Sharp Interface Limits of some Diffused-interface Models

第七届偏微分方程青年学术论坛

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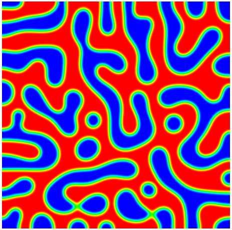
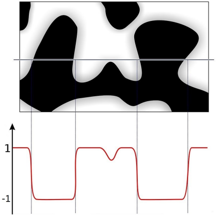
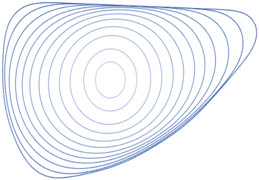
Introduction

The **phase field models** are widely adopted in the description of the evolution of **interfaces** in continuum mechanics. They can be constructed to purposely reproduce a given **sharp interface model** when the thickness of their diffused interface, usually denoted by ε, trends to 0.







Ginzburg-Landau equation

The scalar Ginzburg-Landau equation under diffusive scaling $(x,t) \to (\varepsilon x, \varepsilon^2 t)$:

$$\partial_t c_\varepsilon = \Delta c_\varepsilon - \varepsilon^{-2} W'(c_\varepsilon). \tag{Ginzburg–Landau}$$

is the gradient flow of $\mathcal{E}_\varepsilon(c) = \varepsilon|\nabla c|^2 + \frac{1}{\varepsilon}W(c)$, where $W(c) = (c^2-1)^2$. The energy dissipation $\frac{d}{dt}\int_\Omega \mathcal{E}_\varepsilon(c_\varepsilon)dx = -\int_\Omega \varepsilon|\partial_t c_\varepsilon|^2$.

- Modica-Mortola '77 (static case): $\int_\Omega \mathcal{E}_\varepsilon(c_\varepsilon)\,dx \xrightarrow{Gamma} \mathcal{H}^{N-1}(\Gamma)$.
- De Mottoni & Schatzman '95 (<u>local</u>, asymptotic expansion),
- Evans, Soner, Souganidis '92 (<u>global</u>, convergence to viscosity sol.),
- Ilmanen '93, Chen '96, Röger-Schätzle '06 (<u>global</u>, convergence to Brakke flow, monotonicity formula, rectifiable varifold),

$$\frac{d}{dt}\int_{\Gamma_t} \phi\, d\mathcal{H}^{N-1} \leqslant \int_{\Gamma_t} \partial_t\phi + \int_{\Gamma_t} (\nabla\phi\cdot\nu H - \phi H^2)\, d\mathcal{H}^{N-1}, \forall \phi(t,x) \geqslant 0$$

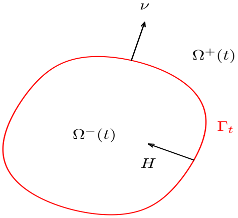
This makes sense for a Radom measure μ_t, generalizing $\mathcal{H}^{N-1}(\Gamma_t)$.

Barrier formulation

Consider the MMC $\{\Gamma_t\}_{t>0}$ parametrized by $X = X(s,t)$ with s being the local coordinate: $\partial_t X(s,t) = -H(X(s,t),t)\nu(X(s,t),t)$. De Giorgi proposed to work with the **signed distance function** $r = d_\Gamma(x,t)$ (negative inside). Differentiating the identity $d_\Gamma(X(s,t) + r\nu(s,t),t) \equiv r$ leads to

$$\nabla d_\Gamma = \nu, \quad \partial_t d_\Gamma = -\partial_t X(s,t) \cdot \nu.$$

Let $\pi(x)$ be the projection of x on Γ_t and $\{\kappa_i\}_{1\leqslant i\leqslant N-1}$ are the principal curvatures,



$$\Delta d_\Gamma(x) = \sum_{i=1}^{N-1} \frac{\kappa_i(\pi(x))}{1+\kappa_i(\pi(x))d_\Gamma} = H - d_\Gamma \sum \kappa_i^2 + o(d_\Gamma).$$

So $(\partial_t - \Delta)d_\Gamma = d_\Gamma \sum_i \kappa_i^2 + o(d_\Gamma)$, called barrier formulation.

