

Some recent studies on steady MHD boundary layers

Zhu ZHANG, City University of Hong Kong

Joint with Prof. Chengjie LIU (SJTU) and Prof. Tong YANG (CityU)

CAMIS, South China Normal University

2020年12月1日

(2D steady MHD system)

$$\begin{cases} \mathbf{U} \cdot \nabla \mathbf{U} + \nabla P - \mathbf{H} \cdot \nabla \mathbf{H} - \mu \varepsilon \Delta \mathbf{U} = \mathbf{F_U}, \\ \mathbf{U} \cdot \nabla \mathbf{H} - \mathbf{H} \cdot \nabla \mathbf{U} - \kappa \varepsilon \Delta \mathbf{H} = \mathbf{F_H}, \quad \text{in } \Omega, \\ \nabla \cdot \mathbf{U} = \nabla \cdot \mathbf{H} = 0. \end{cases}$$

$\Omega = \{(x, y) \mid x \in \mathbb{T}_\varrho, y > 0\}$, with $\mathbb{T}_\varrho = \mathbb{R}/(2\pi\varrho)\mathbb{Z}$, ϱ : Torus length,

$\mathbf{U} = (u, v)$: velocity, $\mathbf{H} = (h, g)$: magnetic field, $P = P_{\mathbf{U}} + \frac{|\mathbf{H}|^2}{2}$: pressure,

$\mathbf{F_U} = (F_{1,\mathbf{U}}, F_{2,\mathbf{U}})$, $\mathbf{F_H} = (F_{1,\mathbf{H}}, F_{2,\mathbf{H}})$ are given external forces,

Boundary conditions : $\mathbf{U}|_{y=0} = (\partial_y h, g)|_{y=0} = \mathbf{0}$,

Compatibility condition : $\nabla \cdot \mathbf{F_H} = 0$, $F_{2,\mathbf{H}}|_{y=0} = 0$.

(2D steady MHD system)

$$\begin{cases} \mathbf{U} \cdot \nabla \mathbf{U} + \nabla P - \mathbf{H} \cdot \nabla \mathbf{H} - \mu \varepsilon \Delta \mathbf{U} = \mathbf{F}_U, \\ \mathbf{U} \cdot \nabla \mathbf{H} - \mathbf{H} \cdot \nabla \mathbf{U} - \kappa \varepsilon \Delta \mathbf{H} = \mathbf{F}_H, \quad \text{in } \Omega, \\ \nabla \cdot \mathbf{U} = \nabla \cdot \mathbf{H} = 0. \end{cases}$$

$\Omega = \{(x, y) \mid x \in \mathbb{T}_\varrho, y > 0\}$, with $\mathbb{T}_\varrho = \mathbb{R}/(2\pi\varrho)\mathbb{Z}$, ϱ : Torus length,

$\mathbf{U} = (u, v)$: velocity, $\mathbf{H} = (h, g)$: magnetic field, $P = P_U + \frac{|\mathbf{H}|^2}{2}$: pressure,

$\mathbf{F}_U = (F_{1,U}, F_{2,U})$, $\mathbf{F}_H = (F_{1,H}, F_{2,H})$ are given external forces,

Boundary conditions : $\mathbf{U}|_{y=0} = (\partial_y h, g)|_{y=0} = \mathbf{0}$,

Compatibility condition : $\nabla \cdot \mathbf{F}_H = 0$, $F_{2,H}|_{y=0} = 0$.

Question

Asymptotic behavior of (\mathbf{U}, \mathbf{H}) as $\varepsilon \rightarrow 0$?



Prandtl's ansatz

- Away from the boundary, $(\mathbf{U}, \mathbf{H})(x, y) \sim (\mathbf{U}', \mathbf{H}')(x, y)$ where $(\mathbf{U}', \mathbf{H}')$ satisfies the ideal MHD system with the boundary conditions $\mathbf{U}' \cdot \vec{n}|_{y=0} = \mathbf{H}' \cdot \vec{n}|_{y=0} = 0$.

Prandtl's ansatz

- Away from the boundary, $(\mathbf{U}, \mathbf{H})(x, y) \sim (\mathbf{U}', \mathbf{H}')(x, y)$ where $(\mathbf{U}', \mathbf{H}')$ satisfies the ideal MHD system with the boundary conditions $\mathbf{U}' \cdot \vec{n}|_{y=0} = \mathbf{H}' \cdot \vec{n}|_{y=0} = 0$.
- Near the boundary
 $(\mathbf{U}, \mathbf{H})(x, y) \sim \left(u^P(x, \frac{y}{\sqrt{\varepsilon}}), \sqrt{\varepsilon} v^P(x, \frac{y}{\sqrt{\varepsilon}}), h^P(x, \frac{y}{\sqrt{\varepsilon}}), \sqrt{\varepsilon} g^P(x, \frac{y}{\sqrt{\varepsilon}}) \right)$
where $(u^P, v^P, h^P, g^P)(x, Y)$ satisfies a Prandtl-type system with the boundary conditions $(u^P, v^P, \partial_Y h^P, g^P)|_{y=0} = \mathbf{0}$ and far-field conditions: $\lim_{Y \rightarrow +\infty} (u^P, h^P)(x, Y)$ matches the trace of tangential components of $(\mathbf{U}', \mathbf{H}')$ on the boundary $\{y = 0\}$.

Prandtl's ansatz

- Away from the boundary, $(\mathbf{U}, \mathbf{H})(x, y) \sim (\mathbf{U}', \mathbf{H}')(x, y)$ where $(\mathbf{U}', \mathbf{H}')$ satisfies the ideal MHD system with the boundary conditions $\mathbf{U}' \cdot \vec{n}|_{y=0} = \mathbf{H}' \cdot \vec{n}|_{y=0} = 0$.
- Near the boundary
 $(\mathbf{U}, \mathbf{H})(x, y) \sim \left(u^P(x, \frac{y}{\sqrt{\varepsilon}}), \sqrt{\varepsilon} v^P(x, \frac{y}{\sqrt{\varepsilon}}), h^P(x, \frac{y}{\sqrt{\varepsilon}}), \sqrt{\varepsilon} g^P(x, \frac{y}{\sqrt{\varepsilon}}) \right)$
where $(u^P, v^P, h^P, g^P)(x, Y)$ satisfies a Prandtl-type system with the boundary conditions $(u^P, v^P, \partial_Y h^P, g^P)|_{y=0} = \mathbf{0}$ and far-field conditions: $\lim_{Y \rightarrow +\infty} (u^P, h^P)(x, Y)$ matches the trace of tangential components of $(\mathbf{U}', \mathbf{H}')$ on the boundary $\{y = 0\}$.

Mathematical questions

- the well-posedness/ill-posedness of the boundary layer system;

Prandtl's ansatz

- Away from the boundary, $(\mathbf{U}, \mathbf{H})(x, y) \sim (\mathbf{U}', \mathbf{H}')(x, y)$ where $(\mathbf{U}', \mathbf{H}')$ satisfies the ideal MHD system with the boundary conditions $\mathbf{U}' \cdot \vec{n}|_{y=0} = \mathbf{H}' \cdot \vec{n}|_{y=0} = 0$.
- Near the boundary
 $(\mathbf{U}, \mathbf{H})(x, y) \sim \left(u^P(x, \frac{y}{\sqrt{\varepsilon}}), \sqrt{\varepsilon} v^P(x, \frac{y}{\sqrt{\varepsilon}}), h^P(x, \frac{y}{\sqrt{\varepsilon}}), \sqrt{\varepsilon} g^P(x, \frac{y}{\sqrt{\varepsilon}}) \right)$
where $(u^P, v^P, h^P, g^P)(x, Y)$ satisfies a Prandtl-type system with the boundary conditions $(u^P, v^P, \partial_Y h^P, g^P)|_{y=0} = \mathbf{0}$ and far-field conditions: $\lim_{Y \rightarrow +\infty} (u^P, h^P)(x, Y)$ matches the trace of tangential components of $(\mathbf{U}', \mathbf{H}')$ on the boundary $\{y = 0\}$.

