Stabilization effect of frictions in three-dimensional steady compressible Euler flows

Qin Zhao

Wuhan University of Technology

Joint with Prof. Hairong Yuan(ECNU)

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Outline

A bow shock generated by a solar explosion

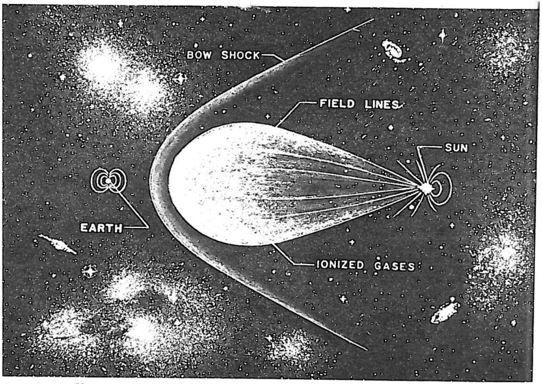
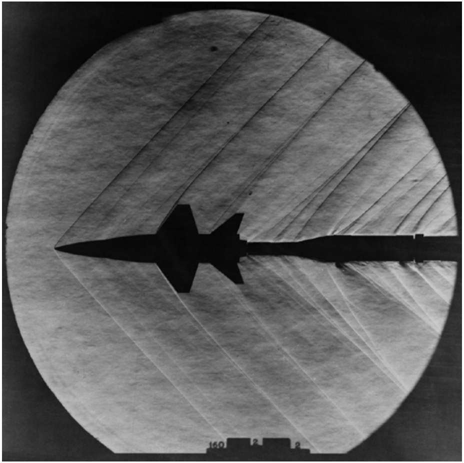


FIG. 50: SOLAR EXPLOSION

A shock wave in space generated by a solar eruption. The sketch shows the fully ionized nucleons attached to the solar magnetic field lines acting as the driving piston for the shock wave. (Courtesy: UTIAS, after Gold, 1962).

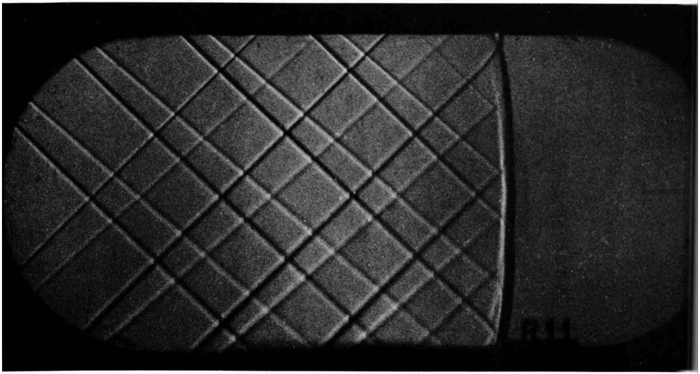
A small scale X-15 placed in a supersonic wind tunnel (from NASA)



Diffraction of a shock wave inside a box (Van Dyke: An album of fluid motion)

240. Diffraction of a shock wave inside a box. A shock wave in nitrogen is diffracted through a window at one end of a rectangular box and reflected from the other end. A shadowgraph shows a remarkable pattern of shock waves, slip lines, and vortices, but one that is altogether determinate and reproducible. The three rope-like traces at the right are slip lines generated as the diffracted shock wave oscillated in shape moving across the box, which have been perturbed by shock waves passing over them roughly at right angles. Several examples of separated boundary-layer flow are also evident. *Photograph by Russell E. Duff in Laporte's laboratory*

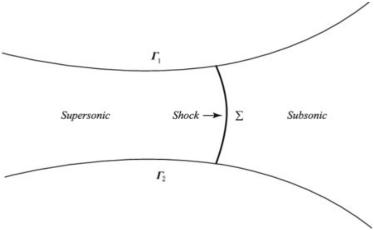
Transonic shocks in nozzles (Van Dyke: An album of fluid motion)



225. Normal shock wave at M=**1.5.** A pattern of pairs of weak oblique shock waves (the N-waves of figures 265 and 269) is produced by strips of tape on the floor and ceiling of a supersonic nozzle. They terminate at an almost straight and normal shock wave, showing that the flow is subsonic downstream. *U.S. Air Force photograph, courtesy of Arnold Engineering Development Center*

Transonic shocks in nozzles

For an uniform supersonic flow entering a de Laval nozzle, Courant and Friedrichs proposed the following problem on transonic shock phenomena: *"......How does the flow, after having attained supersonic speed behind the throat, adjust itself to the prescribed receiver pressure* p_r*?......but at a certain place in the diverging part of the nozzle a shock front intervenes, the gas is compressed and slow down to subsonic speed. From there on the gas is further compressed and slow down; ......The position and strength of the shock front are automatically adjusted so that the end pressure at the exit becomes* p_r*."*

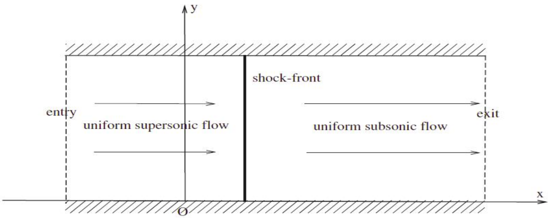


Unsteady compressible Euler equations

Unsteady Euler equations for compressible fluids in $\mathbb{R}^d$:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = 0, \\ \partial_t(\rho E) + \operatorname{div}((\rho E + p)\mathbf{u}) = 0, \end{cases} \tag{1.1}$$

- Piecewise smooth solution: Majda (1983); Alinhac(1989); Coulombel-Secchi(2008); ...
- Persistence of shocks in ducts: Yuan (2012); Fang-Xiang-Xiao (2019)



