Global solutions to compressible Navier-Stokes-Poisson equations on exterior domains

Hairong Liu (刘海蓉)

Nanjing Forestry University （南京林业大学）

Joint work with Prof. Tao Luo and Hua Zhong

The 7th Young Researchers PDE Forum,
South China Normal University

Nov. 30, 2020

Outline

Navier-Stokes-Poisson system

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ \rho(u_t + u\cdot\nabla u) + \nabla p(\rho) = \rho\nabla\Phi + \mathrm{div}S - \alpha\rho u, \\ \triangle\Phi = \rho - \bar{\rho}, \end{cases} \tag{1}$$

- $\rho > 0$ — density, $u = (u^1, \cdots, u^n)$ — velocity,
- p — pressure, $p(\rho) = \rho^\gamma, \gamma \geq 1$,
- Φ — the electrostatic potential.
- The viscous stress tensor S is given by $S = \mu\left(\nabla u + (\nabla u)^T\right) + \lambda \mathrm{div} u I_n$, where μ and λ satisfy the physical conditions $\mu > 0$, $\lambda + \frac{2}{n}\mu \geq 0$.
- $\alpha \geq 0$ is a constant. When $\alpha > 0$, $\tau = \frac{1}{\alpha}$ is momentum relaxation time.
- $\bar{\rho} > 0$ is the background profile. We take $\bar{\rho} = 1$.

Initial boundary conditions

Let Ω be an exterior domain in $\mathbb{R}^n$ with compact smooth boundary. We consider the initial boundary value problem of (1) in the region $x \in \Omega,\ \ t \in [0, +\infty)$ with the following initial data

$$(\rho, u)(x, t = 0) = (\rho_0, u_0)(x), \tag{2}$$

and boundary conditions

$$u|_{\partial\Omega} = 0, \ \ \nabla\Phi \cdot \nu \,|_{\partial\Omega} = 0, \tag{3}$$

where ν is the exterior normal vector.

Related results on compressible Navier-Stokes equations on exterior domains

- The classical global existence of smooth solutions to the initial boundary value problem for 3-D compressible Navier-Stokes equations for initial date being small perturbations of constant states (Matsumura-Nishida, 1983 CMP).
- The decay estimate $\|\partial_x(\rho, v, \theta)\| = O(t^{-1/4})$ (Deckelnick, 1992 Math.Z.)
- The optimal rate of convergence of solutions as $t \to \infty$ under the assumption that the initial data $(\rho - \bar{\rho}, u_0, \theta_0 - \bar{\theta_0}) \in L^1$. (Kobayashi-Shibata, 1999 CMP)
- In the radially symmetric case, for the problem on a domain exterior to a ball, the smallness constraints on the initial perturbations are removed (S.Jiang, 1996 CMP).

Related results on compressible Navier-Stokes-Poisson equations

- The Cauchy problem (initial value problem without boundaries) of the compressible Navier-Stokes-Poisson system:
  - Optimal L^p-decay rate ($p \in [2, \infty]$) of the compressible NSP system in $\mathbb{R}^3$ (H.L.Li-Matsumura-G.J.Zhang, 2010 ARMA).
  - Optimal L^p-decay rate ($p \geq 1$), multi-dimensions ($n \geq 3$), (W.K.Wang-Z.G.Wu, 2010 JDE).
  - Optimal decay rate of the non-isentropic compressible NSP system in $\mathbb{R}^3$ (G.J. Zhang-H.L. Li-C.J. Zhu, 2011 JDE).
  - The global solutions with large data to the Cauchy problem of the 1-D compressible NSP system (Z.Tan-T.Yang-H.J. Zhao-Q.Y. Zou, 2013 SIAM JMA)

- For the asymptotic behavior when some physical parameters tend to zero, such as the quasi-neutral limit when the Debye length goes to zero, with or without physical boundaries. (L.Hsiao-T.Yang 2001; L.Hsiao-P.Markowich-S.Wang 2003; S.Wang-L.Jiang-C.Liu 2019; $\cdots$)
- The global existence of weak solutions of IBV problem of NSP on bounded domain in Sobolev framework (Donatelli 2003).