Asymptotical Analysis

Inner solution is the expansion of c_ε near the interface in a stretched variable $z = d_\Gamma(x,t)/\varepsilon$, which is introduced to relax the sharp transition of c_ε near the interface. Outer solution determines the boundary condition of the inner solution at $z = \pm\infty$. We use the Ansatz

$$c_A(x,t) = c_0(\tfrac{d_\Gamma}{\varepsilon}, x, t) + \varepsilon c_1 + \cdots \text{ near } \Gamma_t \qquad \text{(inner solution)}$$

and look for c_A solving Ginzburg-Landau equations up to a tail:

$$\underbrace{\frac{\partial_t d_\Gamma}{\varepsilon}\partial_z c_0 + \partial_t c_0}_{=\partial_t c_A} \approx \underbrace{\frac{\Delta d_\Gamma}{\varepsilon}\partial_z c_0 + \frac{1}{\varepsilon^2}\partial_z^2 c_0 + \Delta c_0 + \cdots - \frac{1}{\varepsilon^2}W'(c_0)}_{=\Delta c_A - \frac{1}{\varepsilon^2}W'(c_A)}$$

- $O(\varepsilon^{-2})$: we choose $c_0 = \theta(\frac{d_\Gamma}{\varepsilon})$, the optimal profile:

$$\left.\begin{aligned} -\theta''(z) + W'(\theta(z)) &= 0, \forall z \in \mathbb{R}, \\ \theta(0) = 0, \theta(\pm\infty) &= \pm 1. \end{aligned}\right\} \Rightarrow \theta(z) = \tanh(\frac{z}{\sqrt{2}})$$

- $O(\varepsilon^{-1})$: $(\partial_t - \Delta)d_\Gamma = 0$ on Γ_t leads to MMC.

$$(\partial_t - \Delta)c_0 + \frac{1}{\varepsilon^2}W'(c_0) = \frac{\partial_t d_\Gamma - \Delta d_\Gamma}{d_\Gamma}\frac{d_\Gamma}{\varepsilon}\theta'(\frac{d_\Gamma}{\varepsilon}) \approx \sqrt{\varepsilon} \quad \text{in} \quad L^2(\Gamma_t \times (-\delta,\delta))$$

Rigorous Sharp Interface Limit

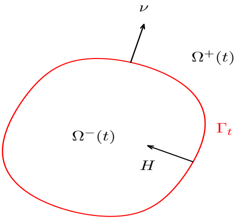
The first results on the convergence of level-set to MMC was due to

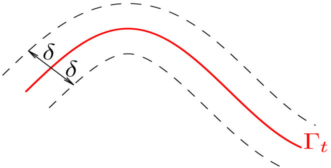
De Mottoni & Schatzman '95

Assume Γ_t evolves by MMC and is smooth on $[0,T]$. Assume $\{c_\varepsilon^0\}_{0<\varepsilon\leqslant 1}$, are well-prepared initial data, i.e. $c_\varepsilon^0(x) = \pm 1$ away from the initial surface Γ_0 and $c_\varepsilon^0(x) = \theta(d_{\Gamma_0}(x)/\varepsilon)$ near it. Then there exists an approximate solution c_A of (Ginzburg–Landau), up to a tail s.t

$$\sup_{0\leqslant t\leqslant T} \|c_\varepsilon(x,t) - c_A(x,t)\|_{L^2(\Omega)} \xrightarrow{\varepsilon\to 0} 0, \text{ where}$$

$$c_A(x,t) = \theta\left(\frac{d_\Gamma(x,t)}{\varepsilon} + O(1)\right) + O(\varepsilon^2) \qquad \text{in } L^\infty((0,T)\times\Omega).$$





