Future stability of the FLRW spacetime for a large class of perfect fluids

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Outline

Einstein-Euler system

The Einstein-Euler system with a positive cosmological constant is given by

$$\begin{cases} \tilde{G}^{\mu\nu} + \Lambda \tilde{g}^{\mu\nu} = \tilde{T}^{\mu\nu}, \\ \tilde{\nabla}_\mu \tilde{T}^{\mu\nu} = 0, \end{cases} \tag{1.1}$$

where $\tilde{\nabla}_\mu$ denotes covariant derivative and Λ the positive cosmological constant, $\tilde{G}^{\mu\nu} = \tilde{R}ic^{\mu\nu} - \frac{1}{2}\tilde{R}\tilde{g}^{\mu\nu}$ is the Einstein tensor of the metric

$$\tilde{g} = \tilde{g}_{\mu\nu} dx^\mu dx^\nu, \tag{1.2}$$

$\tilde{R}ic_{\mu\nu}$, $\tilde{R}$ are the Ricci and scalar curvature of the metric $\tilde{g}$ respectively. $\tilde{T}^{\mu\nu}$ denotes the stress energy tensor

$$\tilde{T}^{\mu\nu} = (\rho + p)\tilde{u}^\mu \tilde{u}^\nu + p\tilde{g}^{\mu\nu} \quad (\tilde{g}_{\mu\nu}\tilde{u}^\mu \tilde{u}^\nu = -1), \tag{1.3}$$

Conformal transformation

We consider the conformal metric

$$g_{\mu\nu} = e^{-2\Phi}\tilde{g}_{\mu\nu}, \quad \text{or} \quad g^{\mu\nu} = e^{2\Phi}\tilde{g}^{\mu\nu}, \tag{1.4}$$

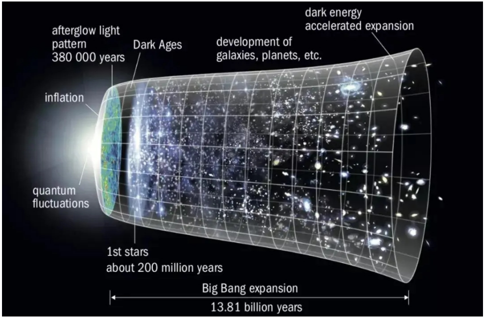
where

$$\Phi = -\ln(\tau). \tag{1.5}$$

Under the conformal transformation (1.4) and (1.5), the equations (1.2) that we consider in this paper is the following Cauchy problem

$$\begin{cases} G^{\mu\nu} = e^{4\Phi}\tilde{T}^{\mu\nu} + 2(\nabla^\mu\nabla^\nu\Phi - \nabla^\mu\Phi\nabla^\nu\Phi) - (2\Box_g\Phi + |\nabla\Phi|_g^2)g^{\mu\nu}, \\ \nabla_\mu\tilde{T}^{\mu\nu} = -6\tilde{T}^{\mu\nu}\nabla_\mu\Phi + g_{\kappa\lambda}\tilde{T}^{\kappa\lambda}g^{\mu\nu}\nabla_\mu\Phi, \\ \tau = 1: \quad g^{\mu\nu} = g_0^{\mu\nu}(x), \quad \partial_\tau g^{\mu\nu} = g_1^{\mu\nu}(x), \\ \qquad\qquad \rho|_{\tau=1} = m(x), \quad v^i|_{\tau=1} = q^i(x). \end{cases} \tag{1.6}$$

Physical Background



- Cosmological evidences show that our universe is undergoing accelerated expansion. (Astrophys. J. 517 565, and Astrophys. J. 116 1109).
- Standard models: "quintessence"、"positive cosmological constant"、"Dark Energy" $\cdots$.

Background (FLRW) solution

- Firemann-Lemaître-Robertson-Walker (FLRW) solutions to (1.2) are spatially homogeneous, isotropy and time dependent only and are used to explain the accelerated expanding of the universe.
- The main motivation of our work is to give a criterion on the fluids such that the Einstein-Euler system with a positive cosmological constant admits a global classical solution when the initial data are small perturbations to the FLRW solutions.

Main Results

Theorem

[Ann. Henri. Poincare, 2020] Suppose $k \in \mathbb{Z}_{\geq 3}$, $\Lambda > 0$, $g_0^{\mu\nu} \in H^{k+1}(\mathbb{T}^3)$, $g_1^{\mu\nu}$, ρ_0, $\nu^\alpha \in H^k(\mathbb{T}^3)$, $\rho_0 > 0$ *for all* $x \in \mathbb{T}^3$ *with*

$$(g^{\mu\nu}, \partial_\tau g^{\mu\nu}, \rho, u^i)|_{\tau=1} = (g_0^{\mu\nu}, g_1^{\mu\nu}, \rho_0, \nu^\alpha) \tag{1.7}$$

which solves the constraint equations

$$(G^{0\mu} - T^{0\mu})|_{\tau=1} = 0 \quad \text{and} \quad Z^\mu|_{\tau=1} = 0.$$

Then there exists a constant $\sigma > 0$, *such that if*

$$\|g_0^{\mu\nu} - \eta^{\mu\nu}(1)\|_{H^{k+1}} + \|g_1^{\mu\nu} - \partial_\tau \eta^{\mu\nu}(1)\|_{H^k} + \|\rho_0 - \bar{\rho}(1)\|_{H^k} + \|\nu^i\|_{H^k} < \sigma,$$

and the fluids satisfy given assumptions in the following

Main Results

Theorem

there exists a unique classical solution $g^{\mu\nu} \in C^2((0,1] \times \mathbb{T}^3)$*,* $\rho,\ v^i \in C^1((0,1] \times \mathbb{T}^3)$ *to the conformal Einstein-Euler system that satisfies the following regularity conditions*

$$(g^{\mu\nu}, u^\mu, \rho) \in \bigcap_{\ell=0}^{2} C^\ell((T_1,1], H^{k+1-\ell}(\mathbb{T}^3)) \times \bigcap_{\ell=0}^{1} C^\ell((T_1,1], H^{k-\ell}(\mathbb{T}^3)) \times \bigcap_{\ell=0}^{1} C^\ell((T_1,1], H^{k-\ell}(\mathbb{T}^3)), \tag{1.8}$$

and the estimates

$$\|g^{\mu\nu}(\tau) - \eta^{\mu\nu}(\tau)\|_{H^{k+1}} + \|\partial_\kappa g^{\mu\nu}(\tau) - \partial_\kappa \eta^{\mu\nu}(\tau)\|_{H^k} + \|\rho(\tau) - \bar{\rho}(\tau)\|_{H^k} + \|u^i(\tau)\|_{H^k} \lesssim \sigma.$$

Basic assumptions on the fluids

Assumption

(The symmetrization condition of Makino-like variable α) There exists an invertible transformation

$$\begin{array}{rcl} C^2 \ni \varphi : [-\infty, +\infty] & \to & (0, +\infty) \\ \alpha(x^\mu) & \mapsto & \rho(x^\mu) \end{array}$$

and a transformation $C^2 \ni \lambda : [-\infty, +\infty] \to [\hat{\delta}, 1/\hat{\delta}]$ for some constant $\hat{\delta} > 0$. such that

$$\frac{d\varphi(\alpha)}{d\alpha} = \frac{\varphi(\alpha) + \mu^* p}{q(\alpha)} \tag{1.9}$$

where

$$q(\alpha) := \frac{s(\alpha)}{\lambda(\alpha)} \quad s(\alpha) := \varphi^*\left(\sqrt{\frac{dp(\rho)}{d\rho}}\right) \tag{1.10}$$