Quasi-one dimensional model

Quasi-one dimensional model including frictions and heating:

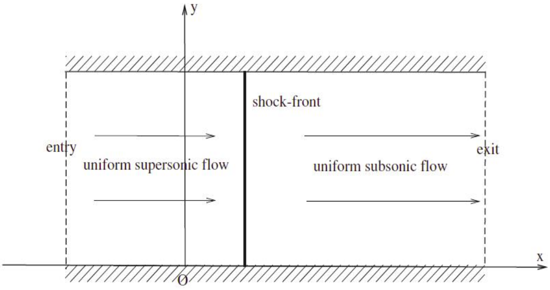
$$
\begin{aligned}
&(A\rho)_t + (A\rho u)_x = 0,\\
&(A\rho u)_t + \Big(A(\rho u^2 + p)\Big)_x = A_x p - \alpha\sqrt{A}\rho u|u|,\\
&(A\rho E)_t + \Big(A(\rho E + p)u\Big)_x = \beta A Q(x) - \alpha\sqrt{A}\rho u^2|u|,
\end{aligned}
$$

$A(x) > 0$ is the cross section of the nozzle, $Q(x)$ is a given function representing the heating effect, $\alpha, \beta > 0$ is the coefficient of frictions and heating, $\beta = 0$ is called Fanno flows, while $\alpha = 0$ is called Rayleigh flows.

- Piecewise smooth solution: Embid-Goodman-Majda (1984); Rauch-Xie-Xin (2013)
- L^∞ solution: Tsuge (2015); Chen-Schrecker (2018); Cao-Huang-Yuan (2019)
- BV solution: Liu (1982); Chou-Hong-Huang-Quita (2018)

Steady flows

- Steady compressible Euler equations
  – free boundary problem of elliptic-hyperbolic composite system
- Isentropic steady irrotational flows (potential flows)
  – free boundary problem of elliptic equation



Subsonic flows

For the 2-D steady case:

- Potential flows: Xie-Xin(2007); Wang-Xin(2019, Smooth transonic flows)
- Euler flows: Chen-Deng-Xiang(2012); Du-Xie-Xin(2014, Large vorticity); Chen-Huang-Wang-Xiang(2019, Large vorticity)

For the M-D steady case:

- Potential flows: Du-Xin-Yan(2011); Liu-Yuan(2014, Largely-open nozzles); Liu(2010, Global uniqueness)
- Euler flows: Chen-Xie(2014); Weng(2015); Du-Duan(2011, Axially symmetric); Deng-Wang-Xiang(2018, Nontrivial swirl); Liu-Xu-Yuan(2016, Spherically symmetric); Chen-Huang-Wang(2016, Subsonic-sonic limt)

Transonic shocks(potential flows)

For the M-D steady case:

- Stability
  Canic-Keyfitz-Lieberman (2000);
  Chen-Feldman (2003; 2004 etc.);
  Xin-Yin (2005; 2008 etc.);
  Bae-Feldman (2011)
- Uniqueness of special solutions
  Chen-Yuan (2009); Liu-Yuan (2009)

Transonic shocks(Euler flows)

For the 2-D steady case:

- Stability
  Chen (2005; 2009); Yuan (2006); Chen-Chen-Song (2006); Chen-Chen-Feldman (2007); Xin-Yan-Yin (2009); Li-Xin-Yin (2009; 2013); Fang-Xin(2019); Gao-Liu-Yuan(2020) etc.
- Uniqueness of transonic shocks
  Fang-Liu-Yuan (2013)

For the 3-D case:

- Stability
  3-D axisymmetric: Li-Xin-Yin(2010); Park-Ryu(2019); Park(2019); Weng-Xie-Xin(2019); Fang-Gao(2020)
  3-D: Chen (2008); Chen-Yuan (2008); Liu-Xu-Yuan (2016)

Mathematical model

Three-dimensional steady non-isentropic compressible Euler system with frictions is governed by

$$\begin{cases} \operatorname{div}(\rho u) = 0, \\ \operatorname{div}(\rho u \otimes u) + \operatorname{grad} p = \rho \mathfrak{b}, \\ \operatorname{div}(\rho u B) = \rho \mathfrak{b} \cdot u, \end{cases} \tag{2.1}$$

Velocity : $u = (u^0, u^1, u^2)^\top$, Pressure : p, Density : ρ,
Bernoulli constant : $B = \frac{1}{2}|u|^2 + \frac{\gamma}{\gamma-1}\frac{p}{\rho}$,
Adiabatic exponent : $\gamma > 1$,
Frictions: $\mathfrak{b} = (-\mu(u^0)^2, 0, 0)^\top$.

Mathematical model

For the polytropic gas, the equation of state are given by

$$p = R\rho T, \quad e = c_v T, \quad \gamma = 1 + R/c_v > 1,$$

and

$$p = p(\rho, S) = \kappa \rho^{\gamma} e^{S/c_v}, \quad e = \frac{\kappa}{\gamma - 1} \rho^{\gamma - 1} e^{S/c_v} = \frac{RT}{\gamma - 1},$$

where R, κ, c_v and γ are all positive constants. Then the sonic speed is given by $c = \sqrt{\gamma p/\rho}$.