Sketched Proof

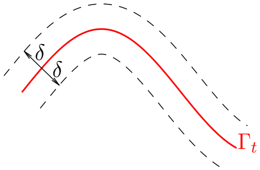
Using the smooth solution of MMC to construct an approximate solution c_A of (Ginzburg–Landau) such that

$$\partial_t c_A - \Delta c_A + \frac{1}{\varepsilon^2} W'(c_A) = r_A.$$

Here r_A can be made as small as possible provided the asymptotic expansion is sufficiently accurate. The difference $\bar{c} = c_\varepsilon - c_A$ satisfies

$$\partial_t \bar{c} - \Delta \bar{c} + \frac{1}{\varepsilon^2} W''(c_A)\bar{c} = \mathcal{N}(\bar{c}) - r_A,$$

where $\mathcal{N}(\bar{c})$ is nonlinear.



De Mottoni-Schatzman '95, Chen '94

There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $\psi \in H^1(\Gamma_t(\delta))$,

$$\int_{\Gamma_t(\delta)} \left(|\nabla \psi|^2 + \frac{1}{\varepsilon^2} W''(c_A)\psi^2 \right) dx \gtrsim - \int_{\Gamma_t(\delta)} \psi^2 dx$$

Spectrum inequality of linearized operator

De Mottoni & Schatzman '95, Chen '94

There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$,

$$\int_{\Gamma_t(\delta)} \left(|\nabla\psi|^2 + \tfrac{1}{\varepsilon^2} W''(c_A)\psi^2\right) dx \gtrsim -\int_{\Gamma_t(\delta)} \psi^2 dx, \quad \psi \in H^1(\Gamma_t(\delta)).$$

Since $c_A = \theta(\frac{d_\Gamma}{\varepsilon}) + O(\varepsilon^2)$, by Taylor's expansion, it suffices to show

$$\int_{\Gamma_t(\delta)} \left(|\nabla_\Gamma\psi|^2 + |\partial_{\mathbf{n}}\psi|^2 + \tfrac{1}{\varepsilon^2} W''(\theta)\psi^2\right) dx \geqslant -C \int_{\Gamma_t(\delta)} \psi^2 dx$$

As $W''(\theta) = \frac{\theta'''}{\theta'}$, by a cut-off near $\Gamma_t(\delta)$ and a change of variable $x \to (s, r) \to (s, \frac{r}{\varepsilon})$

$$\begin{aligned}
&\int_{\Gamma(\delta)} |\partial_{\mathbf{n}}\psi|^2 + \frac{W''(\theta)}{\varepsilon^2}\psi^2 dx \\
=& \int_\Gamma \int_{-\delta}^{\delta} \left(|\partial_r\psi(r,s)|^2 + \frac{W''(\theta)}{\varepsilon^2}\psi^2(r,s)\right) J(r,s) dr \\
=& \frac{1}{\varepsilon}\int_\Gamma \int_{-\delta/\varepsilon}^{\delta/\varepsilon} \left(|\partial_z\psi(\varepsilon z,s)|^2 + \frac{\theta'''}{\theta'}(z)\psi^2(\varepsilon z,s)\right) J(\varepsilon z,s) dz \qquad \text{with } \frac{r}{\varepsilon} = z \\
=& \frac{1}{\varepsilon}\int_\Gamma \int_{-\delta/\varepsilon}^{\delta/\varepsilon} \left(|\partial_z\psi(\varepsilon z,s)|^2 + \psi^2\left(\frac{\theta''}{\theta'}(z)\right)^2 - 2\partial_z\psi\psi\frac{\theta''}{\theta'}\right) J(\varepsilon z,s) dz + \cdots
\end{aligned}$$