Assumption

Suppose $\bar{\rho}$ is the density of the fluid associating with its homogeneous, isotropic state, then we denote

$$\bar{\alpha} := \varphi^{-1}(\bar{\rho}). \tag{1.11}$$

Assume there exists an function $\varrho \in C\big([0,1], C^{\omega}(\mathbb{R})\big)$ *satisfying* $\varrho(\tau, 0) = 0$ *and a rescaling function* $\beta(\tau) \in C[0,1] \cap C^1(0,1]$ *of* α *such that*

$$\varphi(\alpha) - \varphi(\bar{\alpha}) = \tau^{\Theta} \varrho\big(\tau, \beta^{-1}(\tau)(\alpha - \bar{\alpha})\big), \quad \Theta \geq 2. \tag{1.12}$$

Assumptions on β

Assumption

Denote $\bar{s} := s(\bar{\alpha})$, $\bar{\lambda} := \lambda(\bar{\alpha})$ *and* $\bar{q} := q(\bar{\alpha}) = \bar{s}/\bar{\lambda}$. *Suppose* $\bar{s} \lesssim \beta(\tau)$ *and*

$$\left(\partial_\tau \beta(\tau) + \frac{\bar{s}}{\tau}\right)\frac{d\bar{\lambda}}{d\bar{\alpha}} \lesssim 1, \tag{1.13}$$

- *If there is a positive constant* $0 < \hat{\delta} < 1$, *such that* $\beta(\tau)$ *is bounded by*

$$\frac{1}{\hat{\delta}}\tau \le \beta(\tau) \le \frac{1}{\hat{\delta}}\sqrt{\tau}, \chi(\tau) := \tau\partial_\tau \ln \beta(\tau) \ge 0. \tag{1.14}$$

Assumption

and require that

$$1 - 3\bar{s}^2 \geq \chi(\tau) + \hat{\delta}. \tag{1.15}$$

and

$$\frac{1}{3}\chi(\tau) + \frac{1}{\hat{\delta}} \geq q'(\bar{\alpha}) \geq \frac{1}{3}\chi(\tau) + \hat{\delta}, \tag{1.16}$$

holds for all $\tau \in [0, 1]$.

- $\beta \equiv$ *constant* > 0 *and one of the following cases happens*
  - $q = \bar{q}$ *and* $1 - 3\bar{s}^2 \geq \hat{\delta}$;
  - $q = \bar{q}$ *and* $1 - 3s^2 = 0$;
  - $q'(\bar{\alpha}) \geq \hat{\delta}$ *and* $1 - 3\bar{s}^2 \geq \hat{\delta}$.

Fluids satisfy above assumptions

Once the equations of state are given

Corollary

The future nonlinear stability of the FLRW spacetime for a linear equation of state $p = K\rho$, $(K \in (0, \frac{1}{3}])$*, Chaplygin gases* $p = -\frac{\Lambda^{1+\vartheta}}{(\rho+\Lambda)^{\vartheta}}$, $(\vartheta \in (0, \sqrt{\frac{1}{3}}])$ *and polytropic gases* $p = K\rho^{\frac{n+1}{n}}$, $(n \in (1,3))$ *satisfy above assumptions.*

Some known results

Pure Analysis method (wave coordinates)

- When $\widetilde{T}^{\mu\nu} = \widetilde{\partial}^{\mu}\Psi\widetilde{\partial}^{\nu}\Psi - [\frac{1}{2}\widetilde{g}_{\mu\nu}\widetilde{\partial}^{\mu}\Psi\widetilde{\partial}^{\nu}\Psi + V(\Psi)]\widetilde{g}^{\mu\nu}$ and $V(0) > 0$, $V^{'}(0) = 0$, $V^{''}(0) > 0$, H. Ringstrom [Invent. Math, 2008] showed the global non-linear stability of non-vacuum Einstein system.
- When $0 < C_s^2 < \frac{1}{3}$, Rodnianski and Speck [JEMS, 2013] proved the global nonlinear stability of a family of FLRW solutions of the irrotational Euler-Einstein system with $p = C_s^2\rho$.
- When $0 < C_s^2 < \frac{1}{3}$, J. Speck [Selecta Math, 2012] proved the global future stability of Euler-Einstein system with non-zero vorticity when $p = C_s^2\rho$.
- When $C_s = 0$, M. Hadžić and J. Speck [JHDE, 2015] proved the global future stability of Euler-Einstein system with non-zero vorticity.

Some known results

Conformal method

- When $0 < C_s^2 \leq \frac{1}{3}$, T. Oliynyk [CMP, 2016] proved the same result by combining the conformal method and wave coordinates.
- When $0 < C_s^2 \leq \frac{1}{3}$, C. Liu and T. Oliynyk [CMP, 2018; AHP 2018] solved the Nowtonian limit problem (Einstein-Euler and Poisson-Euler)
- When $p = -\frac{1}{\rho^\theta}$ $(\theta \in (0, 1])$ and $\Lambda = 0$, P. LeFloch and Wei [Ann. I. H. Poincare-AN, 2020] proved the global existence of the Einstein-Chaplygin fluids when the fluid is irrotational.

Some known results

In fixed accelerated expanding spacetime
$g = -dt^2 + a^2(t)\sum_{i=1}^{3}(dx^i)^2$

- U. Brauer, A. Rendall and O. Reula [CQG, 1994] showed the global solution for Polytropic gases in Newtonian cosmological spacetime with exponentially expanding rate.
- J. Speck [ARMA, 2013] proved the future stability results for the relativistic Euler equations when $0 \le C_s^2 \le \frac{1}{3}$ under the assumption that $a(t)$ satisfies some time-integrable conditions.
- Wei [JDE, 2018] proved the relationship between the future stability of the fluids and the spacetime when $a(t)$ admits a polynomial expanding rate with $p = C_s^2\rho$ and $p = -\frac{A}{\rho^\alpha}$.
- T. Oliynyk [arXiv: 2002.12526] proved the future stability of the relativistic fluids in exponentially expanding rate when $\frac{1}{3} < C_s^2 \le \frac{1}{2}$. This result found some evidence for the stability of the fluids with parameter $C_s^2 \ge \frac{1}{3}$.

Natural problem

- Does the solution exists globally for general Chaplygin fluids (without irrotational assumption)?
- How about the polytropic gases?
- What is the difference between polytropic gases and the linear case?

In other words, which kind of fluids can ensure the global nonlinear stability of the Einstein-Euler system with a positive cosmological constant when the initial data are small perturbations to the FLRW solutions? (Structural stability of the FLRW-type stabilization with respect to different equations of state.)

