the regular solutions to the Cauchy problem of N-S and Euler equations, respectively. If $(\rho_0^\epsilon, u_0^\epsilon) = (\rho_0, u_0)$, then $(\rho^\epsilon, u^\epsilon)$ converges to (ρ, u) as $\epsilon \rightarrow 0$ in the sense*

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \sup_{0 \leq t \leq T_*} \left(\|(\rho^\epsilon)^{\frac{\gamma-1}{2}} - \rho^{\frac{\gamma-1}{2}}\|_{s'} + \|u^\epsilon - u\|_{s'} \right) &= 0 \\ \sup_{0 \leq t \leq T_*} \left(\|(\rho^\epsilon)^{\frac{\gamma-1}{2}} - \rho^{\frac{\gamma-1}{2}}\|_1 + \|u^\epsilon - u\|_1 \right) &\leq C\epsilon, \\ \sup_{0 \leq t \leq T_*} \left(\|(\rho^\epsilon)^{\frac{\gamma-1}{2}} - \rho^{\frac{\gamma-1}{2}}\|_{D^2} + \|u^\epsilon - u\|_{D^2} \right) &\leq C\sqrt{\epsilon}, \end{aligned}$$

for $s' \in [0, 3)$ and positive constant $C = C(\alpha, \beta, A, \gamma, \delta, T_*, \rho_0, u_0)$.

Main Results

Corollary

If the following condition holds

$$1 < \min\{\delta, \gamma\} \leq \frac{5}{3}, \quad \delta = 2, 3, \quad \gamma = 2, 3,$$

then $(\rho^\epsilon, u^\epsilon)$ converges to (ρ, u) as $\epsilon \rightarrow 0$ in the sense

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \sup_{0 \leq t \leq T_*} \left(\|\rho^\epsilon - \rho\|_{s'} + \|u^\epsilon - u\|_{s'} \right) &= 0 \\ \sup_{0 \leq t \leq T_*} \left(\|\rho^\epsilon - \rho\|_1 + \|u^\epsilon - u\|_1 \right) &\leq C\epsilon, \\ \sup_{0 \leq t \leq T_*} \left(|\rho^\epsilon - \rho|_{D^2} + |u^\epsilon - u|_{D^2} \right) &\leq C\sqrt{\epsilon}, \end{aligned}$$

for positive constant $C = C(\alpha, \beta, A, \gamma, \delta, T_, \rho_0, u_0)$ and $\rho = \varphi^{\frac{2}{\delta-1}}$.*

Step I: Symmetric Reformulation. Introducing

$$\varphi = \rho^{\frac{\delta-1}{2}}, \quad \phi = \rho^{\frac{\gamma-1}{2}},$$

it holds

$$\left\{ \begin{array}{l} \varphi_t + u \cdot \nabla \varphi + \frac{\delta-1}{2} \varphi \operatorname{div} u = 0, \\ \underbrace{A_0 W_t + \sum_{j=1}^3 A_j(W) \partial_j W}_{\text{Symmetric Hyperbolic}} = \underbrace{-\epsilon \varphi^2 L(W)}_{\text{Degenerated Elliptic}} + \underbrace{\epsilon H(\varphi^2) \cdot Q(W)}_{\text{Lower Order Source}}, \end{array} \right. \quad (0.7)$$

where $W = (\phi, u)$ and

$$L(W) = \begin{pmatrix} 0 \\ a_1 L(u) \end{pmatrix}, \quad H(\varphi^2) = \begin{pmatrix} 0 \\ \nabla(\varphi^2) \end{pmatrix},$$

$$Q(W) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{a_1 \delta S(u)}{\delta - 1} \end{pmatrix}, A_0 = \begin{pmatrix} 1 & 0 \\ 0 & a_1 \mathbb{I}_3 \end{pmatrix},$$

with $a_1 = \frac{(\gamma-1)^2}{4A\gamma}$, and

$$A_j = \begin{pmatrix} u^{(j)} & \frac{\gamma-1}{2} \phi e_j \\ \frac{\gamma-1}{2} \phi (e_j)^\top & a_1 u^{(j)} \mathbb{I}_3 \end{pmatrix}, j = 1, 2, 3$$

$e_j = (\delta_{1j}, \delta_{2j}, \delta_{3j})$, δ_{ij} is Kroneckek symbol satisfying $\delta_{ij} = 1, i = j, \delta_{ij} = 0, i \neq j$. For any $\xi \in \mathbb{R}^3$, we have

$$\xi^\top A_0 \xi \geq a_2 |\xi|^2, \quad a_2 = \min \left\{ 1, \frac{(\gamma-1)^2}{4A\gamma} \right\}.$$

Step I: Linearization with an artificial strong elliptic operator.