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Introduction

Consider the Cahn-Hilliard type equation in a domain $\Omega \subset \mathbb{R}^3$:

$$\begin{cases} \varepsilon^3 \partial_t \phi_\varepsilon = \varepsilon^2 \Delta \mu_\varepsilon - W''(\phi_\varepsilon)\mu_\varepsilon, \\ \varepsilon \mu_\varepsilon = -\varepsilon^2 \Delta \phi_\varepsilon + W'(\phi_\varepsilon), \end{cases}$$

where $W(\phi) = (\phi^2 - 1)^2$. [1] It is the gradient flow of

$$\mathcal{E}(\phi) = \frac{1}{2\varepsilon} \int_\Omega \left(\varepsilon \Delta \phi - \frac{W'(\phi)}{\varepsilon} \right)^2 dx \approx \frac{1}{2} \int_\Gamma H^2 dS.$$

For well-prepared initial data, $\lim_{\varepsilon \to 0} (\mu_\varepsilon, \phi_\varepsilon) = (\mu_0, 1_{\Omega^+(t)} - 1_{\Omega^-(t)})$ where the closed surface $\Gamma_t = \partial\Omega^-(t)$ evolves by Willmore flow

$$V = \Delta_\Gamma H + H|A|^2 - H^3/2.$$

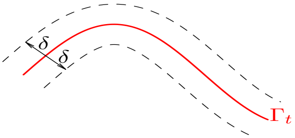
- $V = -\partial_t d_\Gamma \mid_{\Gamma_t}$: normal velocity of $\Gamma = \bigcup_{t \in [0,T]} \Gamma_t$.
- $A = \nabla \nu$: second fundamental form.

[1] The usual Cahn-Hilliard equation is $\partial_t \phi_\varepsilon = \Delta \mu_\varepsilon$.

Asymptotic Expansions

Use Ansatz $\phi_\varepsilon = \theta\left(\frac{d_\Gamma(x,t)}{\varepsilon}\right) + \cdots, \mu_\varepsilon = \cdots$ as inner expansions of

$$\begin{cases} \varepsilon^3 \partial_t \phi_\varepsilon = \varepsilon^2 \Delta \mu_\varepsilon - W''(\phi_\varepsilon)\mu_\varepsilon, \\ \varepsilon \mu_\varepsilon = -\varepsilon^2 \Delta \phi_\varepsilon + W'(\phi_\varepsilon). \end{cases}$$



- ε^0 : $\theta(z) = \tanh(\frac{z}{\sqrt{2}})$ is the optimal profile.
- ε^1 : $\mu_\varepsilon(x,t) = -\Delta d_\Gamma(x,t)\theta'(\frac{d_\Gamma(x,t)}{\varepsilon}) + \cdots$.
- ε^2 : d_Γ satisfies a quasilinear 4-th order equation

$$\partial_t d_\Gamma + \Delta^2 d_\Gamma = \Delta d_\Gamma D + \nabla d_\Gamma \cdot \nabla D \quad \text{on} \quad \Gamma_t,$$

where $D(x,t) = \nabla \Delta d_\Gamma \cdot \nabla d_\Gamma + \frac{1}{2}(\Delta d_\Gamma)^2$. This is the barrier formulation of Willmore flow

$$V = \Delta_\Gamma H + H|A|^2 - H^3/2.$$

Approximate solution and Spectrum inequality

Fei–L. '19 preprint

Let $\Gamma_t \subset \mathbb{R}^3$ evolve by the Willmore flow. Then there exists approximate solutions (ϕ_A, μ_A) such that $\phi_A = \theta(\frac{d_\Gamma(x,t)}{\varepsilon})$ near Γ_t and $\phi_A = \pm 1$ when away from Γ_t, and fulfill the C-H equation up to a tail:

$$\begin{cases} \varepsilon^3 \partial_t \phi_A = \varepsilon^2 \Delta \mu_A - W''(\phi_A)\mu_A + O(\varepsilon^{10}), \\ \varepsilon \mu_A = -\varepsilon^2 \Delta \phi_A + W'(\phi_A) + O(\varepsilon^{10}). \end{cases}$$

Moreover, if $\|\phi_\varepsilon(\cdot,0) - \phi_A(\cdot,0)\|_{L^2(\Omega)} \lesssim \varepsilon^{\frac{7}{2}}$, then

$$\|\phi_\varepsilon - \phi_A\|_{L^\infty(0,T_{\max};L^2(\Omega))} \lesssim \varepsilon^{\frac{7}{2}}.$$