Main idea

The main idea of conformal method is to turn the whole system into a singular symmetric hyperbolic system.

$$B^\mu \partial_\mu u = \frac{1}{t}\mathbf{BP}u + H \quad \text{in } [T_0, T_1] \times \mathbb{T}^n, \tag{3.1}$$

$$u = u_0 \quad \text{in } T_0 \times \mathbb{T}^n, \tag{3.2}$$

where we require the following **Conditions**:

- $T_0 < T_1 \leq 0.$
- $\mathbf{P}$ is a constant, symmetric projection operator, i.e., $\mathbf{P}^2 = \mathbf{P}$, $\mathbf{P}^T = \mathbf{P}$ and $\partial_\mu \mathbf{P} = 0.$
- $u = u(t, x)$ and $H(t, u)$ are $\mathbb{R}^N$-valued maps, $H \in C^0([T_0, 0], C^\infty(\mathbb{R}^N))$ and satisfies $H(t, 0) = 0.$

Singular symmetric system

- $B^\mu = B^\mu(t,u)$ and $\mathbf{B} = \mathbf{B}(t,u)$ are $\mathbb{M}_{N\times N}$-valued maps, and B^μ, $\mathbf{B} \in C^0([T_0,0], C^\infty(\mathbb{R}^N))$, $B^0 \in C^1([T_0,0], C^\infty(\mathbb{R}^N))$ and they satisfy

$$(B^\mu)^T = B^\mu, \quad [\mathbf{P}, \mathbf{B}] = \mathbf{P}\mathbf{B} - \mathbf{B}\mathbf{P} = 0. \tag{3.3}$$

- Suppose

$$B^0 = \mathring{B}^0(t) + \tilde{B}^0(t,u) \tag{3.4}$$

$$\mathbf{B} = \mathring{\mathbf{B}}(t) + \tilde{\mathbf{B}}(t,u) \tag{3.5}$$

where $\tilde{B}^0(t,0) = 0$ and $\tilde{\mathbf{B}}(t,0) = 0$. There exists constants κ, γ_1, γ_2 such that

$$\frac{1}{\gamma_1}\mathbb{I} \le \mathring{B}^0 \le \frac{1}{\kappa}\mathring{\mathbf{B}} \le \gamma_2\mathbb{I} \tag{3.6}$$

for all $t \in [T_0, 0]$.

singular symmetric system

- For all $(t,u) \in [T_0, 0] \times \mathbb{R}^N$, we have

$$\mathbf{P}^{\perp} B^0(t, \mathbf{P}^{\perp} u)\mathbf{P} = \mathbf{P} B^0(t, \mathbf{P}^{\perp} u)\mathbf{P}^{\perp} = 0,$$

  where $\mathbf{P}^{\perp} = \mathbb{I} - \mathbf{P}$ is the complementary projection operator.
- There exists constants θ, β_1 and $\varpi > 0$ such that

$$|\mathbf{P}^{\perp}[D_u B^0(t,u)(B^0)^{-1}\mathbf{B}\mathbf{P}u]\mathbf{P}^{\perp}| \quad \leq \quad |t|\theta + \frac{2\beta_1}{\varpi + |P^{\perp}u|^2}|\mathbf{P}u|^2,$$

Proposition

Proposition

Suppose that $k \geq \frac{n}{2} + 1$, $u_0 \in H^k(\mathbb{T}^n)$ *and above conditions are fulfilled. Then there exists a* $T_* \in (T_0, 0)$, *and a unique classical solution* $u \in C^1([T_0, T_*] \times \mathbb{T}^n)$ *that satisfies* $u \in C^0([T_0, T_*], H^k) \cap C^1([T_0, T_*], H^{k-1})$ *and the energy estimate*

$$\|u(t)\|_{H^k}^2 - \int_{T_0}^{t} \frac{1}{\tau} \|\mathbf{P}u\|_{H^k}^2 \leq Ce^{C(t-T_0)}(\|u(T_0)\|_{H^k}^2)$$

for all $T_0 \leq t < T_*$, *where* $C = C(\|u\|_{L^\infty([T_0,T_*),H^k)}, \gamma_1, \gamma_2, \kappa)$, *and can be uniquely continued to a larger time interval* $[T_0, T^*)$ *for all* $T^* \in (T_*, 0]$ *provided* $\|u\|_{L^\infty([T_0,T_*),W^{1,\infty})} < \infty$.

Energy inequalities

Acting on (3.1) by $D^\alpha \mathbf{B}^{-1}$ to obtain

$$B^\mu \partial_\mu D^\alpha u = \frac{1}{t}\mathbf{B}\mathbf{P} D^\alpha u - \mathbf{B}[D^\alpha, \mathbf{B}^{-1} B^\mu]\partial_\mu u + \mathbf{B} D^\alpha(\mathbf{B}^{-1} H).$$

By tedious computations and standard energy estimates under above assumptions, we can get

$$\partial_t \|u\|_0^2 \lesssim \frac{\tilde{\kappa}}{t}\|\mathbf{P}u\|_0^2 + \gamma(\theta + \|divB\|_{L^\infty})\|u\|_0^2 + 2\sqrt{\gamma}\|H\|_2\|u\|_0,$$

and

$$\partial_t \|u\|_k^2 \lesssim \frac{\tilde{\kappa}}{t}\|\mathbf{P}u\|_k^2 + C\Big[-\frac{1}{t}(\delta\|\mathbf{P}u\|_k^2 + c(\delta)\|\mathbf{P}u\|_0^2) + \|u\|_k^2\Big].$$

Then we have

$$\partial_t\left(\|u\|_k^2 + K\|u\|_0^2 - \int_{T_0}^t \frac{1}{\tau}\|\mathbf{P}u\|_k^2 d\tau\right) \lesssim C\|u\|_k^2.$$

Main difficulties

- The coefficient matrix B^0 is degenerate for polytropic gases
- How to deal with the source terms ($\frac{\rho-\bar{\rho}}{\tau^2}$ is bounded)?
- How to make a balance between the Einstein equations and Euler equations? (The projection matrix for $g^{0\mu}$ is not diagonal)?
- How to ensure the C^1 property of B^0 with respect to τ?
- How to ensure the constraint $\mathbf{P}^{\perp}B^0(t,\mathbf{P}^{\perp}u)\mathbf{P} = \mathbf{P}B^0(t,\mathbf{P}^{\perp}u)\mathbf{P}^{\perp} = 0$? (For Fluids II)

The metrics

- The original metric:

$$\tilde{g} = \tilde{g}_{\mu\nu} dx^\mu dx^\nu.$$

- The original background metric:

$$\tilde{\eta} = \frac{1}{\tau^2}\left(-\frac{1}{w^2} d\tau^2 + \sum_{i=1}^{3} (dx^i)^2\right) = -dt^2 + a^2(t)\sum_{i=1}^{3}(dx^i)^2.$$

- The conformal metric:

$$g = g_{\mu\nu} dx^\mu dx^\nu.$$

- The conformal background metric:

$$\eta = -\frac{1}{w^2} d\tau^2 + \sum_{i=1}^{3} (dx^i)^2.$$

Analysis of the FLRW solution

The time dependent solution satisfies

$$-3\omega^2 - \Omega + \left(\frac{\bar\rho - \bar p}{2} + \Lambda\right) = 0, \tag{4.1}$$

$$-6\Omega - 6\omega^2 + 2\Lambda - (\bar\rho + 3\bar p) = 0. \tag{4.2}$$

$$\partial_0 \bar\rho = \frac{3}{\tau}(\bar\rho + \bar p). \tag{4.3}$$

Solving above and under the assumptions above, we have

$$\tau^4 \bar\rho(1) \leq \bar\rho(\tau) \leq \tau^3 \bar\rho(1), \tag{4.4}$$

$$\frac{1}{3}\bar\rho(1)\tau^4 \leq \omega^2 - \frac{\Lambda}{3} \leq \frac{1}{3}\bar\rho(1)\tau^3, \tag{4.5}$$

$$-\frac{2}{3}\tau^3 \bar\rho(1) \leq \Omega \leq -\frac{1}{2}\tau^4 \bar\rho(1) \tag{4.6}$$

and

$$3\tau^3 \bar\rho(1) \leq \partial_\tau \bar\rho \leq 4\tau^2 \bar\rho(1). \tag{4.7}$$