Elliptic-hyperbolic composite-mixed type

Symmetric form:

$$A_0(U)\partial_0 U + A_1(U)\partial_1 U + A_2(U)\partial_2 U = W(U), \quad U = (p,u,s)^\top \in \mathbb{R}^5.$$

If $u^0 \in (0,c)$ and $|u| < c$, then for any $(\xi,\eta) \in \mathbb{R}^2 \setminus \{(0,0)\}$, $A_0(U)$ is nonsingular and $\det(\lambda A_0 - A_1\xi - A_2\eta) = 0$ has a real eigenvalue $\lambda_0 = \frac{u^1\xi + u^2\eta}{u^0}$ of multiplicity three, and a pair of complex eigenvalues $\lambda_\pm = \lambda_R \pm \sqrt{-1}\lambda_I$ of multiplicity one. Here

$$\lambda_R = \frac{u^0}{(u^0)^2 - c^2}(u^1\xi + u^2\eta)$$

and

$$\lambda_I = \frac{c\sqrt{(c^2-(u^0)^2)(\xi^2+\eta^2) - (u^1\xi + u^2\eta)^2}}{(u^0)^2 - c^2}$$

are real.

Problem (T)

Let $\Omega = \{(x^0, x^1, x^2) : x^0 \in (0, L), x' = (x^1, x^2) \in \mathbb{T}^2\}$ be the duct with $\Sigma_0 = \{0\} \times \mathbb{T}^2$ and $\Sigma_1 = \{L\} \times \mathbb{T}^2$. For the velocity vector $u = (u^0, u^1, u^2)^\top$, we call u^0 the normal velocity and $u' = (u^1, u^2)^\top$ the tangential velocity.

Suppose that

$$S^\psi = \{(x^0, x') \in \Omega : x^0 = \psi(x'),\ x' \in \mathbb{T}^2\}$$

is a surface, where $\psi : \mathbb{T}^2 \to \Omega$ is a C^1 function. The normal vector field on S^ψ is given by

$$n = (1, -\partial_1 \psi, -\partial_2 \psi).$$

Definition 2.1 (Transonic shock)

Let $\psi \in C^1(\mathbb{T}^2)$ and $U^\pm = (p^\pm, s^\pm, B^\pm, (u')^\pm) \in C^1(\Omega_\psi^\pm) \cap C(\overline{\Omega_\psi^\pm})$. We say that $U = (U^-, U^+; \psi)$ is a *transonic shock solution*, if

1) $U^\pm$ solve the system (2.1) in $\Omega_\psi^\pm$ in the classical sense;
2) U^- is supersonic, and U^+ is subsonic;
3) the Rankine–Hugoniot jump conditions hold across S^ψ:

$$[\rho(u \cdot n)u + pn] = 0, \tag{2.2}$$

$$[\rho(u \cdot n)] = 0, \tag{2.3}$$

$$[\rho(u \cdot n)B] = 0, \tag{2.4}$$

4) there holds the physical entropy condition

$$[p] = p^+|_{S^\psi} - p^-|_{S^\psi} > 0. \tag{2.5}$$

Problem (T)

We prescribe the following Cauchy data:

$$U = U_0^-(x') \qquad \text{on} \quad \Sigma_0. \tag{2.6}$$

Here $(u^0)_0^- > c_0^-$. On the exit, we propose the pressure

$$p = p_1(x') \qquad \text{on} \quad \Sigma_1. \tag{2.7}$$

Problem (T): Find a transonic shock solution in Ω which satisfies the boundary conditions (2.6) and (2.7) pointwisely.

Background Solution

Suppose that the flow depends only on x^0, then

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}x^0}(\rho u) = 0, \\ \dfrac{\mathrm{d}}{\mathrm{d}x^0}(\rho u^2 + p) = -\mu \rho u^2, \\ \dfrac{\mathrm{d}}{\mathrm{d}x^0}\Big(\rho(\dfrac{1}{2}u^2 + \dfrac{\gamma p}{(\gamma-1)\rho})u\Big) = -\mu\rho u^3. \end{cases}$$

For Mach number $M \neq 1$, we could solve that

$$\frac{\mathrm{d}u}{\mathrm{d}x^0} = \frac{\mu M^2}{1-M^2}u, \quad \frac{\mathrm{d}\rho}{\mathrm{d}x^0} = \frac{\mu M^2}{M^2-1}\rho, \quad \frac{\mathrm{d}p}{\mathrm{d}x^0} = \frac{\mu\gamma M^2}{M^2-1}p.$$

Background Solution

and

$$\frac{\mathrm{d}B}{\mathrm{d}x^0} = -\frac{2\mu(\gamma-1)M^2}{(\gamma-1)M^2+2}B, \quad \frac{\mathrm{d}s}{\mathrm{d}x^0} = 0,$$
$$\frac{\mathrm{d}T}{\mathrm{d}x^0} = (\gamma-1)\frac{\mu M^2}{M^2-1}T,$$

If the flow is subsonic in the duct, namely $M < 1$ for $x^0 \in [0, L]$,

$$\frac{\mathrm{d}u}{\mathrm{d}x^0} > 0, \quad \frac{\mathrm{d}\rho}{\mathrm{d}x^0} < 0, \quad \frac{\mathrm{d}p}{\mathrm{d}x^0} < 0, \quad \frac{\mathrm{d}M}{\mathrm{d}x^0} > 0, \quad \frac{\mathrm{d}T}{\mathrm{d}x^0} < 0.$$

Background Solution

The equation satisfied by the Mach number is decoupled:

$$\frac{\mathrm{d}M}{\mathrm{d}x^0} = \frac{\gamma+1}{2}\frac{\mu M^3}{1-M^2}. \tag{3.1}$$

Let the Mach number of the flow at the entry $\{x^0 = 0\}$ and the exit $\{x^0 = L\}$ be M_0 and M_1, and $0 < M_0 < M_1 < 1$. Then integrating (3.1) for x^0 from 0 to l yields

$$l = \frac{\frac{1}{M_0^2} - \frac{1}{M_1^2} + \ln M_0^2 - \ln M_1^2}{\mu(\gamma+1)}. \tag{3.2}$$