Mathematical questions

- the well-posedness/ill-posedness of the boundary layer system;
- the rigorous justification $(\mathbf{U}, \mathbf{H}) = (\mathbf{U}', \mathbf{H}') + (\mathbf{U}^P, \mathbf{H}^P) + o(1)$.

Related works

Prandtl system for NS

- Oleinik '63, The Prandtl system in boundary layer theory,
- Sammartino-Caflisch '98, analytic framework,
- Z.P Xin, L.Q. Zhang '04, global weak solution,
- Alexandre, Y.G. Wang, C.J. Xu, T. Yang '15, Masmoudi-Wong '15, Sobolev framework under monotonicity condition,
- Gerard-Masmoudi '15, W.X. Li-T. Yang '18, Dietert-Gerard '19, Gevrey framework in 2D, Li-Masmoudi-Yang '20, Gevrey framework in 3D,
- P.Zhang-Z.F. Zhang '16, Paicu-Zhang '19, long-time well-posedness,
- E.-Engquist '97, blow-up; Dormy-Gerard '10, ill-posedness.

Related works

Convergence results

- Sammartino-Caflisch '98, Wang-Wang-Zhang '17, analytic framework,
- Maekawa '14, Fei-Tao-Zhang '18, vorticity away from boundary,
- Kukavica-Vicol-Wang '20, analytic near boundary,
- Gerard-Maekawa-Masmoudi '18, Gevrey convergence,
- Grenier-Guo-Nguyen '16, instability,
- Guo-Iyer '18, Gao-Zhang '20, steady NS for x in bounded interval,
- Gerard-Maekawa '19, steady NS for $x \in \mathbb{T}$...

Related works

MHD boundary layers

- Liu-Xie-Yang '17, '19, 2D unsteady MHD boundary layer system with non-degenerate tangential magnetic field,
- Wang-Xin '17, S. Wang et al., Convergence results in different settings,
- Ding-Li-Xie '20, 2D steady MHD with moving boundary.

Steady MHD

(2D steady MHD system)

$$\begin{cases} \mathbf{U} \cdot \nabla \mathbf{U} + \nabla P - \mathbf{H} \cdot \nabla \mathbf{H} - \mu \varepsilon \Delta \mathbf{U} = \mathbf{F_U}, \\ \mathbf{U} \cdot \nabla \mathbf{H} - \mathbf{H} \cdot \nabla \mathbf{U} - \kappa \varepsilon \Delta \mathbf{H} = \mathbf{F_H}, \quad \text{in } \Omega = \mathbb{T}_\varrho \times \mathbb{R}_+, \\ \nabla \cdot \mathbf{U} = \nabla \cdot \mathbf{H} = 0, \quad \mathbf{U}|_{y=0} = (\partial_y h, g)|_{y=0} = \mathbf{0} \end{cases}$$

Consider the shear flow of Prandtl type:

$$(\mathbf{U}_s, \mathbf{H}_s)(Y) = (U_s(Y), 0, H_s(Y), 0), \quad Y := \frac{y}{\sqrt{\varepsilon}}$$

Structural Assumptions

- $U_s, H_s \in C^3(\mathbb{R}_+) \cap C^1(\overline{\mathbb{R}_+})$ such that

$$U_s(0) = 0, H'_s(0) = 0, \lim_{Y \rightarrow +\infty} U_s(Y) = U_E, \lim_{Y \rightarrow +\infty} H_s(Y) = H_E \neq 0$$

and

$$\bar{M} := \sum_{1 \leq k \leq 3} \sup_{Y \geq 0} (1 + Y)^3 \left(|\partial_Y^k U_s(Y)| + |\partial_Y^k H_s(Y)| \right) < \infty.$$

Structural Assumptions

- $U_s, H_s \in C^3(\mathbb{R}_+) \cap C^1(\overline{\mathbb{R}_+})$ such that

$$U_s(0) = 0, H'_s(0) = 0, \lim_{Y \rightarrow +\infty} U_s(Y) = U_E, \lim_{Y \rightarrow +\infty} H_s(Y) = H_E \neq 0$$

and

$$\bar{M} := \sum_{1 \leq k \leq 3} \sup_{Y \geq 0} (1+Y)^3 \left(|\partial_Y^k U_s(Y)| + |\partial_Y^k H_s(Y)| \right) < \infty.$$

- **(Strong tangential magnetic field)** There are two positive constants $\underline{\gamma}, \bar{\gamma} > 0$, such that

$$\underline{\gamma} \leq |H_s(Y)| \leq \bar{\gamma}, \text{ for any } Y > 0.$$

and

$$\gamma_0 := \inf_{Y \geq 0} G_s(Y) = \inf_{Y \geq 0} \left(H_s^2(Y) - U_s^2(Y) \right) > 0.$$

Set

$$(\tilde{\mathbf{U}}, \tilde{\mathbf{H}}) = (\mathbf{U}, \mathbf{H}) - (\mathbf{U}_s, \mathbf{H}_s) \triangleq (\tilde{u}, \tilde{v}, \tilde{h}, \tilde{g})$$

be the perturbation of $(\mathbf{U}_s, \mathbf{H}_s)$. the problem for $(\tilde{\mathbf{U}}, \tilde{\mathbf{H}})$ is written as

$$\begin{cases} U_s \partial_x \tilde{\mathbf{U}} + \tilde{v} \partial_y U_s \mathbf{e}_1 - H_s \partial_x \tilde{\mathbf{H}} - \tilde{g} \partial_y H_s \mathbf{e}_1 + \nabla P - \mu \varepsilon \Delta \tilde{\mathbf{U}} = -\tilde{\mathbf{U}} \cdot \nabla \tilde{\mathbf{U}} + \tilde{\mathbf{H}} \cdot \nabla \tilde{\mathbf{H}} + \mathbf{f}_{\mathbf{U}}, \\ U_s \partial_x \tilde{\mathbf{H}} + \tilde{v} \partial_y H_s \mathbf{e}_1 - H_s \partial_x \tilde{\mathbf{U}} - \tilde{g} \partial_y U_s \mathbf{e}_1 - \kappa \varepsilon \Delta \tilde{\mathbf{H}} = -\tilde{\mathbf{U}} \cdot \nabla \tilde{\mathbf{H}} + \tilde{\mathbf{H}} \cdot \nabla \tilde{\mathbf{U}} + \mathbf{f}_{\mathbf{H}}, \\ \nabla \cdot \tilde{\mathbf{U}} = \nabla \cdot \tilde{\mathbf{H}} = 0, \\ \tilde{\mathbf{U}}|_{y=0} = (\partial_y \tilde{h}, \tilde{g})|_{y=0} = \mathbf{0}, \end{cases}$$

where the vector $\mathbf{e}_1 = (1, 0)$, and the source term

$$(\mathbf{f}_{\mathbf{U}}, \mathbf{f}_{\mathbf{H}}) := (\mathbf{F}_{\mathbf{U}}, \mathbf{F}_{\mathbf{H}}) - (\mathbf{F}_{\mathbf{U}_s}, \mathbf{F}_{\mathbf{H}_s}) \triangleq (f_{1,\mathbf{U}}, f_{2,\mathbf{U}}, f_{1,\mathbf{H}}, f_{2,\mathbf{H}})$$

satisfying $\nabla \cdot \mathbf{f}_{\mathbf{H}} = 0$, $f_{2,\mathbf{H}}|_{y=0} = 0$.