Wave coordinates

Define the wave coordinates as

$$Z^\mu = \Gamma^\mu + Y^\mu = \Gamma^\mu + \frac{2}{\tau}\left(g^{\mu 0} + (w^2 + \frac{\dot{w}}{2})\delta_0^\mu\right). \tag{4.8}$$

- $R^{\mu\nu} = -\frac{1}{2} g^{\alpha\beta}\partial_\alpha\partial_\beta g^{\mu\nu} + \nabla^{(\mu}\Gamma^{\nu)} + lower\, order$

Remark

From above, we can see that $Y^\mu = -2\nabla^\mu\Phi - e^{2\Phi}\bar{\tilde{\Gamma}}^\mu$. *For the metric* η, $Z^\mu \equiv 0$. *This fact is very important for the disappear of the linear part of the conformal Einstein and conformal fluid equations* (1.6)*.*

Conformal Einstein equations

With the wave coordinates Z^μ defined by (4.8), we can consider the following equivalently reduced conformal Einstein equation by assuming $Z^\mu|_{\tau=1}=0$

$$\begin{aligned} -2R^{\mu\nu} &+ 2\nabla^{(\mu}Z^{\nu)} + A^{\mu\nu}_{\kappa}Z^\kappa = -4\nabla^\mu\nabla^\nu\Phi + 4\nabla^\mu\Phi\nabla^\nu\Phi \\ &- 2\left[\Box_g\Phi + 2|\nabla\Phi|^2_g + (\frac{\rho-p}{2}+\Lambda)e^{2\Phi}\right]g^{\mu\nu} \\ &- 2e^{2\Phi}(\rho+p)u^\mu u^\nu \end{aligned} \tag{4.9}$$

where

$$A^{\mu\nu}_{\kappa} = -X^{(\mu}\delta^{\nu)}_{\kappa} + Y^{(\mu}\delta^{\nu)}_{\kappa}.$$

Expanding the left hand side of above and subtracting the background metric, we get

$$
\begin{aligned}
- g^{\kappa\lambda}\partial_\kappa\partial_\lambda(g^{\mu\nu}-\eta^{\mu\nu}) \quad = \quad & \frac{2\omega^2}{\tau}\partial_\tau(g^{\mu\nu}-\eta^{\mu\nu}) - \frac{4\omega^2}{\tau^2}(g^{00}+\omega^2)\delta_0^\mu\delta_0^\nu \\
& -\frac{4\omega^2}{\tau^2}g^{0i}\delta_0^{(\mu}\delta_i^{\nu)} - \frac{2}{\tau^2}g^{\mu\nu}(g^{00}+\omega^2) + \mathfrak{H}^{\mu\nu}.
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{H}^{\mu\nu} \quad = \quad & (g^{\kappa\lambda}-\eta^{\kappa\lambda})\partial_\kappa\partial_\lambda\eta^{\mu\nu} - \frac{2\Omega}{\tau^2}(g^{00}+\omega^2)\delta_0^\mu\delta_0^\nu - \frac{\Omega}{\tau}\partial_\tau(g^{\mu\nu}-\eta^{\mu\nu}) \\
& -\frac{2\Omega}{\tau^2}g^{0i}\delta_0^{(\mu}\delta_i^{\nu)} - \frac{\Omega}{\tau^2}(g^{\mu\nu}-\eta^{\mu\nu}) \\
& -\frac{2\partial_\tau\psi(\tau)}{3\tau}\left((g^{\mu0}-\eta^{\mu0})\delta_0^\nu + (g^{\nu0}-\eta^{\nu0})\delta_0^\mu\right) \\
& -\frac{1}{\tau^2}(\rho-\bar\rho-p+\bar p)(g^{\mu\nu}-\eta^{\mu\nu}) - \frac{2}{\tau^2}\left(\frac{\rho-\bar\rho-(p-\bar p)}{2}\right)\eta^{\mu\nu} \\
& -\frac{2}{\tau^2}\Big[(\rho-\bar\rho+p-\bar p)u^\mu u^\nu + (\bar\rho+\bar p)(u^\mu u^\nu - \bar u^\mu\bar u^\nu)\Big] \\
& +Q^{\mu\nu}(g,\partial g) - Q^{\mu\nu}(\eta,\partial\eta)
\end{aligned}
$$

New variables

Define the densitized three metric

$$\mathbf{g}^{ij} = det(\check{g}_{lm})^{\frac{1}{3}} g^{ij}, \tag{4.10}$$

where

$$\check{g}_{lm} = (g^{lm})^{-1},$$

and the variable

$$\mathbf{q} = g^{00} + w^2 - \frac{w^2}{3} \ln(\det(g^{pq})). \tag{4.11}$$

It is easy to check that

$$\partial_\mu \mathbf{g}^{ij} = (\det(\check{g}_{pq}))^{\frac{1}{3}} \mathbf{L}^{ij}_{lm} \partial_\mu g^{lm}, \tag{4.12}$$

where

$$\mathbf{L}^{ij}_{lm} = \delta^i_l \delta^j_m - \frac{1}{3} \check{g}_{lm} g^{ij}.$$

Obviously, $\mathbf{L}^{ij}_{lm}$ is trace-free, i.e.,

$$\mathbf{L}^{ij}_{lm} g^{lm} = 0.$$

New unknowns

$$\mathbf{u}^{0\nu} = \frac{g^{0\nu} - \eta^{0\nu}}{2\tau}, \tag{4.13}$$

$$\mathbf{u}_0^{0\nu} = \partial_\tau (g^{0\nu} - \eta^{0\nu}) - \frac{3(g^{0\nu} - \eta^{0\nu})}{2\tau}, \tag{4.14}$$

$$\mathbf{u}_i^{0\nu} = \partial_i (g^{0\nu} - \eta^{0\nu}), \tag{4.15}$$

$$\mathbf{u}^{ij} = \mathbf{g}^{ij} - \delta^{ij}, \tag{4.16}$$

$$\mathbf{u}_\mu^{ij} = \partial_\mu \mathbf{g}^{ij}, \tag{4.17}$$

$$\mathbf{u} = \mathbf{q}, \tag{4.18}$$

$$\mathbf{u}_\mu = \partial_\mu \mathbf{q}. \tag{4.19}$$

symmetric hyperbolic system

$$A^\kappa \partial_\kappa \begin{pmatrix} \mathbf{u}_0^{0\mu} \\ \mathbf{u}_j^{0\mu} \\ \mathbf{u}^{0\mu} \end{pmatrix} = \frac{1}{\tau}\mathbf{AP} \begin{pmatrix} \mathbf{u}_0^{0\mu} \\ \mathbf{u}_j^{0\mu} \\ \mathbf{u}^{0\mu} \end{pmatrix} + F^{0\mu}, \tag{4.20}$$

$$A^\kappa \partial_\kappa \begin{pmatrix} \mathbf{u}_0^{lm} \\ \mathbf{u}_j^{lm} \\ \mathbf{u}^{lm} \end{pmatrix} = \frac{1}{\tau}(-2g^{00})\Pi \begin{pmatrix} \mathbf{u}_0^{lm} \\ \mathbf{u}_j^{lm} \\ \mathbf{u}^{lm} \end{pmatrix} + F^{lm}, \tag{4.21}$$

and

$$A^\kappa \partial_\kappa \begin{pmatrix} \mathbf{u}_0 \\ \mathbf{u}_j \\ \mathbf{u} \end{pmatrix} = \frac{1}{\tau}(-2g^{00})\Pi \begin{pmatrix} \mathbf{u}_0 \\ \mathbf{u}_j \\ \mathbf{u} \end{pmatrix} + F^{\mathbf{q}}, \tag{4.22}$$