If ω, ψ are known functions and $v = (v^{(1)}, v^{(2)}, v^{(3)})$ is a known vector function, we consider

$$\begin{cases} \varphi_t + v \cdot \nabla \varphi + \frac{\delta - 1}{2} \omega \operatorname{div} v = 0, \\ A_0 W_t + \sum_{j=1}^3 A_j(V) \partial_j W + \epsilon(\varphi^2 + \eta^2) L(W) = \epsilon H(\varphi) \cdot Q(V), \\ (\varphi, W)|_{t=0} = (\varphi_0, W_0), \quad (\varphi, W) \rightarrow (0, 0), \text{ as } |x| \rightarrow +\infty. \end{cases} \quad (0.8)$$

where $\eta \in (0, 1]$ is a constant, $W = (\phi, u)$, $V = (\psi, v)$,
 $(\omega, \psi, v)|_{t=0} = (\varphi_0, \phi_0, u_0)$.

Lemma

Under the assumptions

$$\begin{aligned}\omega &\in C([0, T]; H^3), \quad \omega_t \in C([0, T]; H^2), \quad \psi \in C([0, T]; H^3); \\ \psi_t &\in C([0, T]; H^2), \quad v \in C([0, T]; H^{s'}) \cap L^\infty([0, T]; H^3), \\ \omega \nabla^4 v &\in L^2([0, T]; L^2), \quad v_t \in C([0, T]; H^1) \cap L^2([0, T]; D^2), \\ \varphi_0 &\geq 0, \quad \phi_0 \geq 0, \quad (\varphi_0, W_0) \in H^3, \quad s' \in [2, 3).\end{aligned}$$

then there exists a strong solution to this linearization problem when $\eta > 0$, such that

$$\begin{aligned}\varphi &\in C([0, T]; H^3), \quad \phi \in C([0, T]; H^3); \\ u &\in C([0, T]; H^3) \cap L^2([0, T]; D^4), \\ u_t &\in C([0, T]; H^1) \cap L^2([0, T]; D^2).\end{aligned}$$

Proof Line

Step II: A priori estimate independent of η, ϵ . Fixing $T > 0$ and a positive constant c_0 large enough such that

$$\|\varphi_0\|_3 + \|\phi_0\|_3 + \|u_0\|_3 \leq c_0,$$

$$\begin{aligned} \sup_{0 \leq t \leq T^*} (\|\omega(t)\|_1^2 + \|\psi(t)\|_1^2 + \|v(t)\|_1^2) + \epsilon \int_0^{T^*} |\omega \nabla^2 v|_2^2 dt &\leq c_1^2, \\ \sup_{0 \leq t \leq T^*} (|\omega(t)|_{D^2}^2 + |\psi(t)|_{D^2}^2 + |v(t)|_{D^2}^2) + \epsilon \int_0^{T^*} |\omega \nabla^3 v|_2^2 dt &\leq c_2^2, \\ \text{ess } \sup_{0 \leq t \leq T^*} (|\psi(t)|_{D^3}^2 + |v(t)|_{D^3}^2 + \epsilon |\omega(t)|_{D^3}^2) + \epsilon \int_0^{T^*} |\omega \nabla^4 v|_2^2 dt &\leq c_3^2, \end{aligned} \tag{0.9}$$

for some $T^* \in (0, T)$ and constants $c_i (i = 1, 2, 3)$ such that

$$c_0 \leq c_1 \leq c_2 \leq c_3.$$

The constants c_i and T^* will be determined later and dependent on c_0 and the fixed constants $\alpha, \beta, \gamma, A, \delta, T$.