Therefore, the maximal length of a duct for a subsonic flow is

$$L_{M_0} = \frac{\frac{1}{M_0^2} + \ln M_0^2 - 1}{\mu(\gamma+1)} > 0. \tag{3.3}$$

Background Solution

Let $L_1 < L_{M_0}$ be the distance from the entry to the transonic shock front. Utilizing (3.2), by

$$L_1 = \frac{\frac{1}{M_0^2} - \frac{1}{M_-^2} + \ln M_0^2 - \ln M_-^2}{\mu(\gamma+1)}, \tag{3.4}$$

we could solve M_-. Then by R-H conditions, we get M_+. So the maximal length from the transonic shock front to the exit is

$$L_2 = L_{M_+} = \frac{\frac{1}{M_+^2} + \ln M_+^2 - 1}{\mu(\gamma+1)}. \tag{3.5}$$

Hence for given $M_0 > 1$, suppose that $L < L_1 + L_2$, we may construct a family of special transonic shock solutions.

Theorem 3.1 (Stability of transonic Fanno flow, Yuan-Z., 2020)

Suppose that U_b satisfies the S-Condition, and $\alpha \in (0,1)$. There exist ε_0 and C_* depending only on U_b and γ, α, L such that if

$$\left\|U_0^- - U_b^-\right\|_{C^4(\Sigma_0)} \leq \varepsilon \leq \varepsilon_0, \quad \left\|p_1 - p_b^+\right\|_{C^{3,\alpha}(\Sigma_1)} \leq \varepsilon \leq \varepsilon_0, \tag{3.6}$$

then there exists a transonic shock solution $U = (U^-, U^+; \psi)$ to Problem (T), so that $U^- \in C^4(\overline{\Omega_\psi^-})$, $p^+ \in C^{3,\alpha}(\overline{\Omega_\psi^+})$, $u^+, \rho^+, s^+ \in C^{2,\alpha}(\overline{\Omega_\psi^+})$, $(u^+, \rho^+, s^+)|_{S^\psi} \in C^{3,\alpha}(\mathbb{T}^2)$, $\psi \in C^{4,\alpha}(\mathbb{T}^2)$, and

$$\left\|U^- - U_b^-\right\|_{C^4(\overline{\Omega_\psi^-})} \leq C_* \varepsilon, \tag{3.7}$$

$$\left\|U^+|_{S^\psi} - U_b^+|_{S^\psi}\right\|_{C^{3,\alpha}(\mathbb{T}^2)} + \left\|U^+ - U_b^+\right\|_3 \leq C_* \varepsilon, \tag{3.8}$$

$$\|\psi - r_b\|_{C^{4,\alpha}(\mathbb{T}^2)} \leq C_* \varepsilon. \tag{3.9}$$

Main difficulties

- System of elliptic-hyperbolic composite type
    - No general theory
    - Choosing suitable boundary conditions
- The variable background solution
- New phenomena: Nonlocal elliptic problems

Sketch of Proof

The theorem is proved by solving the following problems:

- A transport equation of Bernoulli constant
- A transport equation of entropy
- A second-order nonlocal elliptic equation of pressure
- Transport equations of tangential velocity
- A div-curl system of the tangential velocity $u'|_{S^\psi}$ on $\mathbb{T}^2$
- The equations for the profile and the position of the surface S^ψ

Proposition 4.1

Suppose that $p \in C^2(\Omega) \cap C^1(\overline{\Omega})$, $\rho, u \in C^1(\overline{\Omega})$, *and* $\rho > 0, u^0 \neq 0$ *in* $\overline{\Omega}$. *Then* p, ρ, u *solve the system* (2.1) *in* Ω *if and only if they satisfy*

$$D_u B - \mathfrak{b} \cdot u = 0, \tag{4.1}$$

$$D_u A(s) = 0, \tag{4.2}$$

$$D_u \left(\frac{1}{\gamma p} D_u p \right) - \operatorname{div} \left(\frac{1}{\rho} \operatorname{grad} p \right) - \partial_j u^k \partial_k u^j + \operatorname{div} \mathfrak{b} = 0, \tag{4.3}$$

$$D_u u^\beta + \frac{1}{\rho} \partial_\beta p = 0, \quad \beta = 1, 2, \tag{4.4}$$

and the boundary condition:

$$\frac{1}{\gamma p} D_u p + \operatorname{div} u = 0. \tag{4.5}$$

Divergence of tangential velocity field on S^{ψ}

Using the commutator relation for a function f on Ω,

$$\partial_\beta(f|_{S^\psi}) = \partial_\beta \psi(\partial_0 f)|_{S^\psi} + (\partial_\beta f)|_{S^\psi}, \quad \beta = 1,2,$$

Then the boundary condition (4.5) on S^{ψ} becomes

$$\left\{\partial_0 \hat{p} + \gamma_1 \hat{p} + \gamma_2 \partial_\beta (u^\beta|_{S^\psi})\right\}\Big|_{S^\psi} + g = 0.$$

which is equivalent to

$$\begin{aligned}\partial_\beta(u^\beta|_{S^\psi}) = {} & \mu_5 (\partial_0 \hat{p})|_{S^\psi} + \mu_6 \psi^p + \mu_6 (r^p - r_b) \\ & + g_5(U, U^-, \psi, DU, D\psi),\end{aligned} \tag{4.6}$$

if we replace $\hat{p}|_{S^\psi}$ by ψ.