Symmetric hyperbolic system

$$A^0 = \begin{pmatrix} -g^{00} & 0 & 0 \\ 0 & g^{ij} & 0 \\ 0 & 0 & -g^{00} \end{pmatrix}, \quad A^k = \begin{pmatrix} -2g^{0k} & -g^{jk} & 0 \\ -g^{ik} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \delta_k^j & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} -g^{00} & 0 & 0 \\ 0 & \frac{3}{2}g^{jk} & 0 \\ 0 & 0 & -g^{00} \end{pmatrix},$$

Symmetric hyperbolic system

$$\Pi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, F^{0\mu} = \begin{pmatrix} 6\mathbf{u}^{0i}\mathbf{u}^{0\mu} + 4\mathbf{u}^{00}\mathbf{u}_0^{0\mu} - 4\mathbf{u}^{00}\mathbf{u}^{0\mu} + \hat{M}^{0\mu} \\ 0 \\ 0 \end{pmatrix}$$

and

$$F^{ij} = \begin{pmatrix} 4\mathbf{u}^{00}\mathbf{u}_0^{ij} + \acute{M}^{ij} \\ 0 \\ g^{00}\mathbf{u}_0^{lm} \end{pmatrix}, \quad F^{\mathbf{q}} = \begin{pmatrix} 4\mathbf{u}^{00}\mathbf{u}_0 - 8(\mathbf{u}^{00})^2 + \hat{R}^{\mathbf{q}} \\ 0 \\ g^{00}\mathbf{u}_0^{lm} \end{pmatrix}.$$

Symmetrize Euler Equations

Define

$$u^\mu = e^\Phi \tilde{u}^\mu, \tag{4.23}$$

Then

$$u^\mu \partial_\mu \rho + (\rho+p) L^\mu_i \nabla_\mu u^i = -3(\rho+p) u^\mu \nabla_\mu \Phi, \tag{4.24}$$

$$M_{ki} u^\mu \partial_\mu u^i + \frac{s^2 L^\mu_i}{\rho+p} \partial_\mu \rho = -L^\mu_k \partial_\mu \Phi., \tag{4.25}$$

where

$$L^\mu_i = \delta^\mu_i - \frac{u_i}{u_0}\delta^\mu_0 \quad \text{and} \quad L_{k\nu} = g_{\nu\lambda} L^\lambda_k \tag{4.26}$$

and

$$M_{ki} = g_{ki} - \frac{u_i}{u_0} g_{0k} - \frac{u_k}{u_0} g_{0i} + \frac{u_i u_k}{u_0^2} g_{00}. \tag{4.27}$$

Symmetrize process

$$u^\mu \frac{d\varphi(\alpha)}{d\alpha}\partial_\mu \alpha + (\rho+p)L_i^\mu \nabla_\mu u^i = -3(\rho+p)u^\mu \nabla_\mu \Phi, \tag{4.28}$$

$$M_{ki}u^\mu \partial_\mu u^i + \frac{s^2 L_i^\mu}{\rho+p}\frac{d\varphi(\alpha)}{d\alpha}\partial_\mu \alpha = -L_k^\mu \partial_\mu \Phi, \tag{4.29}$$

multiplying both sides of (4.28) by $\lambda^2(\alpha)\frac{d\alpha}{d\rho}$, we obtain

$$\lambda^2 u^\mu \partial_\mu \alpha + \lambda^2 \frac{d\alpha}{d\rho}(\rho+p)L_i^\mu \nabla_\mu u^i = -3(\rho+p)\lambda^2 \frac{d\alpha}{d\rho}u^\mu \nabla_\mu \Phi, \tag{4.30}$$

$$M_{ki}u^\mu \partial_\mu u^i + \frac{s^2 L_i^\mu}{\rho+p}\frac{d\varphi}{d\alpha}\partial_\mu \alpha = -L_k^\mu \partial_\mu \Phi. \tag{4.31}$$

The relation (1.9) in Assumption 3 which is

$$\frac{d\varphi(\alpha)}{d\alpha} = \frac{\lambda(\alpha)(\rho+p)}{s(\alpha)}$$

Symmetric Euler

We can get

$$\lambda^2 u^\mu \partial_\mu \alpha + \lambda s L_i^\mu \nabla_\mu u^i = -\, 3\lambda s u^\mu \nabla_\mu \Phi, \tag{4.32}$$
$$\lambda s L_i^\mu \partial_\mu \alpha + M_{ki} u^\mu \nabla_\mu u^k = -\, L_k^\mu \partial_\mu \Phi, \tag{4.33}$$

Remark

For $p = K\rho$ and $p = -\frac{1}{\rho^\vartheta}$, the new density variable is defined by $\xi = \xi(\rho) = \int_{\rho(1)}^{\rho} \frac{dy}{y+p(y)}$. Under this variable transformation, (4.24)–(4.25) *become*

$$s^2 u^\mu \partial_\mu \xi + s^2 L_i^\mu \nabla_\mu u^i = -\, 3s^2 u^\mu \partial_\mu \Phi, \tag{4.34}$$
$$s^2 L_i^\mu \partial_\mu \xi + M_{ij} u^\mu \nabla_\mu u^j = -\, L_i^\mu \partial_\mu \Phi. \tag{4.35}$$

It is evident that (4.32)–(4.33) *coincide with* (4.34)–(4.35) *by choosing $\lambda = s$ and $\alpha = \xi$ provided s is non-degenerate (indeed $s = \sqrt{K}$ for $p = K\rho$).*

Degenerate phenomenon for polytropic gases

When $p = K\rho^{\frac{n+1}{n}}$, we have

$$s^2 = A(1+\frac{1}{n})\frac{C(1,\delta\zeta)\tau^{3/n}}{K+1-KC(1,\delta\zeta)\tau^{3/n}},$$

Fluids (I)

Background equation

$$\partial_\tau \bar{\alpha} = \frac{3\bar{s}}{\tau\bar{\lambda}}. \tag{4.36}$$

Then

$$\lambda^2 u^\mu \partial_\mu(\alpha - \bar{\alpha}) + \lambda s L_i^\mu \partial_\mu u^i = \frac{3}{\tau}\lambda s u^0 - \lambda^2 u^0 \frac{3\bar{s}}{\bar{\lambda}\tau} - \lambda s L_i^\mu \Gamma^i_{\mu\nu} u^\nu, \tag{4.37}$$

$$M_{ki} u^\mu \partial_\mu u^k + \lambda s L_i^\mu \partial_\mu(\alpha - \bar{\alpha}) = L_i^0 \frac{1}{\tau} - \lambda s L_i^0 \frac{3\bar{s}}{\bar{\lambda}\tau} - M_{ki} u^\mu \Gamma^k_{\mu\nu} u^\nu. \tag{4.38}$$