Proof Line

Proof: Applying the operator $\partial_x^\zeta (0 \leq |\zeta| \leq 3)$ to $(9)_1$ and multiplying both sides by $\partial_x^\zeta \varphi$ and integrating over \mathbb{R}^3 by parts, we obtain

$$\frac{1}{2} \frac{d}{dt} |\partial_x^\zeta \varphi|_2^2 \leq C |\nabla \cdot v|_\infty |\partial_x^\zeta \varphi|_2^2 + C \wedge_1^\zeta |\partial_x^\zeta \varphi|_2 + C \wedge_2^\zeta |\partial_x^\zeta \varphi|_2,$$

$$\wedge_1^\zeta = |\partial_x^\zeta (v \cdot \nabla \varphi) - v \cdot \nabla \partial_x^\zeta \varphi|_2, \quad \wedge_2^\zeta = |\partial_x^\zeta (\omega \nabla \cdot v)|_2.$$

Based on Garliardo-Nirenberg inequality, Hölder inequality and Gronwall's inequality, it holds

$$\|\varphi(t)\|_2 \leq (\|\varphi_0\|_2 + c_3^2 t) \exp(Cc_3 t) \leq Cc_0^2,$$

for $0 \leq t \leq T_1 = \min\{T^*, (1 + c_3)^{-2}\}$.

$$\frac{d}{dt} |\varphi|_{D^3} \leq C (\|\nabla v\|_2 |\varphi|_{D^3} + \|\omega \nabla^4 v\|_2 + \|\omega\|_3 \|v\|_3),$$

Using Gronwall's inequality, for $t \leq T_1$, one gets

$$|\varphi(t)|_{D^3} \leq C(c_0 + \epsilon^{-1/2}) \text{ that is } \epsilon |\varphi(t)|_{D^3}^2 \leq Cc_0^2.$$

Similarly, for $(9)_2$, one gets

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \int \left((\partial_x^\zeta W)^\top A_0 \partial_x^\zeta W \right) + a_1 \epsilon \alpha |\sqrt{\varphi^2 + \eta^2} \nabla \partial_x^\zeta u|_2^2 \\
 & \quad + a_1 \epsilon (\alpha + \beta) |\sqrt{\varphi^2 + \eta^2} \operatorname{div} \partial_x^\zeta u|_2^2 \\
 = & \int (\partial_x^\zeta W)^\top \operatorname{div} A(V) \partial_x^\zeta W + a_1 \epsilon \int \left(\nabla \varphi^2 \cdot Q(\partial_x^\zeta v) \right) \cdot \partial_x^\zeta u \\
 & - \frac{\delta - 1}{\delta} a_1 \epsilon \int \left(\nabla (\varphi^2 + \eta^2) \cdot Q(\partial_x^\zeta u) \right) \cdot \partial_x^\zeta u \\
 & - \sum_{j=1}^3 \int \left(\partial_x^\zeta (A_j(V) \partial_j W) - A_j(V) \partial_j \partial_x^\zeta W \right) \cdot \partial_x^\zeta W \\
 & - a_1 \epsilon \int \left(\partial_x^\zeta ((\varphi^2 + \eta^2) L u) - (\varphi^2 + \eta^2) L \partial_x^\zeta u \right) \cdot \partial_x^\zeta u \\
 & + a_1 \epsilon \int \left(\partial_x^\zeta (\nabla \varphi^2 \cdot Q(v)) - \nabla \varphi^2 \cdot Q(\partial_x^\zeta v) \right) \cdot \partial_x^\zeta u.
 \end{aligned}$$