Functional spaces

For any x -dependent function $f(x) \in L^2(\mathbb{T}_\varrho)$, we denote by f_n its n -th Fourier coefficient, i.e.,

$$f_n = \frac{1}{2\pi\varrho} \int_0^{2\pi\varrho} e^{-i\tilde{n}x} f(x) dx, \quad n \in \mathbb{Z}, \quad \tilde{n} = \frac{n}{\varrho},$$

$$\mathcal{P}_n f = f_n e^{i\tilde{n}x} \text{ and } \mathcal{Q}_0 f = (I - \mathcal{P}_0)f.$$

Define the solution norm with some weight function $Z(y) \sim y$ near $y = 0$.

$$\begin{aligned} \|(\tilde{\mathbf{U}}, \tilde{\mathbf{H}})\|_{\mathcal{X}} := & \sum_n \|(\tilde{\mathbf{U}}_n, \tilde{\mathbf{H}}_n)\|_{L^\infty(\mathbb{R}_+)} + \varepsilon^{\frac{1}{4}} \|(\partial_y \tilde{u}_0, \partial_y \tilde{h}_0)\|_{L^2(\mathbb{R}_+)} + \|Z^{\frac{1}{2}}(\partial_y \tilde{u}_0, \partial_y \tilde{h}_0)\|_{L^2(\mathbb{R}_+)} \\ & + \varepsilon^{-\frac{1}{4}} \|(\mathcal{Q}_0 \tilde{\mathbf{U}}, \mathcal{Q}_0 \tilde{\mathbf{H}})\|_{L^2(\Omega)} + \varepsilon^{-\frac{1}{2}} \|Z^{\frac{1}{2}}(\mathcal{Q}_0 \tilde{\mathbf{U}}, \mathcal{Q}_0 \tilde{\mathbf{H}})\|_{L^2(\Omega)} \\ & + \varepsilon^{\frac{1}{4}} \|(\nabla \mathcal{Q}_0 \tilde{\mathbf{U}}, \nabla \mathcal{Q}_0 \tilde{\mathbf{H}})\|_{L^2(\Omega)} + \|Z^{\frac{1}{2}}(\nabla \mathcal{Q}_0 \tilde{\mathbf{U}}, \nabla \mathcal{Q}_0 \tilde{\mathbf{H}})\|_{L^2(\Omega)}. \end{aligned}$$

Theorem(C.J. Liu, T. Yang & Z.'20)

There exist positive constants δ_1, δ_2 and ε_0 , such that for any $\varepsilon \in (0, \varepsilon_0)$ and $\eta > 0$, if

$$\varrho(\bar{M} + \bar{M}^4) \in (0, \delta_1), \quad \|(\mathbf{f}_{\mathbf{U}}, \mathbf{f}_{\mathbf{H}})\|_{L^2(\Omega)} + \varepsilon^{-\frac{1}{4}} \|Z^{\frac{1}{2}}(\mathbf{f}_{\mathbf{U}}, \mathbf{f}_{\mathbf{H}})\|_{L^2(\Omega)} \leq \frac{\delta_2 \varepsilon^{\frac{3}{4}}}{|\log \varepsilon|^{3+\eta}},$$

then the steady MHD system admits a unique solution

$(\tilde{\mathbf{U}}, \tilde{\mathbf{H}}, \nabla P) : (\tilde{\mathbf{U}}, \tilde{\mathbf{H}}) \in \mathcal{X} \cap H_{loc}^2(\Omega), \nabla P \in L^2(\Omega)$ that satisfies the estimate:

$$\|(\tilde{\mathbf{U}}, \tilde{\mathbf{H}})\|_{\mathcal{X}} \leq C \varepsilon^{-\frac{1}{4}} |\log \varepsilon|^{\frac{3+\eta}{2}} \left(\|(\mathbf{f}_{\mathbf{U}}, \mathbf{f}_{\mathbf{H}})\|_{L^2(\Omega)} + \varepsilon^{-\frac{1}{4}} \|Z^{\frac{1}{2}}(\mathbf{f}_{\mathbf{U}}, \mathbf{f}_{\mathbf{H}})\|_{L^2(\Omega)} \right),$$

where C is independent of ε .



Remark

- Neither monotonicity nor positivity assumption on the velocity field is need.

Remark

- Neither monotonicity nor positivity assumption on the velocity field is need.
- The length ϱ of torus is allowed to be large for boundary layer with suitably small amplitude.

Main Ingredients of Proof

Linearized system-Mode by mode analysis

$$\begin{cases} i\tilde{n}U_s \mathbf{U}_n + \textcolor{red}{v_n} \partial_y U_s \mathbf{e}_1 - i\tilde{n}H_s \mathbf{H}_n - \textcolor{red}{g_n} \partial_y H_s \mathbf{e}_1 + (i\tilde{n}P_n, \partial_y P_n) - \mu\varepsilon \Delta_n \mathbf{U}_n = \mathbf{f}_n, \\ i\tilde{n}U_s \mathbf{H}_n + \textcolor{red}{v_n} \partial_y H_s \mathbf{e}_1 - i\tilde{n}H_s \mathbf{U}_n - \textcolor{red}{g_n} \partial_y U_s \mathbf{e}_1 - \kappa\varepsilon \Delta_n \mathbf{H}_n = \mathbf{q}_n, \\ i\tilde{n}u_n + \partial_y v_n = i\tilde{n}h_n + \partial_y g_n = 0, \\ (u_n, v_n, \partial_y h_n, g_n)|_{y=0} = \mathbf{0}. \end{cases}$$

Here $n \in \mathbb{Z}$, $\tilde{n} = \frac{n}{\varrho}$ and $\Delta_n := \partial_y^2 - \tilde{n}^2$.

Main Ingredients of Proof

Linearized system-Mode by mode analysis

$$\begin{cases} i\tilde{n}U_s \mathbf{U}_n + \textcolor{red}{v_n \partial_y U_s \mathbf{e}_1} - i\tilde{n}H_s \mathbf{H}_n - \textcolor{red}{g_n \partial_y H_s \mathbf{e}_1} + (i\tilde{n}P_n, \partial_y P_n) - \mu\varepsilon \Delta_n \mathbf{U}_n = \mathbf{f}_n, \\ i\tilde{n}U_s \mathbf{H}_n + \textcolor{red}{v_n \partial_y H_s \mathbf{e}_1} - i\tilde{n}H_s \mathbf{U}_n - \textcolor{red}{g_n \partial_y U_s \mathbf{e}_1} - \kappa\varepsilon \Delta_n \mathbf{H}_n = \mathbf{q}_n, \\ i\tilde{n}u_n + \partial_y v_n = i\tilde{n}h_n + \partial_y g_n = 0, \\ (u_n, v_n, \partial_y h_n, g_n)|_{y=0} = \mathbf{0}. \end{cases}$$

Here $n \in \mathbb{Z}$, $\tilde{n} = \frac{n}{\varrho}$ and $\Delta_n := \partial_y^2 - \tilde{n}^2$.

Difficulty 1: Large stretching terms

$$v_n \partial_y U_s - g_n \partial_y H_s = \varepsilon^{-\frac{1}{2}} (v_n \partial_Y U_s - g_n \partial_Y H_s),$$

$$v_n \partial_y H_s - g_n \partial_y U_s = \varepsilon^{-\frac{1}{2}} (v_n \partial_Y H_s - g_n \partial_Y U_s).$$

Stream function $\psi(x, y)$ of magnetic field: $h = \partial_y \psi$, $g = -\partial_x \psi$, $\psi|_{y=0} = 0$.

Define $a_p(Y) = \frac{U_s(Y)}{H_s(Y)}$, $b_p(Y) = \frac{\partial_Y H_s(Y)}{H_s(Y)}$ and $\widehat{\mathbf{W}} = (\hat{u}, \hat{v}, \hat{h}, \hat{g})$

Transformation (Liu-Xie-Yang 19')

$$\left\{ \begin{array}{l} \hat{u}(x, y) = u(x, y) - \partial_y(a_p(Y)\psi(x, y)), \\ \hat{v}(x, y) = v(x, y) + \partial_x(a_p(Y)\psi(x, y)), \\ \hat{h}(x, y) = \partial_y\left(\frac{\psi(x, y)}{H_s(Y)}\right) = \frac{1}{H_s(Y)}\left(h(x, y) - \varepsilon^{-\frac{1}{2}}b_p(Y)\psi(x, y)\right), \\ \hat{g}(x, y) = -\partial_x\left(\frac{\psi(x, y)}{H_s(Y)}\right) = \frac{g(x, y)}{H_s(Y)}, \\ \hat{\psi}(x, y) = \frac{\psi(x, y)}{H_s(Y)}, \quad (\nabla \cdot \widehat{\mathbf{U}} = \nabla \cdot \widehat{\mathbf{H}} = 0) \end{array} \right.$$

Stream function $\psi(x, y)$ of magnetic field: $h = \partial_y \psi$, $g = -\partial_x \psi$, $\psi|_{y=0} = 0$.