Introduce

$$\alpha = \beta(\tau)\zeta, \quad \bar{\alpha} = \beta(\tau)\bar{\zeta}, \quad u^i = \beta(\tau) v^i \quad \text{and} \quad \delta\zeta = \zeta - \bar{\zeta}, \tag{4.39}$$

New variables for Fluids (I)

Then

$$\lambda^2 u^\mu \partial_\mu \delta\zeta + \lambda s L_i^\mu \partial_\mu v^i = S, \tag{4.40}$$

$$\lambda s L_i^\mu \partial_\mu \delta\zeta + M_{ki} u^\mu \partial_\mu v^k = S_i. \tag{4.41}$$

Where

$$S = \frac{1}{\tau}\left(3\lambda u^0\left[\Xi - \frac{\bar{s}}{\bar{\lambda}}\Upsilon\right] - \chi(\tau)\lambda^2 u^0\right)\delta\zeta + \frac{1}{\tau}\chi(\tau)\left(\frac{\lambda s g_{ij}\beta(\tau)v^j}{u_0}\right)v^i + \frac{\lambda s \beta'(\tau) g_{0i} u^0}{\beta(\tau) u_0} v^i - \lambda \frac{s}{\beta(\tau)}\frac{u^0}{2} g^{ik}(\partial_i g_{k0} + \partial_\tau g_{ki} - \partial_k g_{i0}) - \mathtt{S}(\tau, \mathbf{U}) \tag{4.42}$$

and

$$S_i = \frac{1}{\tau}\left(-\frac{g_{ik}}{u_0}\left(1 - \frac{3\lambda s \bar{s}}{\bar{\lambda}} - \chi(\tau)\beta(\tau)\lambda s \delta\zeta\right) - \chi(\tau) M_{ki} u^0\right) v^k - 2\left(\frac{3(\lambda s - \bar{\lambda}\bar{s})\bar{s}}{\bar{\lambda}\beta(\tau)} + \chi(\tau)\lambda s \delta\zeta\right) g_{ij}\mathbf{u}^{0j} - \frac{1}{\tau}\frac{\tau g_{ij}}{\beta(\tau)}\left((1 + 6\bar{s}^2)\mathbf{u}^{0j} + \mathbf{u}_0^{0j}\right) + \frac{\eta_{00}\mathbf{u}_i^{00}}{2\beta(\tau)} + \mathtt{S}_i(\tau, \mathbf{U}, \mathbf{V}). \tag{4.43}$$

New transformation

Define

$$\mathbf{v}^k = v^k - Ag^{0k} = v^k - 2\tau A\mathbf{u}^{0k}, \tag{4.44}$$

$$A = A(\tau) = -\frac{3s^2[\bar{\alpha}(\tau)]}{\sqrt{-\eta^{00}}\beta(\tau)} = -\frac{3\bar{s}^2}{\omega\beta(\tau)}. \tag{4.45}$$

Then

$$\begin{aligned}\lambda s L_i^\mu \partial_\mu \delta\zeta + M_{ki} u^\mu \partial_\mu (\mathbf{v}^k) \quad = \quad & \frac{1}{\tau}\left(-\frac{g_{ik}}{u_0}\left(1 - \frac{3\lambda s\bar{s}}{\bar{\lambda}} - \chi(\tau)\beta(\tau)\lambda s\delta\zeta\right) - \chi(\tau)g_{ki}u^0\right)\mathbf{v}^k \\ & -\frac{g_{ij}}{\tau}\left(\frac{\tau}{\beta(\tau)}(1-3\bar{s}^2)(\mathbf{u}_0^{0j} + \mathbf{u}^{0j})\right) + \frac{1}{\tau}\frac{\tau\eta_{00}\mathbf{u}_i^{00}}{2\beta(\tau)} \\ & +\hat{\mathtt{S}}_i(\tau, \mathbf{U}, \tilde{\mathbf{V}}) \end{aligned} \tag{4.46}$$

and

$$\begin{aligned}\lambda^2 u^\mu \partial_\mu \delta\zeta + \lambda s L_i^\mu \partial_\mu \mathbf{v}^i \quad = \quad & \frac{1}{\tau}\left[3\lambda u^0\left(\Xi - \frac{\bar{s}}{\bar{\lambda}}\Upsilon\right) - \chi(\tau)\lambda^2 u^0\right]\delta\zeta \\ & +\frac{\chi(\tau)}{\tau}\left(\frac{\beta(\tau)\lambda s g_{ij}\mathbf{v}^j}{u_0}\right)\mathbf{v}^i + \hat{\mathtt{F}}(\tau, \mathbf{U}, \tilde{\mathbf{V}}), \end{aligned} \tag{4.47}$$

Final form

We get

$$D^{\mu}\partial_{\mu}\tilde{\mathbf{V}} = \frac{1}{\tau}\mathbf{D}\mathbf{P}^{\dagger}\tilde{\mathbf{V}} + \frac{1}{\tau}(\mathbf{E}_0\delta^0_{\mu} + \mathbf{E}_q\delta^q_{\mu})\mathbf{U}^{\mu} + F(\tau, \tilde{\mathbf{V}}, \mathbf{U}), \tag{4.48}$$

where

$$\tilde{\mathbf{V}} = (\delta\zeta, \mathbf{v}^p)^T, \qquad \mathbf{U}^{\mu} = (\mathbf{u}^{0\mu}_0, \mathbf{u}^{0\mu}_j, \mathbf{u}^{0\mu})^T \tag{4.49}$$

and

$$D^{\mu} = \begin{pmatrix} \lambda^2 u^{\mu} & \lambda s L^{\mu}_p \\ \lambda s L^{\mu}_r & M_{rp}u^{\mu} \end{pmatrix}, \quad \mathbf{E}_0 = \frac{\tau\eta_{00}}{2\beta(\tau)}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta^j_r & 0 \end{pmatrix},$$

$$\mathbf{D} = \begin{pmatrix} 3\lambda u^0[\Xi - \frac{\bar{s}}{\bar{\lambda}}\Upsilon] - \chi(\tau)\lambda^2 u^0 & \frac{\chi(\tau)\beta(\tau)\lambda s g_{ij}\mathbf{v}^j}{u_0} \\ \frac{\chi(\tau)\beta(\tau)\lambda s g_{rj}\mathbf{v}^j}{u_0} & -\frac{g_{ir}}{u_0}(1 - \frac{3\lambda s\bar{s}}{\bar{\lambda}}) - \chi(\tau)g_{ri}u^0, \end{pmatrix},$$

$$\mathbf{E}_q = -\frac{\tau}{\beta(\tau)}(1 - 3\bar{s}^2)\begin{pmatrix} 0 & 0 & 0 \\ g_{rq} & 0 & g_{rq} \end{pmatrix} \quad \text{and} \quad \mathbf{P}^{\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & \delta^i_p \end{pmatrix}.$$

and $F = (\hat{\mathtt{F}}, \hat{\mathtt{S}}_i)^T$.