The equation of $\phi = \phi_\varepsilon - \phi_A$ writes, up to some tails terms

$$\partial_t \phi = -\Big(\Delta - \varepsilon^{-2} W''(\phi_A)\Big)^2 \phi - \varepsilon^{-3} W'''(\phi_A)\mu_A\, \phi + \text{higher order terms.}$$

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Phase field model with fluid

Consider the coupled system with Stokes flow

$$
\begin{gathered}
-\Delta v_\varepsilon + \nabla p_\varepsilon = -\varepsilon \operatorname{div}(\nabla c_\varepsilon \otimes \nabla c_\varepsilon), \qquad \operatorname{div} v_\varepsilon = 0, \\
\partial_t c_\varepsilon + v_\varepsilon \cdot \nabla c_\varepsilon = \Delta c_\varepsilon - \tfrac{1}{\varepsilon^2} W'(c_\varepsilon), \\
v_\varepsilon|_{\partial\Omega} = 0, \qquad c_\varepsilon|_{\partial\Omega} = -1.
\end{gathered}
$$

Assume $(v_\varepsilon, c_\varepsilon) \to (v_0, 1_{\Omega^+(t)} - 1_{\Omega^-(t)})$ and let $z = \frac{d_\Gamma(x,t)}{\varepsilon}$ be the fast variable. We shall construct approximate solution with form

$$
c_A(x,t) = c_0(z,x,t) + \varepsilon c_1 + \cdots, \quad v_A(x,t) = v_0(z,x,t) + \varepsilon v_1 + \cdots.
$$

$$
\overbrace{\frac{\partial_t d_\Gamma}{\varepsilon}\partial_z c_0 + \partial_t c_0}^{=\partial_t c_A} + \overbrace{\left(\frac{\nu}{\varepsilon}\partial_z c_0 + \nabla c_0\right) \cdot v_0}^{=v_A \cdot \nabla c_A} \approx \overbrace{\frac{\Delta d_\Gamma}{\varepsilon}\partial_z c_0 + \frac{1}{\varepsilon^2}\partial_z^2 c_0 + \cdots - \frac{1}{\varepsilon^2} W'(c_0)}^{=\Delta c_A - \varepsilon^{-2} W'(c_A)}
$$

As before, the ε^{-1} order gives the MMC with convection

$$
\partial_t d_\Gamma + v_0 \cdot \nu = \Delta d_\Gamma.
$$

Expansion for Stokes system near the interface

Near the interface $c_\varepsilon \approx \theta(\frac{d_\Gamma}{\varepsilon})$. So $\nabla c_\varepsilon \approx \frac{\nu}{\varepsilon}\theta'(\frac{d_\Gamma}{\varepsilon})$. Tested by a solenoidal vector field $\varphi(x)$, the fluid equation writes

$$\begin{aligned}\int_\Omega \operatorname{div}(-2Dv_\varepsilon + p_\varepsilon \mathbf{I}) \cdot \varphi &= -\int_\Omega \varepsilon \operatorname{div}(\nabla c_\varepsilon \otimes \nabla c_\varepsilon) \cdot \varphi \\ &= \int_\Omega \varepsilon (\nabla c_\varepsilon \otimes \nabla c_\varepsilon) : \nabla \varphi \\ &\approx \int_{\Gamma_t(\delta)} \frac{\theta'(\frac{d_\Gamma}{\varepsilon})^2}{\varepsilon} (\nu \otimes \nu - I_d) : \nabla \varphi \xrightarrow{\varepsilon \to 0} -\sigma \int_{\Gamma_t} \operatorname{div}_{\Gamma_t} \varphi\end{aligned}$$

$$\text{Integration by parts: } \int_{\Gamma_t} [\![-2Dv_\varepsilon + p_\varepsilon \mathbf{I}]\!] : \nu \otimes \varphi \xrightarrow{\varepsilon \to 0} \sigma \int_{\Gamma_t} \varphi \cdot \nu H,$$

which implies the jump condition of the stress tensor:

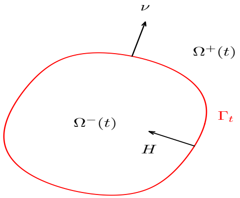
$$[\![2Dv_0 - p_0 \mathbf{I}]\!] \cdot \nu = -\sigma \nu H.$$

Away from Γ_t outer expansion leads to Stokes flow.