Proof Line

Using Young inequality, Hölder inequality and Gronwall's inequality, one gets

$$\begin{aligned} & \|W(t)\|_1^2 + \epsilon \int_0^t |\sqrt{\varphi^2 + \eta^2} \nabla^2 u|_2^2 ds \\ & \leq C(\|W_0\|_1^2 + c_3^2 \epsilon t) \exp(C(c_3^2 + c_3^4 \epsilon)t) \leq Cc_0^2, \\ & |W(t)|_{D^2}^2 + \epsilon \int_0^t |\sqrt{\varphi^2 + \eta^2} \nabla \partial_x^\zeta u|_2^2 ds \\ & \leq C(|W_0|_{D^2}^2 + c_3^2(1 + \epsilon)t) \exp(Cc_3^4 \epsilon t) \leq Cc_0^2, \\ & |W(t)|_{D^3}^2 + \epsilon \int_0^t |\sqrt{\varphi^2 + \eta^2} \nabla^4 u|_2^2 ds \\ & \leq C(|W_0|_{D^3}^2 + c_3^4 t) \exp(Cc_3^4 t) \leq Cc_0^2, \end{aligned}$$

for $0 \leq t \leq T_2 = \min\{T_1, (1 + c_3)^{-4}\}$.

Proof Line

Similarly, to get the continuity of solution, for $0 \leq t \leq T_2 = \min\{T_1, (1 + c_3)^{-4}\}$, we have the following estimates

$$\begin{aligned} |\varphi_t(t)|_2^2 &\leq Cc_1^4, \quad |\varphi_t(t)|_{D^1}^2 \leq Cc_2^4, \quad \epsilon |\varphi_t(t)|_{D^2}^2 \leq Cc_3^4, \\ |W_t(t)|_2^2 + |\phi_t(t)|_{D^1}^2 + \int_0^t |\nabla u_t|_2^2 ds &\leq Cc_2^4, \\ |u_t(t)|_{D^1}^2 + |\phi_t(t)|_{D^2}^2 + \int_0^t |\nabla^2 u_t|_2^2 ds &\leq Cc_3^4, \end{aligned}$$

In other words, given fixed c_0 and T , there exist positive constants T^* and c_i ($i = 1, 2, 3$), depending only on c_0 , T and the generic constant C , independent of ϵ, η , such that if (0.10) holds for ω and V , then the following holds for the strong solution in $[0, T^*] \times \mathbb{R}^3$.

Proof Line

$$\begin{aligned} & \sup_{0 \leq t \leq T^*} (\|\varphi(t)\|_1^2 + \|\phi(t)\|_1^2 + \|u(t)\|_1^2) + \epsilon \int_0^{T^*} \epsilon |\omega \nabla^2 u|_2^2 dt \leq c_1^2, \\ & \sup_{0 \leq t \leq T^*} (|\varphi(t)|_{D^2}^2 + |\phi(t)|_{D^2}^2 + |u(t)|_{D^2}^2) + \epsilon \int_0^{T^*} \epsilon |\omega \nabla^3 u|_2^2 dt \leq c_2^2, \\ & \text{ess} \sup_{0 \leq t \leq T^*} (|\phi(t)|_{D^3}^2 + |u(t)|_{D^3}^2 + \epsilon |\varphi(t)|_{D^3}^2) + \epsilon \int_0^{T^*} \epsilon |\omega \nabla^4 u|_2^2 dt \leq c_3^2, \\ & \text{ess} \sup_{0 \leq t \leq T^*} (\|W^\epsilon(t)\|_1^2 + |\phi(t)|_{D^2}^2 + \epsilon |\varphi_t(t)|_{D^2}^2) + \int_0^{T^*} \epsilon |u_t|_{D^2}^2 dt \leq c_3^6. \end{aligned} \tag{0.10}$$

where we defined

$$c_1 = c_2 = c_3 = C^{\frac{1}{2}} c_0, \quad T^* = \min\{T, (1 + c_3)^{-4}\}.$$

Proof Line

Thus, we have the existence to the Cauchy problem: ($\eta \rightarrow 0$).

$$\begin{cases} \varphi_t + \mathbf{v} \cdot \nabla \varphi + \frac{\delta - 1}{2} \omega \operatorname{div} \mathbf{v} = 0, \\ A_0 W_t + \sum_{j=1}^3 A_j(\mathbf{V}) \partial_j W + \epsilon \varphi^2 L(W) = \epsilon H(\varphi) \cdot Q(\mathbf{V}), \\ (\varphi, W)|_{t=0} = (\varphi_0, W_0), \quad (\varphi, W) \rightarrow (0, 0), \text{ as } |x| \rightarrow +\infty. \end{cases}$$

Lemma

Assume that the initial data satisfy (0.6). Then there exists $T^ > 0$ and a unique strong solution (φ, W) such that*

$$\begin{aligned} (\varphi, \phi) &\in C([0, T^*]; H^3), \quad u \in C([0, T^*]; H^{s'}) \cap L^\infty([0, T^*]; H^3), \\ \varphi \nabla^4 u &\in L^2([0, T^*]; L^2), \quad u_t \in C([0, T^*]; H^1) \cap L^2([0, T^*]; D^2), \end{aligned}$$

for $s' \in [2, 3)$. Moreover, (φ, W) also satisfies (0.10).