Define $a_p(Y) = \frac{U_s(Y)}{H_s(Y)}$, $b_p(Y) = \frac{\partial_Y H_s(Y)}{H_s(Y)}$ and $\widehat{\mathbf{W}} = (\hat{u}, \hat{v}, \hat{h}, \hat{g})$

Transformation (Liu-Xie-Yang 19')

$$\left\{ \begin{array}{l} \hat{u}(x, y) = u(x, y) - \partial_y(a_p(Y)\psi(x, y)), \\ \hat{v}(x, y) = v(x, y) + \partial_x(a_p(Y)\psi(x, y)), \\ \hat{h}(x, y) = \partial_y \left(\frac{\psi(x, y)}{H_s(Y)} \right) = \frac{1}{H_s(Y)} \left(h(x, y) - \varepsilon^{-\frac{1}{2}} b_p(Y)\psi(x, y) \right), \\ \hat{g}(x, y) = -\partial_x \left(\frac{\psi(x, y)}{H_s(Y)} \right) = \frac{g(x, y)}{H_s(Y)}, \\ \hat{\psi}(x, y) = \frac{\psi(x, y)}{H_s(Y)}, \quad (\nabla \cdot \widehat{\mathbf{U}} = \nabla \cdot \widehat{\mathbf{H}} = 0) \end{array} \right.$$

$$-H_s \partial_x h - g \partial_y H_s = -H_s^2 \partial_x \hat{h} - \partial_y H_s (\cancel{\partial_x \psi + g})$$

$$-H_s \partial_x u - g \partial_y U_s = -H_s \partial_x \hat{u} + H_s \partial_y (U_s \hat{g}) - g \partial_y U_s$$

$$= -H_s \partial_x \hat{u} - H_s^2 a_p \partial_x \hat{h} + \partial_y U_s (\cancel{H_s \hat{g} - g}).$$

The system of $\widehat{\mathbf{W}}_n = (\hat{u}_n, \hat{v}_n, \hat{h}_n, \hat{g}_n)$ reads

$$\begin{cases} i\tilde{n} \left[(1 + \frac{\mu}{\kappa}) U_s \widehat{\mathbf{U}}_n - \textcolor{red}{G}_s \widehat{\mathbf{H}}_n + \varepsilon^{\frac{1}{2}} \mathbf{A}_{\mathbf{U}} \widehat{\mathbf{H}}_n \right] + \varepsilon^{\frac{1}{2}} \mathbf{B}_{\mathbf{U}} \partial_y \widehat{\mathbf{H}}_n + \mathbf{C}_{\mathbf{U}} \widehat{\mathbf{H}}_n + \varepsilon^{-\frac{1}{2}} \hat{\psi}_n \mathbf{D}_{\mathbf{U}} \\ \quad + (i\tilde{n} p_n, \partial_y p_n)^T - \mu \varepsilon \Delta_n \widehat{\mathbf{U}}_n = \mathbf{R}_{\mathbf{U},n}, \\ -i\tilde{n} \widehat{\mathbf{U}}_n - 2\kappa \varepsilon^{\frac{1}{2}} b_p \partial_y \widehat{\mathbf{H}}_n + \mathbf{C}_{\mathbf{H}} \widehat{\mathbf{H}}_n + \varepsilon^{-\frac{1}{2}} \hat{\psi}_n \mathbf{D}_{\mathbf{H}} - \kappa \varepsilon \Delta_n \widehat{\mathbf{H}}_n = \mathbf{R}_{\mathbf{H},n}, \\ i\tilde{n} \hat{u}_n + \partial_y \hat{v}_n = i\tilde{n} \hat{h}_n + \partial_y \hat{g}_n = 0, \quad \widehat{\mathbf{U}}_n|_{y=0} = (\partial_y \hat{h}_n, \hat{g}_n)|_{y=0} = \mathbf{0}, \end{cases}$$

The system of $\widehat{\mathbf{W}}_n = (\hat{u}_n, \hat{v}_n, \hat{h}_n, \hat{g}_n)$ reads

$$\begin{cases} i\tilde{n} \left[(1 + \frac{\mu}{\kappa}) U_s \widehat{\mathbf{U}}_n - \textcolor{red}{G_s} \widehat{\mathbf{H}}_n + \varepsilon^{\frac{1}{2}} \mathbf{A}_{\mathbf{U}} \widehat{\mathbf{H}}_n \right] + \varepsilon^{\frac{1}{2}} \mathbf{B}_{\mathbf{U}} \partial_y \widehat{\mathbf{H}}_n + \mathbf{C}_{\mathbf{U}} \widehat{\mathbf{H}}_n + \varepsilon^{-\frac{1}{2}} \hat{\psi}_n \mathbf{D}_{\mathbf{U}} \\ \quad + (i\tilde{n} p_n, \partial_y p_n)^T - \mu \varepsilon \Delta_n \widehat{\mathbf{U}}_n = \mathbf{R}_{\mathbf{U},n}, \\ -i\tilde{n} \widehat{\mathbf{U}}_n - 2\kappa \varepsilon^{\frac{1}{2}} b_p \partial_y \widehat{\mathbf{H}}_n + \mathbf{C}_{\mathbf{H}} \widehat{\mathbf{H}}_n + \varepsilon^{-\frac{1}{2}} \hat{\psi}_n \mathbf{D}_{\mathbf{H}} - \kappa \varepsilon \Delta_n \widehat{\mathbf{H}}_n = \mathbf{R}_{\mathbf{H},n}, \\ i\tilde{n} \hat{u}_n + \partial_y \hat{v}_n = i\tilde{n} \hat{h}_n + \partial_y \hat{g}_n = 0, \quad \widehat{\mathbf{U}}_n|_{y=0} = (\partial_y \hat{h}_n, \hat{g}_n)|_{y=0} = \mathbf{0}, \end{cases}$$

Observations:

- $\widehat{\mathbf{W}}_n \sim \mathbf{W}_n$
- $\mathbf{A}_{\mathbf{U}}, \mathbf{B}_{\mathbf{U}}, \mathbf{C}_{\mathbf{U}}, \mathbf{D}_{\mathbf{U}}, \mathbf{C}_{\mathbf{H}}, \mathbf{D}_{\mathbf{H}} \sim O(1)$, and all elements in these matrices or vectors involve Y -derivative of boundary layer profiles.