Fluids (II)

We use lower index for the velocity and get

$$
\begin{aligned}
\lambda^2 u^\mu \partial_\mu(\alpha-\bar{\alpha}) + \lambda s L_i^\mu J^{iq}\partial_\mu u_q &= \frac{3}{\tau}\lambda^2 u^0\Big(\frac{s}{\lambda}-\frac{\bar{s}}{\bar{\lambda}}\Big) \\
&\quad - \lambda s L_i^\mu \left(\frac{\partial u^i}{\partial(g^{\alpha\beta})}\partial_\mu g^{\alpha\beta} + \Gamma^i_{\mu\nu}u^\nu\right), \qquad (4.50)
\end{aligned}
$$

$$
\begin{aligned}
M_{ki}u^\mu J^{kj}J^{iq}\partial_\mu u_q + \lambda s J^{ij}L_i^\mu \partial_\mu(\alpha-\bar{\alpha}) &= J^{jq}\Big[\frac{1}{\tau}\Big(-\frac{1}{u_0}+\frac{1}{u_0}\big(\frac{3\lambda s\bar{s}}{\bar{\lambda}}\big)\Big)u_q \\
&\quad - M_{ki}u^\mu\Big(\frac{\partial u^k}{\partial(g^{\alpha\beta})}\partial_\mu g^{\alpha\beta} + \Gamma^k_{\mu\nu}u^\nu\Big)\Big]. \qquad (4.51)
\end{aligned}
$$

In above,

$$J^{ij} = \frac{\partial u^i}{\partial u_j}.$$

Fluids (II)

$$\hat{D}^\mu \partial_\mu \hat{\mathbf{V}} = \frac{1}{\tau} \hat{\mathbf{D}} \hat{\mathbf{P}}^\dagger \hat{\mathbf{V}} + \hat{H}(\tau, \mathbf{U}, \hat{\mathbf{V}}), \tag{4.52}$$

where $\hat{\mathbf{V}} = (\delta\zeta, u_q)^T$, $\hat{D}^\mu$ and $\hat{H}$ are given by

$$\hat{D}^\mu = \frac{1}{\lambda^2 u^0} \begin{pmatrix} \lambda^2 u^\mu & \lambda s J^{iq} L_i^\mu \\ \lambda s J^{ij} L_i^\mu & M_{ki} J^{kj} J^{iq} u^\mu \end{pmatrix},$$

and

$$\hat{H} = \frac{1}{\lambda^2 u^0} \begin{pmatrix} -\lambda s L_i^\mu \left(\frac{\partial u^i}{\partial (g^{\alpha\beta})} \partial_\mu g^{\alpha\beta} + \Gamma^i_{\mu\nu} u^\nu \right) \\ -J^{ij} M_{ki} u^\mu \left(\frac{\partial u^k}{\partial (g^{\alpha\beta})} \partial_\mu g^{\alpha\beta} + \Gamma^k_{\mu\nu} u^\nu \right) \end{pmatrix}.$$

Decomposition

1. If $\frac{s}{\lambda} = \frac{\bar{s}}{\bar{\lambda}}$ and $1 - 3s^2 \geq \hat{\delta}$ hold, then set

$$\hat{\mathbf{D}} = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{\lambda^2 u^0 u_0}(1 - \frac{3\lambda s \bar{s}}{\bar{\lambda}})J^{ij} \end{pmatrix} \quad \text{and} \quad \hat{\mathbf{P}}^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & \delta_i^q \end{pmatrix}. \tag{4.53}$$

2. If $\frac{s}{\lambda} = \frac{\bar{s}}{\bar{\lambda}}$ and $1 - 3s^2 \equiv 0$ hold, then set

$$\hat{\mathbf{D}} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\lambda^2}\delta^{iq} \end{pmatrix} \quad \text{and} \quad \hat{\mathbf{P}}^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.54}$$

3. If $\frac{d}{d\bar{\alpha}}\left(\frac{\bar{s}}{\bar{\lambda}}\right) \geq \hat{\delta} > 0$, then take

$$\hat{\mathbf{D}} = \begin{pmatrix} 3\left[\frac{d}{d\bar{\alpha}}\left(\frac{\bar{s}}{\bar{\lambda}}\right) + \frac{d^2}{d\alpha^2}\left(\frac{s}{\lambda}\right)[\bar{\alpha} + K_8(\alpha - \bar{\alpha})](\delta\zeta)\right] & 0 \\ 0 & -\frac{1}{\lambda^2 u^0 u_0}(1 - \frac{3\lambda s \bar{s}}{\bar{\lambda}})J^{iq} \end{pmatrix} \tag{4.55}$$

$$\hat{\mathbf{P}}^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & \delta_i^j \end{pmatrix}. \tag{4.56}$$

The origin of the assumptions

$$
\begin{aligned}
g_{00} =& \eta_{00} + \tau \mathcal{S}_{00}(\tau, \mathbf{U}),\\
g_{0i} =& \tau S_{0i}(\tau, \mathbf{U}),\\
u^0 =& -\sqrt{-\eta^{00}} + \tau \mathcal{S}(\tau, \mathbf{U}) + \beta^2(\tau)\mathcal{W}(\tau, \mathbf{U}, \mathbf{V}) + \tau\beta(\tau)\mathcal{V}(\tau, \mathbf{U}, \mathbf{V}),\\
u_0 =& \frac{1}{\sqrt{-\eta^{00}}} + \tau \mathcal{S}(\tau, \mathbf{U}) + \beta^2(\tau)\mathcal{W}(\tau, \mathbf{U}, \mathbf{V}) + \tau\beta(\tau)\mathcal{V}(\tau, \mathbf{U}, \mathbf{V}),\\
u_i =& \beta(\tau) g_{ij} v^j + 2\tau \mathbf{u}^{0j} \frac{g_{kj} u^0}{g_{il} g^{0i} g^{0l} - g^{00}}\\
=& \beta(\tau) g_{ij} v^j + \tau \mathcal{S}_i(\tau, \mathbf{U}) + \beta^2(\tau)\mathcal{W}_i(\tau, \mathbf{U}, \mathbf{V}) + \tau\beta(\tau)\mathcal{V}_i(\tau, \mathbf{U}, \mathbf{V}).\\
s(\alpha) - s(\bar{\alpha}) =& \Xi(\tau, \alpha - \bar{\alpha})(\alpha - \bar{\alpha}) = \beta(\tau)\Xi(\tau, \beta(\tau)\delta\zeta)\delta\zeta\\
\lambda(\alpha) - \lambda(\bar{\alpha}) =& \Upsilon(\tau, \alpha - \bar{\alpha})(\alpha - \bar{\alpha}) = \beta(\tau)\Upsilon(\tau, \beta(\tau)\delta\zeta)\delta\zeta\\
M_{ik} =& g_{ki} + \beta^2(\tau)\mathcal{W}_{ki}(\tau, \mathbf{U}, \mathbf{V}) + \beta^3(\tau)\mathcal{U}_{ki}(\tau, \mathbf{U}, \mathbf{V}) + \tau\beta(\tau)\mathcal{V}_{ki}(\tau, \mathbf{U}, \mathbf{V})\\
L^0_i =& -\omega\beta\delta_{ij} v^j + \beta \mathcal{T}_i(\tau, \mathbf{U}, \mathbf{V}) + \tau \mathcal{S}_i(\tau, \mathbf{U}, \mathbf{V}) + \beta^2 \mathcal{W}_i(\tau, \mathbf{U}, \mathbf{V}) + \tau\beta\mathcal{V}_i(\tau, \mathbf{U}, \mathbf{V})
\end{aligned}
$$

Application to polytropic gas

We introduce the standard relationship between ρ and α which are

$$\rho = \varphi(\alpha) = \frac{1}{\left(4Kn(n+1)\right)^n}\alpha^{2n} \tag{4.57}$$

and

$$\lambda = \lambda(\alpha) = \left(1 + \frac{1}{4n(n+1)}\alpha^2\right)^{-1}. \tag{4.58}$$