Proof:

- Existence: Strong and weak convergence;
- Uniqueness: Energy estimates to $\varphi_1 - \varphi_2, W_1 - W_2$,
- Time Continuity: Regularity of $\varphi, \varphi_t, \phi, \phi_t, u, u_t$.

III: The existence of nonlinear system. Consider (φ^{k+1}, W^{k+1}) be the unique solution as follows

$$\left\{ \begin{array}{l} \varphi_t^{k+1} + \mathbf{u}^k \cdot \nabla \varphi^{k+1} + \frac{\delta-1}{2} \varphi^k \operatorname{div} \mathbf{u}^k = 0, \\ A_0 W_t^{k+1} + \sum_{j=1}^3 A_j(W^k) \partial_j W^{k+1} + \epsilon (\varphi^{k+1})^2 L(W^{k+1}) \\ = \epsilon H(\varphi^{k+1}) \cdot Q(\mathbf{u}^k), \\ (\varphi^{k+1}, W^{k+1})|_{t=0} = (\varphi_0, W_0), \\ (\varphi^{k+1}, W^{k+1}) \rightarrow (0, 0), \text{ as } |x| \rightarrow +\infty. \end{array} \right.$$

Proof Line

To prove the convergence, let

$\bar{\varphi}^{k+1} = \varphi^{k+1} - \varphi^k$, $\bar{W}^{k+1} = W^{k+1} - W^k$, then one has

$$\left\{ \begin{array}{l} \bar{\varphi}^{k+1}_t + u^k \cdot \nabla \bar{\varphi}^{k+1} + \bar{u}^k \cdot \nabla \bar{\varphi}^k + \frac{\delta - 1}{2} (\bar{\varphi}^k \operatorname{div} u^{k-1} + \varphi^k \operatorname{div} \bar{u}^k) = 0, \\ A_0 \bar{W}^{k+1}_t + \sum_{j=1}^3 A_j(W^k) \partial_j \bar{W}^{k+1} + \epsilon (\varphi^{k+1})^2 L(\bar{W}^{k+1}) \\ = \sum_{j=1}^3 A_j(\bar{W}^k) \partial_j W^k - \epsilon \bar{\varphi}^{k+1} (\varphi^{k+1} + \varphi^k) L(W^k) \\ + \epsilon (H(\varphi^{k+1}) - H(\varphi^k)) \cdot Q(u^k) + \epsilon H(\varphi^{k+1}) \cdot Q(u^k). \end{array} \right.$$

Using energy estimate, it holds ($T_* \in (0, \min\{1, T^*\})$)

$$\sum_{k=1}^{\infty} \left(\sup_{t \in [0, T_*]} |(\bar{\varphi}^{k+1}, \bar{W}^{k+1})|_2^2 + \int_0^{T_*} \alpha \epsilon |\varphi^{k+1} \nabla \bar{u}^{k+1}|_2^2 dt \right) \leq C \leq \infty.$$

Step IV: The uniform of local-in-time well-posedness

- Existence of regular solution: The regularity estimates of (0.10) and the strong convergence of

$$(\varphi^k, W^k) \rightarrow (\varphi, u) \text{ in } L^\infty([0, T_*]; H^2),$$

$$(\rho^{\frac{\delta-1}{2}}, \rho^{\frac{\gamma-1}{2}}) \in C^1((0, T_*) \times \mathbb{R}^3), \quad (u, \nabla u) \in C((0, T_*); \times \mathbb{R}^3),$$