The system of $\widehat{\mathbf{W}}_n = (\hat{u}_n, \hat{v}_n, \hat{h}_n, \hat{g}_n)$ reads

$$\begin{cases} i\tilde{n} \left[(1 + \frac{\mu}{\kappa}) U_s \widehat{\mathbf{U}}_n - \textcolor{red}{G_s} \widehat{\mathbf{H}}_n + \varepsilon^{\frac{1}{2}} \mathbf{A}_{\mathbf{U}} \widehat{\mathbf{H}}_n \right] + \varepsilon^{\frac{1}{2}} \mathbf{B}_{\mathbf{U}} \partial_y \widehat{\mathbf{H}}_n + \mathbf{C}_{\mathbf{U}} \widehat{\mathbf{H}}_n + \varepsilon^{-\frac{1}{2}} \hat{\psi}_n \mathbf{D}_{\mathbf{U}} \\ \quad + (i\tilde{n} p_n, \partial_y p_n)^T - \mu \varepsilon \Delta_n \widehat{\mathbf{U}}_n = \mathbf{R}_{\mathbf{U},n}, \\ -i\tilde{n} \widehat{\mathbf{U}}_n - 2\kappa \varepsilon^{\frac{1}{2}} b_p \partial_y \widehat{\mathbf{H}}_n + \mathbf{C}_{\mathbf{H}} \widehat{\mathbf{H}}_n + \varepsilon^{-\frac{1}{2}} \hat{\psi}_n \mathbf{D}_{\mathbf{H}} - \kappa \varepsilon \Delta_n \widehat{\mathbf{H}}_n = \mathbf{R}_{\mathbf{H},n}, \\ i\tilde{n} \hat{u}_n + \partial_y \hat{v}_n = i\tilde{n} \hat{h}_n + \partial_y \hat{g}_n = 0, \quad \widehat{\mathbf{U}}_n|_{y=0} = (\partial_y \hat{h}_n, \hat{g}_n)|_{y=0} = \mathbf{0}, \end{cases}$$

Observations:

- $\widehat{\mathbf{W}}_n \sim \mathbf{W}_n$
- $\mathbf{A}_{\mathbf{U}}, \mathbf{B}_{\mathbf{U}}, \mathbf{C}_{\mathbf{U}}, \mathbf{D}_{\mathbf{U}}, \mathbf{C}_{\mathbf{H}}, \mathbf{D}_{\mathbf{H}} \sim O(1)$, and all elements in these matrices or vectors involve Y -derivative of boundary layer profiles.

Step 1: Bound on derivatives

$$\sqrt{\varepsilon} \left(\|\partial_y \widehat{\mathbf{W}}_n\|_{L^2} + |\tilde{n}| \|\widehat{\mathbf{W}}_n\|_{L^2} \right) \lesssim \bar{M}^{\frac{1}{2}} (1 + \bar{M}^{\frac{1}{2}}) \|\widehat{\mathbf{W}}_n\|_{L^2} + \dots$$

The system of $\widehat{\mathbf{W}}_n = (\hat{u}_n, \hat{v}_n, \hat{h}_n, \hat{g}_n)$ reads

$$\begin{cases} i\tilde{n} \left[(1 + \frac{\mu}{\kappa}) U_s \widehat{\mathbf{U}}_n - \mathbf{G}_s \widehat{\mathbf{H}}_n + \varepsilon^{\frac{1}{2}} \mathbf{A}_{\mathbf{U}} \widehat{\mathbf{H}}_n \right] + \varepsilon^{\frac{1}{2}} \mathbf{B}_{\mathbf{U}} \partial_y \widehat{\mathbf{H}}_n + \mathbf{C}_{\mathbf{U}} \widehat{\mathbf{H}}_n + \varepsilon^{-\frac{1}{2}} \hat{\psi}_n \mathbf{D}_{\mathbf{U}} \\ \quad + (i\tilde{n} p_n, \partial_y p_n)^T - \mu \varepsilon \Delta_n \widehat{\mathbf{U}}_n = \mathbf{R}_{\mathbf{U},n}, \\ -i\tilde{n} \widehat{\mathbf{U}}_n - 2\kappa \varepsilon^{\frac{1}{2}} b_p \partial_y \widehat{\mathbf{H}}_n + \mathbf{C}_{\mathbf{H}} \widehat{\mathbf{H}}_n + \varepsilon^{-\frac{1}{2}} \hat{\psi}_n \mathbf{D}_{\mathbf{H}} - \kappa \varepsilon \Delta_n \widehat{\mathbf{H}}_n = \mathbf{R}_{\mathbf{H},n}, \\ i\tilde{n} \hat{u}_n + \partial_y \hat{v}_n = i\tilde{n} \hat{h}_n + \partial_y \hat{g}_n = 0, \quad \widehat{\mathbf{U}}_n|_{y=0} = (\partial_y \hat{h}_n, \hat{g}_n)|_{y=0} = \mathbf{0}, \end{cases}$$

Observations:

- $\widehat{\mathbf{W}}_n \sim \mathbf{W}_n$
- $\mathbf{A}_{\mathbf{U}}, \mathbf{B}_{\mathbf{U}}, \mathbf{C}_{\mathbf{U}}, \mathbf{D}_{\mathbf{U}}, \mathbf{C}_{\mathbf{H}}, \mathbf{D}_{\mathbf{H}} \sim O(1)$, and all elements in these matrices or vectors involve Y -derivative of boundary layer profiles.

Step 1: Bound on derivatives

$$\sqrt{\varepsilon} \left(\|\partial_y \widehat{\mathbf{W}}_n\|_{L^2} + |\tilde{n}| \|\widehat{\mathbf{W}}_n\|_{L^2} \right) \lesssim \bar{M}^{\frac{1}{2}} (1 + \bar{M}^{\frac{1}{2}}) \|\widehat{\mathbf{W}}_n\|_{L^2} + \dots$$

Q: How to obtain $\|\widehat{\mathbf{W}}_n\|_{L^2}$? Lack of Gronwall's inequality

L^2 -coercivity

Main part of linearized equation:

$$\begin{aligned} -i\tilde{\eta}G_s\left(\frac{y}{\sqrt{\varepsilon}}\right)\widehat{\mathbf{H}}_n + (i\tilde{\eta}p_n, \partial_y p_n) - \varepsilon\mu\Delta_n\widehat{\mathbf{U}}_n &= \cdots, \\ -i\tilde{\eta}\widehat{\mathbf{U}}_n - \varepsilon\kappa\Delta_n\widehat{\mathbf{H}}_n &= \cdots, \end{aligned}$$

where $G_s(Y) = H_s^2(Y) - U_s^2(Y) \geq \gamma_0 > 0$.

L^2 -coercivity

Main part of linearized equation:

$$\begin{aligned} -i\tilde{n}G_s\left(\frac{y}{\sqrt{\varepsilon}}\right)\widehat{\mathbf{H}}_n + (i\tilde{n}p_n, \partial_y p_n) - \varepsilon\mu\Delta_n\widehat{\mathbf{U}}_n &= \cdots, \\ -i\tilde{n}\widehat{\mathbf{U}}_n - \varepsilon\kappa\Delta_n\widehat{\mathbf{H}}_n &= \cdots, \end{aligned}$$

where $G_s(Y) = H_s^2(Y) - U_s^2(Y) \geq \gamma_0 > 0$.

Step 2: Uniform-in- ε estimate on velocity

Notice that $\hat{u}_n\partial_y\hat{h}_n|_{y=0} = 0$, multiplying the second equation by $\widehat{\mathbf{U}}_n$ gives

$$|\tilde{n}|^{\frac{1}{2}}\|\widehat{\mathbf{U}}_n\|_{L^2} \lesssim \bar{M}^{\frac{1}{2}}(1 + \bar{M}^{\frac{1}{2}})\|[\widehat{\mathbf{U}}_n, \widehat{\mathbf{H}}_n]\|_{L^2} + \cdots$$

Does NOT work for $\widehat{\mathbf{H}}_n$ due to boundary term $\varepsilon\hat{h}_n\partial_y\hat{u}_n|_{y=0}$.

Weighted L^2 -estimate

$$\begin{aligned}\hat{\phi} : \hat{\phi}_y &= \hat{u}, \quad -\hat{\phi}_x = \hat{v}, \quad \hat{\phi}|_{y=0} = 0, \quad \omega_u = \Delta \hat{\phi}, \\ \hat{\psi} : \hat{\phi}_y &= \hat{h}, \quad -\hat{\phi}_x = \hat{g}, \quad \hat{\psi}|_{y=0} = 0, \quad \omega_h = \Delta \hat{\psi},\end{aligned}$$

Vorticity formulation

$$-i\tilde{\eta} \operatorname{curl}(G_s \widehat{\mathbf{H}}_n) - \varepsilon\mu \Delta_n \omega_{u,n} = \operatorname{curl} \widetilde{\mathbf{R}}_{\mathbf{U},n},$$

$$-i\tilde{\eta} \omega_{u,n} - \varepsilon\kappa \Delta_n \omega_{h,n} = \operatorname{curl} \widetilde{\mathbf{R}}_{\mathbf{H},n},$$

Need to construct a suitable weight function $Z(y)$ compatible with the vorticity equation.