Rigorous Sharp Interface Limit

The *inner expansion* gives:

$$
\begin{aligned}
[\![2Dv_0 - p_0\mathbf{I}]\!] \cdot \nu &= -\sigma H \nu && \text{on } \Gamma_t \\
V + v_0 \cdot \nu &= H, \quad [\![v_0]\!] = 0, && \text{on } \Gamma_t
\end{aligned}
$$



The *outer expansion* leads to

$$
\begin{cases}
-\Delta v_0 + \nabla p_0 = 0 & \text{in } \Omega^\pm(t) \\
\operatorname{div} v_0 = 0 & \text{in } \Omega^\pm(t)
\end{cases}
$$

Abels–L. '18

Assume a smooth solution on $[0, T_0]$ and that $c_\varepsilon^0(x) \approx \theta\left(d_{\Gamma_0}(x)/\varepsilon\right)$ near Γ_t and $c_\varepsilon^0(x) = \pm 1$ away from Γ_t. Then

$$
\sup_{0 \leqslant t \leqslant T} \|c_\varepsilon(t) - c_A(t)\|_{L^2(\Omega)} = O(\varepsilon^2), \; \|v_\varepsilon - v_A\|_{L^{2-}((0,T)\times\Omega))} = O(\varepsilon),
$$

$$
c_A(x,t) = \theta\left(\frac{d_\Gamma(x,t)}{\varepsilon} + O(1)\right) + O(\varepsilon^2) \text{ near } \Gamma_t
$$

$$
v_A(x,t) = v_0(x,t) + O(\varepsilon).
$$

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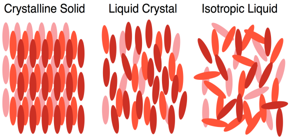
Nematic-Isotropic phase transition

Using spectrum stability argument, Fei-Wang-Zhang-Zhang '19 studied the isotropic-nematic phase transition modeled by the matrix-valued Ginzburg-Landau equation:



$$\partial_t Q_\varepsilon = \Delta Q_\varepsilon - \frac{1}{\varepsilon^2} W'(Q_\varepsilon), \quad Q_\varepsilon : \Omega \to \mathbb{Q},$$
$$\mathbb{Q} = \{Q \in \mathbb{R}^{3\times 3} : Q = Q^T, \operatorname{tr} Q = 0\}$$

where $W(Q) = \frac{a}{2}\operatorname{tr}(Q^2) - \frac{b}{3}\operatorname{tr}(Q^3) + \frac{c}{4}\left(\operatorname{tr}(Q^2)\right)^2$ is the Landau's expansion of the free energy. Here a, b, c are temperature dependent constant, and satisfies $b^2 = 27ac$ at the critical temperature:
$\operatorname{Argmin} W(Q) = \{s_\pm(\mathbf{n}\otimes\mathbf{n} - \frac{1}{3}I_3) : \mathbf{n} \in \mathbb{S}^2\}$, $s_- = 0, s_+ = \frac{b+\sqrt{b^2-24ac}}{4c}$.
Rubinstein-Sternberg-Keller '89. Lin-Pan-Wang, '13: steady case.
As a toy model one can consider the vector-valued G–L equation with $W(u) = |u|^2(1-|u|^2)^2$.