Background solution

$$\bar{\alpha}(\tau) = \tau^{\frac{3}{2n}}\left(\frac{1}{\bar{\alpha}^2(1)} + \frac{1}{4n(n+1)} - \frac{1}{4n(n+1)}\tau^{\frac{3}{n}}\right)^{-\frac{1}{2}} \in [0, \bar{\alpha}(1)] \tag{4.59}$$

for $\tau \in [0, 1]$. Which shows that

$$\bar{\rho} \sim \tau^3.$$

Applications for Polytropic gases

♣ Choose $\beta(\tau) = \tau^{(3-\varepsilon)/(2n)} \in C[0,1] \cap C^1(0,1]$ for $\varepsilon \in (0,1]$

$$\partial_\tau \beta(\tau)\Big(\frac{d\big(\lambda(\alpha)+\beta(\tau)s(\alpha)\big)}{d\alpha} + s(\alpha)\Big)\Big|_{\alpha=\beta(\tau)y}$$
$$= \frac{3-\varepsilon}{2n}\tau^{\frac{3-\varepsilon}{n}-1}\Big(-\Big(1+\frac{1}{4n(n+1)}\tau^{(3-\varepsilon)/n}y^2\Big)^{-2}\frac{1}{2n(n+1)}y + \frac{1}{2n} + \frac{1}{2n}y\Big) \lesssim 1$$

$$\bar\lambda = \lambda(\bar\alpha) = \Big(1+\frac{1}{4n(n+1)}\bar\alpha^2\Big)^{-1} \quad \text{and} \quad \bar s = s(\bar\alpha) = \frac{\bar\alpha}{2n}. \tag{4.60}$$

♣Then, calculate quantity

$$3\Big(\frac{d\bar s}{d\bar\alpha} - \frac{\bar s}{\bar\lambda}\frac{d\bar\lambda}{d\bar\alpha}\Big) = \frac{3}{2n} + \Big(\frac{n}{3} + \frac{1}{\frac{3}{4n^2(n+1)}\bar\alpha^2}\Big)^{-1} \geq \frac{3}{2n} = \chi(\tau) + \frac{\varepsilon}{2n}, \tag{4.61}$$

$$1-3\bar s^2 = 1 - \frac{3\bar\alpha^2}{4n^2} \geq 1 - \frac{3\bar\alpha^2(1)}{4n^2}\begin{cases} = \chi(\tau) = \frac{3-\varepsilon}{2n}, & \text{if} \quad \bar\alpha(1) = 2n\sqrt{\frac{1}{3}\big(1-\frac{3-\varepsilon}{2n}\big)}, \\ > \chi(\tau) = \frac{3-\varepsilon}{2n}, & \text{if} \quad 0 < \bar\alpha(1) < 2n\sqrt{\frac{1}{3}\big(1-\frac{3-\varepsilon}{2n}\big)}. \end{cases}$$

Index n

In conclusion, we need $\beta^2(\tau) \leq \tau \leq \beta(\tau)$ and $1 - \frac{3-\varepsilon}{2n} > 0$, then

$$\frac{3-\epsilon}{2} < n \leq 3 - \epsilon.$$

Since $\varepsilon \in (0, 1]$, so $n \in (1, 3)$, i.e., $\gamma \in (\frac{4}{3}, 2)$.

Application to Chaplygin fluids

We take $\{\lambda(\alpha), \varrho, \Theta, \beta(\tau)\}$ as

$$\alpha(\rho) = \int_{\bar\rho(1)}^{\rho} \frac{d\xi}{\xi + p(\xi)} = \frac{1}{\vartheta+1} \ln \frac{(\rho+\Lambda)^{\vartheta+1} - \Lambda^{\vartheta+1}}{(\bar\rho(1)+\Lambda)^{\vartheta+1} - \Lambda^{\vartheta+1}}, \tag{4.62}$$

$$\lambda(\alpha) = \frac{\vartheta\Lambda^{1+\vartheta}}{\Lambda^{\vartheta+1} + [(\bar\rho(1)+\Lambda)^{\vartheta+1} - \Lambda^{\vartheta+1}]e^{(\vartheta+1)\alpha}} \tag{4.63}$$

$$\beta(\tau) = 1, \quad \Theta = 3(1+\vartheta) \quad \text{and} \quad \varrho(\tau, \alpha-\bar\alpha) = \tau^{-3(1+\vartheta)}(\mu(\alpha) - \mu(\bar\alpha)). \tag{4.64}$$

Thus,

$$\bar\alpha = \alpha(\bar\rho) = \frac{1}{\vartheta+1} \ln \frac{(\bar\rho+\Lambda)^{\vartheta+1} - \Lambda^{\vartheta+1}}{(\bar\rho(1)+\Lambda)^{\vartheta+1} - \Lambda^{\vartheta+1}} = \ln \tau^3, \tag{4.65}$$

$$\varphi(\alpha) = \Lambda\left\{1 + \left[(1+\frac{\bar\rho(1)}{\Lambda})^{\vartheta+1} - 1\right]e^{(\vartheta+1)\alpha}\right\}^{\frac{1}{\vartheta+1}} - \Lambda, \tag{4.66}$$

$$\varphi(\bar\alpha) = \Lambda\left\{1 + \left[(1+\frac{\bar\rho(1)}{\Lambda})^{\vartheta+1} - 1\right]\tau^{3(\vartheta+1)}\right\}^{\frac{1}{\vartheta+1}} - \Lambda, \tag{4.67}$$

Applications to Chaplygin fluids

$$
\begin{aligned}
s(\alpha) =&\lambda(\alpha) = \sqrt{p'(\rho)} = \frac{\vartheta\Lambda^{1+\vartheta}}{\Lambda^{\vartheta+1} + [(\bar{\rho}(1)+\Lambda)^{\vartheta+1} - \Lambda^{\vartheta+1}]e^{(\vartheta+1)\alpha}} \\
>&\frac{\vartheta}{1 + [(\frac{\bar{\rho}(1)}{\Lambda}+1)^{\vartheta+1} - 1]e^{(\vartheta+1)K_0}} > 0, \\
&3\Big(\frac{d\bar{s}}{d\bar{\alpha}} - \frac{\bar{s}}{\bar{\lambda}}\frac{d\bar{\lambda}}{d\bar{\alpha}}\Big) = 0, \qquad \chi(\tau) \equiv 0,
\end{aligned}
$$

and

$$
\begin{aligned}
1 - 3\bar{s}^2 =&1 - 3\Big(\frac{\vartheta\Lambda^{1+\vartheta}}{\Lambda^{\vartheta+1} + \tau^{3(1+\vartheta)}[(\bar{\rho}(1)+\Lambda)^{\vartheta+1} - \Lambda^{\vartheta+1}]}\Big)^2 \\
\geq&1 - 3\vartheta^2 \\
=&\begin{cases} 0, & \text{if} \quad \vartheta = \sqrt{\frac{1}{3}}, \\ 1 - 3\vartheta^2 > 0, & \text{if} \quad 0 < \vartheta < \sqrt{\frac{1}{3}}. \end{cases}
\end{aligned}
$$

Remaining problems (Global solution or blow up?)

- The gap of $n \in [3, +\infty)$ for polytropic gases.
- The case of $\vartheta \in (\sqrt{\frac{1}{3}}, 1]$.
- The case of $C_s^2 \in (\frac{1}{3}, 1]$

Thanks for your attention!