The Weight function $Z(y)$

We construct a C^1 -function $\tilde{G}(y)$, $y \in \mathbb{R}_+$ satisfying

$$\tilde{G}(y) := \begin{cases} \frac{1}{G_s(y/\sqrt{\varepsilon})}, & 0 \leq y \leq 1, \\ 0, & y \geq 2, \end{cases}$$

and

$$\frac{1}{2\bar{\gamma}^2} \leq \tilde{G}(y) \leq \frac{2}{\gamma_0}, \quad \left| \tilde{G}'(y) \right| \lesssim \bar{M}\varepsilon, \text{ for } y \in \left[1, \frac{3}{2}\right]; \quad \tilde{G}'(y) \leq 0, \text{ for } y \in \left[\frac{3}{2}, 2\right].$$

It is not difficult to know such function $\tilde{G}(y)$ exists due to the fact

$$\left. \left(\frac{1}{G_s(y/\sqrt{\varepsilon})} \right)' \right|_{y=1} = \varepsilon \cdot \left. \left(-\frac{Y^3 G'_s(Y)}{G_s^2(Y)} \right) \right|_{Y=\frac{1}{\sqrt{\varepsilon}}} \lesssim \bar{M}\varepsilon.$$

The weight function $Z(y)$ is

$$Z(y) := \int_0^y \tilde{G}(y') dy'.$$

Properties of $Z(y)$

- $C_0^{-1}y \leq Z(y) \leq C_0y$ for $y \in [0, 2]$, $Z(y) \equiv \int_0^2 \tilde{G}(y')dy' \triangleq \bar{Z}$ for $y \geq 2$.
- $G_s(\frac{y}{\sqrt{\varepsilon}})Z'(y) \equiv 1$ for $y \in [0, 1]$, $|y^k Z''(y)| \leq C_0 \bar{M} \varepsilon^{\frac{k-1}{2}}$ for $y \in [0, \frac{3}{2}]$,
- $-\left(G_s(\frac{y}{\sqrt{\varepsilon}})Z'(y)\right)' \geq -C_0 \bar{M} \varepsilon$ for $y \geq 1$; $Z''(y) \leq 0$, for $y \geq \frac{3}{2}$.

Properties of $Z(y)$

- $C_0^{-1}y \leq Z(y) \leq C_0y$ for $y \in [0, 2]$, $Z(y) \equiv \int_0^2 \tilde{G}(y')dy' \triangleq \bar{Z}$ for $y \geq 2$.
- $G_s(\frac{y}{\sqrt{\varepsilon}})Z'(y) \equiv 1$ for $y \in [0, 1]$, $|y^k Z''(y)| \leq C_0 \bar{M} \varepsilon^{\frac{k-1}{2}}$ for $y \in [0, \frac{3}{2}]$,
- $-\left(G_s(\frac{y}{\sqrt{\varepsilon}})Z'(y)\right)' \geq -C_0 \bar{M} \varepsilon$ for $y \geq 1$; $Z''(y) \leq 0$, for $y \geq \frac{3}{2}$.

Step 3: Weighted L^2 -estimate:

$$|\tilde{n}|^{\frac{1}{2}} \|Z^{\frac{1}{2}} \widehat{\mathbf{W}}_n\|_{L^2} \lesssim \varepsilon^{\frac{1}{4}} \bar{M}^{\frac{1}{4}} (1 + \bar{M}^{\frac{5}{4}}) \|\widehat{\mathbf{W}}_n\|_{L^2} + |\tilde{n}|^{-\frac{1}{2}} |\log \varepsilon|^{1+} \|Z^{\frac{1}{2}} \mathbf{R}_n\|_{L^2} + \dots$$

Properties of $Z(y)$

- $C_0^{-1}y \leq Z(y) \leq C_0 y$ for $y \in [0, 2]$, $Z(y) \equiv \int_0^2 \tilde{G}(y') dy' \triangleq \bar{Z}$ for $y \geq 2$.
- $G_s(\frac{y}{\sqrt{\varepsilon}})Z'(y) \equiv 1$ for $y \in [0, 1]$, $|y^k Z''(y)| \leq C_0 \bar{M} \varepsilon^{\frac{k-1}{2}}$ for $y \in [0, \frac{3}{2}]$,
- $-\left(G_s(\frac{y}{\sqrt{\varepsilon}})Z'(y)\right)' \geq -C_0 \bar{M} \varepsilon$ for $y \geq 1$; $Z''(y) \leq 0$, for $y \geq \frac{3}{2}$.

Step 3: Weighted L^2 -estimate:

$$|\tilde{n}|^{\frac{1}{2}} \|Z^{\frac{1}{2}} \widehat{\mathbf{W}}_n\|_{L^2} \lesssim \varepsilon^{\frac{1}{4}} \bar{M}^{\frac{1}{4}} (1 + \bar{M}^{\frac{5}{4}}) \|\widehat{\mathbf{W}}_n\|_{L^2} + |\tilde{n}|^{-\frac{1}{2}} |\log \varepsilon|^{1+} \|Z^{\frac{1}{2}} \mathbf{R}_n\|_{L^2} + \dots$$

Estimate of $\|\widehat{\mathbf{H}}_n\|_{L^2}$ can be obtained by a standard interpolation:

$$\tilde{n}^{\frac{1}{3}} \|\widehat{\mathbf{H}}_n\|_{L^2} \lesssim C \left(\tilde{n}^{\frac{1}{2}} \varepsilon^{-\frac{1}{4}} \|Z^{\frac{1}{2}} \widehat{\mathbf{H}}_n\|_{L^2} \right)^{\frac{2}{3}} \left(\varepsilon^{\frac{1}{2}} \|\partial_y \widehat{\mathbf{H}}_n\|_{L^2} \right)^{\frac{1}{3}} + \|Z^{\frac{1}{2}} \widehat{\mathbf{H}}_n\|_{L^2}$$

Advantage of the chosen weight:

$$\begin{aligned} & \operatorname{Im} \langle -i\tilde{n} \operatorname{curl}(G_s \hat{H}_n), Z \hat{\psi}_n \rangle \\ &= \int_0^\infty \tilde{n} G_s(Y) Z(y) |\hat{\mathbf{H}}_n|^2 dy + \tilde{n} \operatorname{Re} \int_0^\infty \partial_y Z G_s(Y) \partial_y \hat{\psi}_n \overline{\hat{\psi}_n} dy \\ &= \underbrace{\tilde{n} \int_0^\infty G_s(Y) Z(y) |\hat{\mathbf{H}}_n|^2 dy}_{\mathcal{J}_1} + \underbrace{\frac{\tilde{n}}{2} \int_0^\infty \frac{-\partial_y [G_s(Y) \partial_y Z]}{2} |\hat{\psi}_n|^2 dy}_{\mathcal{J}_2}. \end{aligned}$$

with $\mathcal{J}_1 \sim \tilde{n} \|Z^{\frac{1}{2}} \hat{\mathbf{H}}_n\|_{L^2}^2$ and

$$\mathcal{J}_2 \geq -C_0 \bar{M} \varepsilon |\tilde{n}| \|\hat{h}_n\|_{L^2}^2 \geq -C_0 \bar{M} \varepsilon \|\widehat{\partial_x \mathbf{H}_n}\|_{L^2} \|\hat{\mathbf{H}}_n\|_{L^2}$$