Decomposition of R-H conditions

Let $m \triangleq \rho(u\cdot n)|_{S^\psi} = \rho^-(u^-\cdot n)|_{S^\psi} \neq 0$ be the mass flux, then the R-H conditions (2.2)-(2.4) may be written as

$$[m] = 0, \quad [B] = 0, \tag{4.7}$$

and

$$[mu^0 + p] = 0, \tag{4.8}$$

$$[mu^\beta - p\,\partial_\beta\psi] = 0, \qquad \beta = 1,2. \tag{4.9}$$

If $[p] > 0$, from (4.9), we solve that

$$\partial_\beta\psi = \frac{m[u^\beta]}{[p]}\bigg|_{S^\psi} = \mu_0(u^\beta|_{S^\psi}) + g_0^\beta, \tag{4.10}$$

with

$$\mu_0 = \frac{(\rho u^0)_b}{p_b^+ - p_b^-}\bigg|_{x^0=r_b} > 0.$$

Decomposition of R-H conditions

It is necessary that $\partial_2\partial_1\psi - \partial_1\partial_2\psi = 0$, which implies

$$\partial_2(u^1|_{S^\psi}) - \partial_1(u^2|_{S^\psi}) = -\frac{1}{\mu_0}(\partial_2 g_0^1 - \partial_1 g_0^2). \tag{4.11}$$

Since ψ is well defined on $\mathbb{T}^2$, then

$$\begin{cases} \int_0^{2\pi} \Big(\mu_0(u^1|_{S^\psi}) + g_0^1\Big)(\psi(s,x^2),s,x^2)ds = 0, \\ \int_0^{2\pi} \Big(\mu_0(u^2|_{S^\psi}) + g_0^2\Big)(\psi(x^1,s),x^1,s)ds = 0. \end{cases} \tag{4.12}$$

On the contrary, (4.11)-(4.12) are also sufficient for the existence of a function ψ^p on $\mathbb{T}^2$ so that (4.10) holds, and $\int_{\mathbb{T}^2} \psi^p \mathrm{d}x^1\mathrm{d}x^2 = 0$.

Linearization of R-H conditions

We linearize the R–H conditions:

$$G_i(V,V^-) = \Psi_i(U,U^-,D\psi), \qquad i = 1,2,3,$$

with

$$\begin{aligned} G_1 &= [\rho(u^0)^2 + p], & \Psi_1 &= \partial_1\psi[\rho u^0 u^1] + \partial_2\psi[\rho u^0 u^2], \\ G_2 &= [\rho u^0], & \Psi_2 &= \partial_1\psi[\rho u^1] + \partial_2\psi[\rho u^2], \\ G_3 &= [B], & \Psi_3 &= 0. \end{aligned}$$

One can obtain

$$\widehat{u^0}|_{S^\psi} = \mu_1(\psi^p + r^p - r_b) + g_1(U,U^-,\psi,D\psi), \tag{4.13}$$

$$\hat{p}|_{S^\psi} = \mu_2(\psi^p + r^p - r_b) + g_2(U,U^-,\psi,D\psi), \tag{4.14}$$

$$\hat{\rho}|_{S^\psi} = \mu_3(\psi^p + r^p - r_b) + g_3(U,U^-,\psi,D\psi). \tag{4.15}$$

Using $A(S) = p\rho^{-\gamma}$, we also obtain

$$\widehat{A(S)}|_{S^\psi} = \mu_4(\psi^p + r^p - r_b) + g_4(U,U^-,\psi,D\psi).$$

Problem (T1)

Problem (T1): Find $\psi = \psi^p + r^p$ and $\hat{U} = U - U_b^+$ in Ω_ψ^+ solving the following problems (4.16)–(4.19).

The Cauchy problem of transport equations for $\hat{B}$:

$$\begin{cases} D_u \hat{B} + 2\mu u^0 \hat{B} = \dfrac{2\mu u^0}{\gamma - 1} \rho_b^{\gamma-1} \widehat{A(s)} + \dfrac{2\mu u^0}{\rho_b} \hat{p} + h(U) & \text{in} \quad \Omega_\psi^+, \\ B = B^- & \text{on} \quad S^\psi. \end{cases} \tag{4.16}$$

Problem (T1)

A second order elliptic equation for $\hat{p}$:

$$\begin{cases} L(\hat{p}) = f(U, DU, D^2 p) & \text{in} \quad \Omega_\psi^+, \\ \hat{p} = p_1 - p_b^+ & \text{on} \quad \Sigma_1, \\ \hat{p} = \mu_2 (\psi^p + r^p - r_b) + g_2(U, U^-, \psi, D\psi) & \text{on} \quad S^\psi, \end{cases} \tag{4.17}$$

where

$$\begin{aligned} L(\hat{p}) \quad \triangleq \quad & (t-1)\partial_0^2 \hat{p} - \partial_1^2 \hat{p} - \partial_2^2 \hat{p} + \mu d_1(t) \partial_0 \hat{p} + \mu^2 d_2(t) \hat{p} \\ & + \mu^2 \rho_b d_3(t) \hat{B} + \mu^2 \rho_b^\gamma d_4(t) \widehat{A(s)}. \end{aligned}$$

Problem (T1)

The Cauchy problems of transport equations for $\widehat{A(s)}$ and $u^\beta\,(\beta=1,2)$:

$$\begin{cases} D_u A(s) = 0 & \text{in} \quad \Omega_\psi^+, \\ \widehat{A(S)} = \mu_4\,(\psi^p + r^p - r_b) + g_4(U, U^-, \psi, D\psi) & \text{on} \quad S^\psi, \end{cases} \tag{4.18}$$

and

$$\begin{cases} D_u u^\beta = -\dfrac{1}{\rho}\partial_\beta p & \text{in} \quad \Omega_\psi^+, \\ u^\beta = u_0^\beta & \text{on} \quad S^\psi. \end{cases} \tag{4.19}$$

Problem (T1)