Nematic-Isotropic phase transition

Fei–Wang–Zhang–Zhang '18

Assume Γ_t is a MMC and is smooth on $[0,T]$. $\mathbf{n}:\Omega^+(t)\mapsto\mathbb{S}^2$ is a smooth harmonic heat flow in $\Omega^+(t)$ with $\partial_\nu\mathbf{n}=0$ on Γ_t: $\partial_t\mathbf{n}=\Delta\mathbf{n}+|\nabla\mathbf{n}|^2\mathbf{n}$. Then for any $k\geqslant 1$ there exists approximate $Q^{[k]}$ which is close to $s_+(\mathbf{n}\otimes\mathbf{n}-\frac{1}{3}I_3)$ in $\Omega^+(t)\backslash\Gamma_t(\delta)$ and 0 in $\Omega^-(t)\backslash\Gamma_t(\delta)$, and

$$\partial_t Q^{[k]}-\Delta Q^{[k]}+\frac{1}{\varepsilon^2}W'(Q^{[k]})=O(\varepsilon^{k-1}).$$

Moreover, if the initial data satisfies $\|Q^0_\varepsilon(x)-Q^{[k]}(0,x)\|_{H^2}\lesssim\varepsilon^9$, then

$$\sup_{t\in[0,T]}\|Q_\varepsilon(t,x)-Q^{[k]}(t,x)\|_{H^2}\lesssim\varepsilon^9.$$

The proof is based on asymptotic expansions and **spectrum inequality** of the linearized operator at $Q^{[k]}$.

Modulated energy method: distance function

Bronsard-Kohn '91: Recall the energy dissipation

$$\int_\Omega \left(\tfrac{\varepsilon}{2}|\nabla c_\varepsilon|^2 + \tfrac{1}{\varepsilon}W(c_\varepsilon)\right) dx + \int_{\Omega_t} \varepsilon|\partial_t c_\varepsilon|^2 = \mathcal{E}_\varepsilon(c_{in}).$$

Let $G(\tau) = \tau^3/3 - \tau$. Then $\nabla_{x,t}G(c_\varepsilon) = (c_\varepsilon^2 - 1)\nabla_{x,t}c_\varepsilon$, thus $G(c_\varepsilon)$ is bounded in BV. Motivated by Lin '96 on the Ginzburg-Landau vortices, one can construct a weight function $\phi(x,t) = d_\Gamma(x,t)^2$ near Γ_t and constant away from it. This function will cut-off the singularity and lead to

$$\begin{aligned}
&\frac{d}{dt}\int \phi(x,t)\left(\tfrac{1}{2}|\nabla c_\varepsilon|^2 + \tfrac{W(c_\varepsilon)}{\varepsilon^2}\right) dx \\
&\lesssim \int \phi(x,t)\left(\tfrac{1}{2}|\nabla c_\varepsilon|^2 + \tfrac{W(c_\varepsilon)}{\varepsilon^2}\right) dx + \int \left(\tfrac{1}{2}|\nabla c_\varepsilon|^2 - \tfrac{W(c_\varepsilon)}{\varepsilon^2}\right) dx.
\end{aligned}$$

Modica '85, Ilmanen '93 showed that the discrepancy $|\nabla c_\varepsilon|^2 - \frac{W(c_\varepsilon)}{\varepsilon^2}$ preserves negativity. This seems only valid for scalar equation.

Modulated energy method: Tilt-excess

Motivated by Jerrard–Smets '15 on the binormal curve flow, and Fischer–Laux–Simon '20 preprint on the scalar case, we define

$$E_\varepsilon[Q_\varepsilon|\Gamma](t) := \int_\Omega \left(\frac{\varepsilon}{2} |\nabla Q_\varepsilon|^2 + \frac{1}{\varepsilon} W(Q_\varepsilon) - \boldsymbol{\xi} \cdot \nabla \psi_\varepsilon \right) dx, \tag{2}$$

where $\boldsymbol{\xi}$ is an appropriate extension of the normal of Γ_t,
$\psi_\varepsilon(x,t) := d^W_\varepsilon \circ Q_\varepsilon(x,t)$ and $d^W_\varepsilon(Q) := (\phi_\varepsilon * d^W)(Q)$ and

$$d^W(Q) := \inf_{\gamma(0)\in\{s^+(\mathbf{n}\otimes\mathbf{n}-I_3/3)\},\gamma(1)=Q} \int_0^1 \sqrt{2W(\gamma(t))}|\gamma'(t)|\, dt. \tag{3}$$