- Smoothness of regular solutions: Sobolev imbedding theorem and energy estimate lead to

$$(\rho, \nabla \rho, u, \nabla u, u_t, \operatorname{div} T) \in C((0, T_*) \times \mathbb{R}^3).$$

- The proof of $\rho \in C([0, T_*]; H^3) \cap C^1([0, T_*]; H^2)$:
 $\rho = \varphi^{\frac{2}{\delta-1}}, \quad \frac{2}{\delta-1} \geq 3, \text{ if } 1 < \delta \leq \frac{5}{3}.$

Step V: Vanishing viscosity limit. Denote $\overline{W}^\epsilon = W^\epsilon - W$, it satisfies the error system

$$\begin{cases} A_0 \overline{W}^\epsilon_t + \sum_{j=1}^3 A_j(W^\epsilon) \partial_j \overline{W}^\epsilon = - \sum_{j=1}^3 A_j(\overline{W}^\epsilon) \partial_j W - \epsilon(\varphi^\epsilon)^2 L(W^\epsilon). \\ \overline{W}^\epsilon|_{t=0} = (\overline{\rho}^\epsilon, \overline{u}^\epsilon)|_{t=0} = (0, 0). \end{cases} \quad (0.11)$$

Lemma

If W^ϵ and W are the regular solution to the Cauchy problem of N-S and Euler equations, respectively, then we have

$$\|\overline{W}^\epsilon\|_1 \leq C\epsilon, \quad |\overline{W}^\epsilon|_{D^2} \leq C\epsilon^{1/2}.$$

where C is independent of ϵ .

Proof Line

Applying the operator ∂_x^ζ on (0.11), then multiplying it by $2\partial_x^\zeta \overline{W}^\epsilon$ and integrating by parts, using the uniform regularity estimates on $W^\epsilon, W, \varphi^\epsilon$, it holds

$$\begin{aligned} a_2 \frac{d}{dt} \|\overline{W}^\epsilon\|_1^2 &\leq \frac{d}{dt} \int (\partial_x^\zeta \overline{W}^\epsilon)^\top A_0 (\partial_x^\zeta \overline{W}^\epsilon) \\ &\leq C \|\overline{W}^\epsilon\|_1^2 + C\epsilon^2, \quad |\zeta| = 0, 1, \\ a_2 \frac{d}{dt} |\overline{W}^\epsilon|_{D^2}^2 &\leq \frac{d}{dt} \int (\partial_x^\zeta \overline{W}^\epsilon)^\top A_0 (\partial_x^\zeta \overline{W}^\epsilon) \\ &\leq C |\overline{W}^\epsilon|_{D^1}^2 + C\epsilon + C\epsilon^2 |\varphi^\epsilon \nabla^4 u^\epsilon|_2^2, \quad |\zeta| = 2. \end{aligned} \tag{0.12}$$

According to Gronwall's inequality, the Lemma is proved. If $s' \in (2, 3)$, we have

$$\|\overline{W}^\epsilon(t)\|_{s'} \leq C \|\overline{W}^\epsilon(t)\|_0^{1-\frac{s'}{3}} \|\overline{W}^\epsilon(t)\|_3^{\frac{s'}{3}} \leq C\epsilon^{1-\frac{s'}{3}}.$$

If

$$1 < \min\{\delta, \gamma\} \leq \frac{5}{3},$$

one has $\frac{2}{\gamma-1} \geq 3$, $\rho = \phi^{\frac{2}{\gamma-1}}$, from the above inequalities, so one gets

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \sup_{0 \leq t \leq T_*} (\|(\rho^\epsilon - \rho)(t)\|_{s'} + \|(u^\epsilon - u)(t)\|_{s'}) &= 0, \\ \sup_{0 \leq t \leq T_*} (\|(\rho^\epsilon - \rho)(t)\|_1 + \|(u^\epsilon - u)(t)\|_1) &\leq C\epsilon, \\ \sup_{0 \leq t \leq T_*} (|(\rho^\epsilon - \rho)(t)|_{D^2} + |(u^\epsilon - u)(t)|_{D^2}) &\leq C\sqrt{\epsilon}. \end{aligned} \quad (0.13)$$

Thus, we proved the inviscid limit Theorem.

Thank You !!