Advantage of the chosen weight:

$$\begin{aligned}
 & \operatorname{Im} \langle -i\tilde{n} \operatorname{curl}(G_s \hat{H}_n), Z \hat{\psi}_n \rangle \\
 &= \int_0^\infty \tilde{n} G_s(Y) Z(y) |\hat{\mathbf{H}}_n|^2 dy + \tilde{n} \operatorname{Re} \int_0^\infty \partial_y Z G_s(Y) \partial_y \hat{\psi}_n \overline{\hat{\psi}_n} dy \\
 &= \underbrace{\tilde{n} \int_0^\infty G_s(Y) Z(y) |\hat{\mathbf{H}}_n|^2 dy}_{\mathcal{J}_1} + \underbrace{\frac{\tilde{n}}{2} \int_0^\infty \frac{-\partial_y [G_s(Y) \partial_y Z]}{2} |\hat{\psi}_n|^2 dy}_{\mathcal{J}_2}.
 \end{aligned}$$

with $\mathcal{J}_1 \sim \tilde{n} \|Z^{\frac{1}{2}} \hat{\mathbf{H}}_n\|_{L^2}^2$ and

$$\mathcal{J}_2 \geq -C_0 \bar{M} \varepsilon |\tilde{n}| \|\hat{h}_n\|_{L^2}^2 \geq -C_0 \bar{M} \varepsilon \|\widehat{\partial_x \mathbf{H}}_n\|_{L^2} \|\hat{\mathbf{H}}_n\|_{L^2}$$

The weight Z is NOT compatible with the magnetic equation:

$$\begin{aligned}
 \operatorname{Im} \langle -i\tilde{n} \omega_{u,n}, Z \hat{\phi}_n \rangle &= \tilde{n} \|Z^{\frac{1}{2}} \widehat{\mathbf{U}}_n\|_{L^2}^2 + \frac{\tilde{n}}{2} \int_0^\infty -y^2 \partial_y^2 Z |y^{-1} \hat{\phi}_n|^2 dy \\
 &\geq \tilde{n} \|Z^{\frac{1}{2}} \widehat{\mathbf{U}}_n\|_{L^2}^2 - \underbrace{\tilde{n} \sqrt{\varepsilon} \|\widehat{\mathbf{U}}_n\|_{L^2}^2}_{\text{obtained in Step 2!}}
 \end{aligned}$$

Gain of $\varepsilon^{\frac{1}{4}}$:

- When Z hitting on the terms involving derivatives of boundary layer,

eg.

$$ZH'_s\left(\frac{y}{\sqrt{\varepsilon}}\right) = \sqrt{\varepsilon} \frac{y}{\sqrt{\varepsilon}} H'_s\left(\frac{y}{\sqrt{\varepsilon}}\right) \frac{Z}{y} \sim O(\sqrt{\varepsilon})$$
$$\Rightarrow \langle H'_s\left(\frac{y}{\sqrt{\varepsilon}}\right) \hat{\mathbf{H}}_n, Z \hat{\mathbf{H}}_n \rangle \lesssim \varepsilon^{\frac{1}{2}} \|\hat{\mathbf{H}}_n\|_{L^2}^2.$$

- Terms involving commutator $[\text{curl}, Z] \sim 1$ near the boundary:

$$|\langle R, [\text{curl}, Z] \partial_y^{-1} \hat{h}_n \rangle| \lesssim |\langle \sqrt{y} R, [\text{curl}, Z] \frac{\partial_y^{-1} \hat{h}_n}{\sqrt{y}} \rangle|$$

Gain of $\varepsilon^{\frac{1}{4}}$:

- When Z hitting on the terms involving derivatives of boundary layer,

eg. $ZH'_s\left(\frac{y}{\sqrt{\varepsilon}}\right) = \sqrt{\varepsilon} \frac{y}{\sqrt{\varepsilon}} H'_s\left(\frac{y}{\sqrt{\varepsilon}}\right) \frac{Z}{y} \sim O(\sqrt{\varepsilon})$
 $\Rightarrow \langle H'_s\left(\frac{y}{\sqrt{\varepsilon}}\right) \hat{\mathbf{H}}_n, Z \hat{\mathbf{H}}_n \rangle \lesssim \varepsilon^{\frac{1}{2}} \|\hat{\mathbf{H}}_n\|_{L^2}^2.$

- Terms involving commutator $[\text{curl}, Z] \sim 1$ near the boundary:

$$|\langle R, [\text{curl}, Z] \partial_y^{-1} \hat{h}_n \rangle| \lesssim |\langle \sqrt{y} R, [\text{curl}, Z] \frac{\partial_y^{-1} \hat{h}_n}{\sqrt{y}} \rangle|$$

Critical Hardy : $\left\| [\text{curl}, Z] \frac{\partial_y^{-1} \hat{h}_n}{\sqrt{y}} \right\|_{L^2} \lesssim \|Z^{\frac{1}{2}} |\log Z|^{1+} \hat{h}_n\|_{L^2}.$

$$\Rightarrow |\langle R, [\text{curl}, Z] \partial_y^{-1} \hat{h}_n \rangle| \lesssim \|Z^{\frac{1}{2}} R\|_{L^2} \|Z^{\frac{1}{2}} |\log Z|^{1+} \hat{h}_n\|_{L^2}$$

Gain of $\varepsilon^{\frac{1}{4}}$:

- When Z hitting on the terms involving derivatives of boundary layer,

eg. $ZH'_s\left(\frac{y}{\sqrt{\varepsilon}}\right) = \sqrt{\varepsilon} \frac{y}{\sqrt{\varepsilon}} H'_s\left(\frac{y}{\sqrt{\varepsilon}}\right) \frac{Z}{y} \sim O(\sqrt{\varepsilon})$
 $\Rightarrow \langle H'_s\left(\frac{y}{\sqrt{\varepsilon}}\right) \hat{\mathbf{H}}_n, Z \hat{\mathbf{H}}_n \rangle \lesssim \varepsilon^{\frac{1}{2}} \|\hat{\mathbf{H}}_n\|_{L^2}^2.$

- Terms involving commutator $[\text{curl}, Z] \sim 1$ near the boundary:

$$|\langle R, [\text{curl}, Z] \partial_y^{-1} \hat{h}_n \rangle| \lesssim |\langle \sqrt{y} R, [\text{curl}, Z] \frac{\partial_y^{-1} \hat{h}_n}{\sqrt{y}} \rangle|$$

Critical Hardy : $\left\| [\text{curl}, Z] \frac{\partial_y^{-1} \hat{h}_n}{\sqrt{y}} \right\|_{L^2} \lesssim \|Z^{\frac{1}{2}} |\log Z|^{1+} \hat{h}_n\|_{L^2}.$

$$\Rightarrow |\langle R, [\text{curl}, Z] \partial_y^{-1} \hat{h}_n \rangle| \lesssim \|Z^{\frac{1}{2}} R\|_{L^2} \|Z^{\frac{1}{2}} |\log Z|^{1+} \hat{h}_n\|_{L^2}$$
$$\|Z^{\frac{1}{2}} |\log Z|^{1+} \hat{h}_n\|_{L^2} \lesssim |\log \varepsilon|^{1+} \left(\|Z^{\frac{1}{2}} h\|_{L^2} + \varepsilon^{\frac{1}{4}} \|h\|_{L^2} \right)$$

Synthesis: since $\tilde{n} = \frac{n}{\varrho}$ with $n \neq 0$,

$$\begin{aligned}\|\widehat{\mathbf{W}}_n\|_{L^2} &\leq \left[\frac{1}{4} + O(1) \underbrace{|\tilde{n}|^{-\frac{1}{3}} \bar{M}^{\frac{1}{3}} (1 + \bar{M})}_{\leq \varrho^{\frac{1}{3}} \bar{M}^{\frac{1}{3}} (1 + \bar{M}) \ll 1} + O(\varepsilon^{\frac{1}{4}}) \right] \|\widehat{\mathbf{W}}_n\|_{L^2} \\ &\quad + C |\tilde{n}|^{-\frac{2}{3}} (1 + |\tilde{n}|^{-\frac{4}{3}}) |\log \varepsilon|^{1+} \left(\varepsilon^{-\frac{1}{4}} \|Z^{\frac{1}{2}} \mathbf{R}_n\|_{L^2} + \|\mathbf{R}_n\|_{L^2} \right),\end{aligned}$$