The equation for the surface S^ψ and the div-curl system of the tangential velocity $u'|_{S^\psi}$ on $\mathbb{T}^2$;

$$\begin{cases} \partial_\beta \psi = \mu_0(u^\beta|_{S^\psi}) + g_0^\beta(U, U^-, D\psi), \qquad \beta = 1, 2, \\ \partial_\beta(u^\beta|_{S^\psi}) = \mu_5\,(\partial_0 \hat{p}|_{S^\psi}) + \mu_6\,\psi^p + \mu_6\,(r^p - r_b) \\ \qquad\qquad +g_5(U, U^-, \psi, DU, D\psi), \\ \partial_2(u^1|_{S^\psi}) - \partial_1(u^2|_{S^\psi}) = -\dfrac{1}{\mu_0}(\partial_2 g_0^1 - \partial_1 g_0^2), \\ \displaystyle\int_0^{2\pi} \Big(\mu_0(u^1|_{S^\psi}) + g_0^1\Big)(\psi(s, x^2), s, x^2)ds = 0, \\ \displaystyle\int_0^{2\pi} \Big(\mu_0(u^2|_{S^\psi}) + g_0^2\Big)(\psi(x^1, s), x^1, s)ds = 0. \end{cases} \tag{4.20}$$

Problem (T2)

Acting the divergence operator to the first equation in (4.20) and using the second equation, we derive that

$$\begin{aligned}\Delta'\psi^p+\mu_7\psi^p=&\ \mu_0\mu_6(r^p-r_b)+\mu_0\mu_5\left(\partial_0\hat{p}|_{S^\psi}\right)\\&+g_6(U,U^-,\psi,DU^-,DU,D\psi,D^2\psi),\end{aligned}$$

Here Δ' is the standard Laplace operator on $\mathbb{T}^2$.

Then we get

$$\begin{aligned}&\Delta'(\hat{p}|_{S^\psi})+\mu_7(\hat{p}|_{S^\psi})+\mu_8(\partial_0\hat{p}|_{S^\psi})\\=&\ g_8(U,U^-,\psi,DU,DU^-,D\psi,D^2U,D^2U^-,D^2\psi,D^3\psi),\end{aligned}$$

if we replace ψ by $\hat{p}|_{S^\psi}$.

Problem (T2)

By the second equation in (4.20), using the divergence theorem, and recall that $\int_{\mathbb{T}^2} \psi^p \mathrm{d}x^1 \mathrm{d}x^2 = 0$, we have

$$r^p - r_b = -\frac{1}{4\pi^2 \mu_6} \int_{\mathbb{T}^2} \Big(\mu_5 \left(\partial_0 \hat{p}|_{S^\psi} \right) + g_5(U, U^-, \psi, DU, D\psi) \Big) \mathrm{d}x^1 \mathrm{d}x^2.$$

Then we obtain

$$\psi^p = \frac{1}{\mu_2} \left((\hat{p}|_{S^\psi}) - \mu_9 \int_{\mathbb{T}^2} (\partial_0 \hat{p}|_{S^\psi}) \, \mathrm{d}x^1 \mathrm{d}x^2 + g_9(U, U^-, \psi, D\psi) \right).$$

Problem (T2)

Problem (T2): Find $\psi = \psi^p + r^p$ and $\hat{U} = U - U_b^+$ that solve (4.16), (4.21), (4.22), (4.18), (4.23) and (4.19).

The mixed boundary value problem of a second order elliptic equation with a Venttsel boundary condition for $\hat{p}$:

$$\begin{cases} L(\hat{p}) = f & \text{in} \quad \Omega_\psi^+, \\ \hat{p} = p_1 - p_b^+ & \text{on} \quad \Sigma_1, \\ \Delta'(\hat{p}|_{S^\psi}) + \mu_7(\hat{p}|_{S^\psi}) + \mu_8(\partial_0 \hat{p}|_{S^\psi}) = g_8 & \text{on} \quad S^\psi. \end{cases} \tag{4.21}$$

Problem (T2)

The position r^p and the profile ψ^p of the surface S^ψ is determined by

$$\begin{cases} r^p - r_b = -\dfrac{1}{4\pi^2\mu_6}\displaystyle\int_{\mathbb{T}^2}\Big(\mu_5\left(\partial_0\hat{p}|_{S^\psi}\right) + g_5\Big)\,\mathrm{d}x^1\mathrm{d}x^2, \\ \psi^p = \dfrac{1}{\mu_2}\Big(\left(\hat{p}|_{S^\psi}\right) - \mu_9\displaystyle\int_{\mathbb{T}^2}\left(\partial_0\hat{p}|_{S^\psi}\right)\mathrm{d}x^1\mathrm{d}x^2 + g_9\Big), \\ \psi = \psi^p + r^p. \end{cases} \tag{4.22}$$

Problem (T2)

The div-curl system of the tangential velocity $u'|_{S^\psi}$ on $\mathbb{T}^2$:

$$
\begin{cases}
\partial_2(u^1|_{S^\psi}) - \partial_1(u^2|_{S^\psi}) = -\dfrac{1}{\mu_0}(\partial_2 g_0^1 - \partial_1 g_0^2), \\
\partial_\beta(u^\beta|_{S^\psi}) = \mu_5\,(\partial_0 \hat{p}|_{S^\psi}) + \mu_6\,\psi^p + \mu_6\,(r^p - r_b) + g_5, \\
\displaystyle\int_0^{2\pi} \Big(\mu_0(u^1|_{S^\psi}) + g_0^1\Big)(\psi(s,x^2),s,x^2)ds = 0, \\
\displaystyle\int_0^{2\pi} \Big(\mu_0(u^2|_{S^\psi}) + g_0^2\Big)(\psi(x^1,s),x^1,s)ds = 0.
\end{cases}
\tag{4.23}
$$