Laux–L. ' 20 preprint

Let Γ_t be a smooth MMC on $[0,T]$ and $\varepsilon\|Q^{in}_\varepsilon\|_{L^\infty(\Omega)} + E_\varepsilon[Q_\varepsilon|\Gamma](0) \lesssim \varepsilon$, then $\sup_{t\in[0,T]} E_\varepsilon[Q_\varepsilon|\Gamma](t) \lesssim \varepsilon$ and $Q_{\varepsilon_k} \xrightarrow{k\to\infty} Q = s_\pm \left(\mathbf{n}\otimes\mathbf{n} - \frac{1}{3}I_3\right)$ strongly in $L^\infty_t(H^1_{loc}(\Omega^\pm(t)))$ where $\mathbf{n} : \Omega^+(t) \mapsto \mathbb{S}^2$ is a weak solution of the harmonic map heat flow with 0-Neumann boundary condition on $\partial\Omega^+(t) = \Gamma_t$.
$s_- = 0, s_+ = \frac{b+\sqrt{b^2-24ac}}{4c} > 0$.

Controls of tilt-excess and discrepancy

In the scalar case $\psi_\varepsilon(x,t) = \int_0^{c_\varepsilon(x,t)} \sqrt{2W(z)}\,dz$ and $\mathrm{n}_\varepsilon = \frac{\nabla\psi_\varepsilon}{|\nabla\psi_\varepsilon|}$, thus

$$E_\varepsilon[c_\varepsilon|\Gamma](t) = \int_\Omega \left(\frac{\varepsilon}{2}|\nabla c_\varepsilon|^2 + \frac{1}{\varepsilon}W(c_\varepsilon) - \boldsymbol{\xi}\cdot\frac{\nabla\psi_\varepsilon}{|\nabla\psi_\varepsilon|}|\nabla c_\varepsilon|\sqrt{2W(c_\varepsilon)}\right)dx.$$

In the vectorial case, we introduce an orthogonal projection

$$\Pi_{Q_\varepsilon}\partial_{x_i}Q_\varepsilon = \left(\partial_{x_i}Q_\varepsilon : \frac{\nabla_q d_\varepsilon^W(Q_\varepsilon)}{|\nabla_q d_\varepsilon^W(Q_\varepsilon)|}\right)\frac{\nabla_q d_\varepsilon^W(Q_\varepsilon)}{|\nabla_q d_\varepsilon^W(Q_\varepsilon)|}, \text{ with } x_0 := t.$$

Grönwall' inequality
$$\int_\Omega \left(\frac{1}{2}|\nabla Q_\varepsilon|^2 + \frac{1}{\varepsilon^2}W(Q_\varepsilon)\right)(1-\boldsymbol{\xi}\cdot\mathrm{n}_\varepsilon) + \int_\Omega \left(|\nabla Q_\varepsilon - \Pi_{Q_\varepsilon}\nabla Q_\varepsilon|^2\right) \lesssim \varepsilon^{-1}E_\varepsilon[Q_\varepsilon|\Gamma](t) \lesssim 1.$$

As $d_\varepsilon^W(Q)$ is isotropic, the commutator $[\Pi_{Q_\varepsilon}\partial_{x_i}Q_\varepsilon, Q_\varepsilon] = 0$. So

$$[\partial_t Q_\varepsilon - \Pi_{Q_\varepsilon}\partial_t Q_\varepsilon, Q_\varepsilon] = \nabla\cdot[\nabla Q_\varepsilon - \Pi_{Q_\varepsilon}\nabla Q_\varepsilon, Q_\varepsilon] \qquad \text{(weak compactness)}$$

Thank you

谢谢！

Thank you !