Synthesis: since $\tilde{n} = \frac{n}{\varrho}$ with $n \neq 0$,

$$\begin{aligned} \|\widehat{\mathbf{W}}_n\|_{L^2} &\leq \left[\frac{1}{4} + O(1) \underbrace{|\tilde{n}|^{-\frac{1}{3}} \bar{M}^{\frac{1}{3}} (1 + \bar{M})}_{\leq \varrho^{\frac{1}{3}} \bar{M}^{\frac{1}{3}} (1 + \bar{M}) \ll 1} + O(\varepsilon^{\frac{1}{4}}) \right] \|\widehat{\mathbf{W}}_n\|_{L^2} \\ &\quad + C |\tilde{n}|^{-\frac{2}{3}} (1 + |\tilde{n}|^{-\frac{4}{3}}) |\log \varepsilon|^{1+} \left(\varepsilon^{-\frac{1}{4}} \|Z^{\frac{1}{2}} \mathbf{R}_n\|_{L^2} + \|\mathbf{R}_n\|_{L^2} \right), \end{aligned}$$

$$\begin{aligned} \Rightarrow |\tilde{n}|^{\frac{2}{3}} \|\widehat{\mathbf{W}}_n\|_{L^2} &+ \varepsilon^{-\frac{1}{4}} |\tilde{n}|^{\frac{5}{6}} \|Z^{\frac{1}{2}} \widehat{\mathbf{W}}_n\|_{L^2} + \varepsilon^{\frac{1}{2}} |\tilde{n}|^{\frac{1}{3}} \left(\|\partial_y \widehat{\mathbf{W}}_n\|_{L^2} + |\tilde{n}| \|\widehat{\mathbf{W}}_n\|_{L^2} \right) \\ &+ \varepsilon^{\frac{1}{4}} |\tilde{n}|^{\frac{1}{3}} \left(\|Z^{\frac{1}{2}} \partial_y \widehat{\mathbf{W}}_n\|_{L^2} + |\tilde{n}| \|Z^{\frac{1}{2}} \widehat{\mathbf{W}}_n\|_{L^2} \right) \\ &\lesssim |\log \varepsilon|^{1+} \left(\|\mathbf{R}_n\|_{L^2} + \varepsilon^{-\frac{1}{4}} \|Z^{\frac{1}{2}} \mathbf{R}_n\|_{L^2} \right), \end{aligned}$$

Synthesis: since $\tilde{n} = \frac{n}{\varrho}$ with $n \neq 0$,

$$\begin{aligned}\|\widehat{\mathbf{W}}_n\|_{L^2} &\leq \left[\frac{1}{4} + O(1) \underbrace{|\tilde{n}|^{-\frac{1}{3}} \bar{M}^{\frac{1}{3}} (1 + \bar{M})}_{\leq \varrho^{\frac{1}{3}} \bar{M}^{\frac{1}{3}} (1 + \bar{M}) \ll 1} + O(\varepsilon^{\frac{1}{4}}) \right] \|\widehat{\mathbf{W}}_n\|_{L^2} \\ &\quad + C |\tilde{n}|^{-\frac{2}{3}} (1 + |\tilde{n}|^{-\frac{4}{3}}) |\log \varepsilon|^{1+} \left(\varepsilon^{-\frac{1}{4}} \|Z^{\frac{1}{2}} \mathbf{R}_n\|_{L^2} + \|\mathbf{R}_n\|_{L^2} \right),\end{aligned}$$

$$\begin{aligned}&\Rightarrow |\tilde{n}|^{\frac{2}{3}} \|\widehat{\mathbf{W}}_n\|_{L^2} + \varepsilon^{-\frac{1}{4}} |\tilde{n}|^{\frac{5}{6}} \|Z^{\frac{1}{2}} \widehat{\mathbf{W}}_n\|_{L^2} + \varepsilon^{\frac{1}{2}} |\tilde{n}|^{\frac{1}{3}} \left(\|\partial_y \widehat{\mathbf{W}}_n\|_{L^2} + |\tilde{n}| \|\widehat{\mathbf{W}}_n\|_{L^2} \right) \\ &\quad + \varepsilon^{\frac{1}{4}} |\tilde{n}|^{\frac{1}{3}} \left(\|Z^{\frac{1}{2}} \partial_y \widehat{\mathbf{W}}_n\|_{L^2} + |\tilde{n}| \|Z^{\frac{1}{2}} \widehat{\mathbf{W}}_n\|_{L^2} \right) \\ &\lesssim |\log \varepsilon|^{1+} \left(\|\mathbf{R}_n\|_{L^2} + \varepsilon^{-\frac{1}{4}} \|Z^{\frac{1}{2}} \mathbf{R}_n\|_{L^2} \right),\end{aligned}$$

L^∞ -estimate:

$$\begin{aligned}\|\widehat{\mathbf{W}}_n\|_{L^\infty} &\lesssim |\tilde{n}|^{-\frac{1}{2}} \varepsilon^{-\frac{1}{4}} \left(|\tilde{n}|^{\frac{1}{3}} \varepsilon^{\frac{1}{2}} \|\partial_y \widehat{\mathbf{W}}_n\|_{L^2} \right)^{\frac{1}{2}} \left(|\tilde{n}|^{\frac{2}{3}} \|\widehat{\mathbf{W}}_n\|_{L^2} \right)^{\frac{1}{2}} \\ &\Rightarrow \sum_{|n| \leq \varepsilon^{-1}} \|\widehat{\mathbf{W}}_n\|_{L^\infty} \lesssim \varepsilon^{-\frac{1}{4}} |\log \varepsilon|^{\frac{3}{2}+} \left(\|\mathbf{R}\|_{L^2(\Omega)} + \varepsilon^{-\frac{1}{4}} \|Z^{\frac{1}{2}} \mathbf{R}\|_{L^2(\Omega)} \right);\end{aligned}$$

For $|n| \geq \varepsilon^{-1}$,

$$\begin{aligned}
\|\widehat{\mathbf{W}}_n\|_{L^\infty} &\lesssim |\tilde{n}|^{-\frac{5}{6}} \varepsilon^{-\frac{1}{2}} \left(|\tilde{n}|^{\frac{1}{3}} \varepsilon^{\frac{1}{2}} \|\partial_y \widehat{\mathbf{W}}_n\|_{L^2} \right)^{\frac{1}{2}} \left(\varepsilon^{\frac{1}{2}} |\tilde{n}| \|\widehat{\mathbf{W}}_n\|_{L^2} \right)^{\frac{1}{2}} \\
&\lesssim \varepsilon^{-\frac{1}{4}} \underbrace{|n|^{-\frac{7}{12}}}_{\in L^2} |\log \varepsilon|^{\frac{3}{2}+} \left(\|\mathbf{R}_n\|_{L^2} + \varepsilon^{-\frac{1}{4}} \|Z^{\frac{1}{2}} \mathbf{R}_n\|_{L^2} \right); \\
\Rightarrow \sum_{|n| \geq \varepsilon^{-1}} \|\widehat{\mathbf{W}}_n\|_{L^\infty} &\lesssim \varepsilon^{-\frac{1}{4}} |\log \varepsilon|^{\frac{3}{2}+} \left(\|\mathbf{R}\|_{L^2(\Omega)} + \varepsilon^{-\frac{1}{4}} \|Z^{\frac{1}{2}} \mathbf{R}\|_{L^2(\Omega)} \right) \\
\Rightarrow \|\mathbf{W}\|_{\mathcal{X}} &\leq C \varepsilon^{-\frac{1}{4}} |\log \varepsilon|^{\frac{3}{2}+} \left[\|\mathcal{Q}_0(\mathbf{f}, \mathbf{q})\|_{L^2(\Omega)} + \varepsilon^{-\frac{1}{4}} \|Z^{\frac{1}{2}} \mathcal{Q}_0(\mathbf{f}, \mathbf{q})\|_{L^2(\Omega)} \right].
\end{aligned}$$

THANK YOU!