Problem (T3)

Let $\mathscr{M}=(r_b,L)\times\mathbb{T}^2, \mathscr{M}_0=\{r_b\}\times\mathbb{T}^2, \mathscr{M}_1=\{L\}\times\mathbb{T}^2$. We consider the transport equations of Bernoulli constant and the entropy:

$$\begin{cases} D_u\hat{B}+2\mu u^0\hat{B}=\frac{2\mu u^0}{\gamma-1}\rho_b^{\gamma-1}\widehat{A(s)}+\frac{2\mu u^0}{\rho_b}\hat{p}+\bar{h} & \text{in} \quad \mathscr{M}, \\ \hat{B}=B^- - B_b^- & \text{on} \quad \mathscr{M}_0; \end{cases} \tag{4.24}$$

$$\begin{cases} D_uA(s)=0 & \text{in} \quad \mathscr{M}, \\ i^*(\widehat{A(s)})=\frac{\mu_4}{\mu_2}i^*(\hat{p})+\bar{g}_4-\frac{\mu_4}{\mu_2}\bar{g}_2 & \text{on} \quad \mathscr{M}_0. \end{cases} \tag{4.25}$$

For the vector field $\frac{u'}{u^0}(x^0,x')$ defined for $x'=(x^1,x^2)\in\mathbb{T}^2$ and $x^0\in[0,L]$, we write the integral curve passing $(0,\bar{x})$ as $x'=\varphi(x^0,\bar{x}), \bar{x}\in\mathbb{T}^2$. For fixed x^0, the map $\varphi_{x^0}:\mathbb{T}^2\to\mathbb{T}^2,\ \bar{x}\mapsto x'=\varphi(x^0,\bar{x})$.

Problem (T3)

We write the unique solution to the linear transport equation as follows:

$$\begin{aligned}\hat{B}(y) &= \hat{B}(y^0, y') = e^{2\mu(r_b - y^0)}(B^- - B_b^-)(\bar{y}) \\ &\quad + \int_{r_b}^{y^0} e^{2\mu(\tau - y^0)}\left(\frac{2\mu}{\gamma - 1}\rho_b^{\gamma-1}\widehat{A(s)} + \frac{2\mu}{\rho_b}\hat{p} + \frac{1}{u^0}\bar{h}\right)(\tau, \varphi_\tau(\bar{y}))\,\mathrm{d}\tau.\end{aligned}$$

and

$$A(s)(y) = A(s)(y^0, y') = (i^* A(s))(\bar{y}).$$

Hence, recall that the entropy is a constant behind the shock-front for the background solution, we have

$$\begin{aligned}\widehat{A(s)}(y) &= i^*(\widehat{A(s)})(y') + \Big((i^* A(s))(\bar{y}) - (i^* A(s))(y')\Big) \\ &= \frac{\mu_4}{\mu_2} i^*(\hat{p}) + \bar{g}_4 - \frac{\mu_4}{\mu_2}\bar{g}_2 + \Big((i^* A(s))(\bar{y}) - (i^* A(s))(y')\Big).\end{aligned}$$

Problem (T3)

The second order nonlocal elliptic equation of pressure subjected to a Venttsel boundary condition on $\mathscr{M}_0$:

$$\begin{cases} \mathfrak{L}(\hat{p}) = e_6(y^0)\hat{E}^- + F & \text{in} \quad \mathscr{M}, \\ \hat{p} = p_1 - p_b^+ & \text{on} \quad \mathscr{M}_1, \\ \Delta' \hat{p} + \mu_7 \hat{p} + \mu_8 \partial_0 \hat{p} = G & \text{on} \quad \mathscr{M}_0. \end{cases} \tag{4.26}$$

where

$$\begin{aligned} \mathfrak{L}(\hat{p}) \quad \triangleq \quad & e_1(y^0)\partial_0^2 \hat{p} - \partial_1^2 \hat{p} - \partial_2^2 \hat{p} + e_2(y^0)\partial_0 \hat{p} + e_3(y^0)\hat{p} \\ & + e_4(y^0) \int_{r_b}^{y^0} b(\mu, \tau)\hat{p}(\tau, y')\,\mathrm{d}\tau + e_5(y^0)(\hat{p}|_{\mathscr{M}_0}). \end{aligned}$$

A strong solution $\hat{p}$ in Sobolev space $H^2(\mathscr{M})$ with $i^*\hat{p} \in H^2(\mathscr{M}_0)$ to problem (4.26) is unique ?

Uniqueness of solutions in Sobolev spaces

We consider the homogeneous problem

$$\begin{cases} \mathfrak{L}(\hat{p}) = 0 & \text{in} \quad \mathscr{M}, \\ \hat{p} = 0 & \text{on} \quad \mathscr{M}_1, \\ \Delta'(i^*\hat{p}) + \mu_7(i^*\hat{p}) + \mu_8(i^*\partial_0\hat{p}) = 0 & \text{on} \quad \mathscr{M}_0. \end{cases} \tag{4.27}$$

Denote $m = (m_1, m_2)$, we could write

$$\hat{p}(y) = \sum_{m_1,m_2=0}^{\infty} \lambda_m \{ p_{1,m}(y^0)\cos(m_1y^1)\cos(m_2y^2) + p_{2,m}(y^0)\sin(m_1y^1)\cos(m_2y^2) + p_{3,m}(y^0)\cos(m_1y^1)\sin(m_2y^2) + p_{4,m}(y^0)\sin(m_1y^1)\sin(m_2y^2) \},$$

where

$$\lambda_m = \begin{cases} \frac{1}{4} & \text{if} \quad m_1 = m_2 = 0, \\ \frac{1}{2} & \text{if only one of} \ \ m_1, m_2 \ \ \text{is} \ \ 0, \\ 1 & \text{if} \quad m_1 > 0, m_2 > 0. \end{cases}$$

Uniqueness of solutions in Sobolev spaces

For $y^0 \in [r_b, L]$, each $p_{i,m}(y^0)$ solves the following nonlocal ordinary differential equation

$$\begin{aligned} e_1(y^0)p''_{i,m} + e_2(y^0)p'_{i,m} + (e_3(y^0) + |m|^2)p_{i,m} \\ + e_4(y^0)\int_{r_b}^{y^0} b(\mu,\tau)p_{i,m}(\tau)\,\mathrm{d}\tau + e_5(y^0)p_{i,m}(r_b) = 0, \end{aligned} \tag{4.28}$$

subjected to the two-point boundary conditions:

$$p'_{i,m}(r_b) + \frac{\mu_7 - |m|^2}{\mu_8}p_{i,m}(r_b) = 0, \qquad p_{i,m}(L) = 0. \tag{4.29}$$

Uniqueness of solutions in Sobolev spaces

Supposing that $p_{i,m}(r_b) = 0$, we set $\mathscr{P}_{i,m}(y^0) = \int_{r_b}^{y^0} b(\mu,\tau) p_{i,m}(\tau)\,\mathrm{d}\tau$. Then problem (4.28) and (4.29) can be written as

$$\begin{cases} \tilde{e}_1(y^0)\mathscr{P}_{i,m}''' + \tilde{e}_2(y^0)\mathscr{P}_{i,m}'' + \tilde{e}_3(y^0)\mathscr{P}_{i,m}' + e_4(y^0)\mathscr{P}_{i,m} = 0, \\ \mathscr{P}_{i,m}(r_b) = \mathscr{P}_{i,m}'(r_b) = \mathscr{P}_{i,m}''(r_b) = 0, \\ \mathscr{P}_{i,m}'(L) = 0. \end{cases} \tag{4.30}$$

Here we define

$$\tilde{e}_1(y^0) = \frac{e_1(y^0)}{b(\mu,y^0)} < 0,$$

$$\tilde{e}_2(y^0) = \left(\frac{e_2(y^0)}{b(\mu,y^0)} - \frac{2e_1(y^0)b'(\mu,y^0)}{b^2(\mu,y^0)} \right),$$

$$\tilde{e}_3(y^0) = \left(\frac{(e_3(y^0) + |m|^2)}{b(\mu,y^0)} - \frac{e_2(y^0)b'(\mu,y^0) + e_1(y^0)b''(\mu,y^0)}{b^2(\mu,y^0)} + \frac{2e_1(y^0)(b'(\mu,y^0))^2}{b^3(\mu,y^0)} \right),$$

S-Condition

If $p_{i,m}(r_b) \neq 0$ for some i,m, we set $w_{i,m}(y^0) = \frac{p_{i,m}(y^0)}{p_{i,m}(r_b)}$ and $\mathscr{W}_{i,m}(y^0) = \int_{r_b}^{y^0} b(\mu,\tau) w_{i,m}(\tau)\,\mathrm{d}\tau$. Then it solves

$$\begin{cases} \tilde{e}_1(y^0)\mathscr{W}_{i,m}''' + \tilde{e}_2(y^0)\mathscr{W}_{i,m}'' + \tilde{e}_3(y^0)\mathscr{W}_{i,m}' + e_4(y^0)\mathscr{W}_{i,m} + e_5(y^0) = 0, \\ \mathscr{W}_{i,m}(r_b) = 0, \quad \mathscr{W}_{i,m}'(r_b) = b(\mu, r_b), \quad \mathscr{W}_{i,m}''(r_b) = b'(\mu, r_b) - \frac{\mu_7 - |m|^2}{\mu_8} b(\mu, r_b), \\ \mathscr{W}_{i,m}'(L) = 0. \end{cases} \tag{4.31}$$

Definition 4.1

We say a background solution U_b satisfies the *S-Condition*, if for each $i = 1,2,3,4$ and $m \in \mathbb{Z}^2$, problem (4.31) does *not* have a classical solution.

S-Condition

Recall that a background solution is determined by the parameters: $\gamma > 1$, $\mu > 0, r_b \in (0, L)$, $p_b^+(L) > 0$, $\rho_b^+(r_b) > 0$, $M_b^+(r_b) \in (0, 1)$. Our purpose below is to show theoretically that almost all background solutions satisfy the S-Condition.

Lemma 4.2

There exists a set $\mathscr{S} \subset (0, +\infty)$ of at most countable infinite points such that the background solutions U_b determined by $\mu \in (0, +\infty) \setminus \mathscr{S}$ satisfy the S-Condition.

Existence

We now use Fourier series to establish a family of approximate solutions to problem (4.26). Therefore, we proved the following lemma.

Lemma 4.3

Suppose that the S-Condition holds. Then problem (4.26) *has one and only one solution in* $C^{k,\alpha}(\overline{\mathscr{M}})$*, and it satisfies the following estimate*

$$\|\hat{p}\|_{C^{k,\alpha}(\overline{\mathscr{M}})} \leq C\Big(\|h_0\|_{C^{k-2,\alpha}(\mathbb{T}^2)} + \|h_1\|_{C^{k,\alpha}(\mathbb{T}^2)} + \|f\|_{C^{k-2,\alpha}(\overline{\mathscr{M}})} \Big) \qquad (4.32)$$

Further Discussions

- 3-D transonic shock problem;
- The structural stability of more complex configuration;
- ……

[1] Z. Xin: On the Courant-Friedrichs' transonic shock wave in a nozzle, 2018.

Thank You!