❷The radially symmetric solutions to NSP systems with large initial data

The radially symmetric formulation of NSP

Let $\Omega \equiv \{x \in \mathbb{R}^n, |x| > a\}$,

$$u^i(x,t) = \frac{x_i}{r}u(r,t),\ \ i = 1,\cdots,n,\ \ \rho(x,t) = \rho(r,t),\ \ \Phi(x,t) = \Phi(r,t).$$

Assume that

$$\rho_0(x) = \rho_0(r),\ \ u_0(x) = \frac{xu_0(r)}{r},$$

Let $\beta := \lambda + 2\mu$, IBV problem (1)-(3) are reduced as:

$$\begin{cases} \rho_t + \partial_r(\rho u) + \frac{n-1}{r}\rho u = 0, \quad r \in (a,+\infty), \quad t > 0, \\ \rho(u_t + uu_r) + p'(\rho)\rho_r = \beta(u_r + \frac{n-1}{r}u)_r + \rho\Phi_r, \\ \Phi_{rr} + \frac{n-1}{r}\Phi_r = \rho - 1, \\ \rho(r,0) = \rho_0(r),\ \ u(r,0) = u_0(r),\ \ r \in [a,+\infty), \\ u(a,t) = 0,\ \ \Phi_r(a,t) = 0,\ \ t \geq 0. \end{cases} \tag{4}$$

Lagrangian coordinates

The Eulerian coordinates (r, t) are connected to the Lagrangian coordinates (ζ, t) by the following relation

$$r(\zeta, t) := r_0(\zeta) + \int_0^t \tilde{u}(\zeta, \tau)d\tau,$$

where $\tilde{u}(\zeta, t) := u(r(\zeta, t), t)$ and

$$r_0(\zeta) := \eta^{-1}(\zeta), \ \eta(r) := \int_a^r s^{n-1}\rho_0(s)ds, \ r \in [a, +\infty).$$

For convenience and without the danger of confusion, $(\tilde{\rho}, \tilde{u}, \tilde{\Phi})$ is still denoted by (ρ, u, Φ) and (ζ, t) by (x, t).

The NSP systems in Lagrangian coordinates

In Lagrangian coordinates (x, t), let $v = \frac{1}{\rho}$, then (4) reads as

$$\begin{cases} v_t = (r^{n-1}u)_x, \\ u_t + r^{n-1}p_x = \beta r^{n-1}\left[\frac{(r^{n-1}u)_x}{v}\right]_x + \frac{r^{n-1}\Phi_x}{v}, \quad x \in (0, +\infty), \quad t > 0, \\ \left(\frac{r^{2n-2}\Phi_x}{v}\right)_x = 1 - v, \end{cases} \tag{5}$$

where $p(v) = v^{-\gamma}$, with the initial data and boundary condition

$$v(x, 0) = v_0(x), \;\; u(x, 0) = u_0(x), \;\; x \in [0, +\infty) \tag{6}$$

$$u(0, t) = 0, \;\; \Phi_x(0, t) = 0, \;\; t \geq 0, \tag{7}$$

$$v(x, t) \to 1, \;\; \Phi(x, t) \to 0, \quad \text{as} \quad x \to +\infty, \tag{8}$$

where $v_0 := \frac{1}{\rho_0}$.

$r = r(x,t)$ is determined by

$$r(x,t) = r_0(x) + \int_0^t u(x,\tau)d\tau, \quad x \in [0,+\infty), \quad t \geq 0,$$

$$r_0(x) = \left(a^n + n\int_0^x v_0(y)dy\right)^{1/n}.$$

The facts:

$$r_t(x,t) = u(x,t),$$

$$r^{n-1}(x,t)r_x(x,t) = v(x,t), \quad x \in [0,+\infty), \quad t \geq 0,$$

$$r(x,t) \geq r(0,t) = a > 0, \quad x \in [0,+\infty), \quad t \geq 0.$$

Main result: the global existence

Assume that

$$\left(r_0^{n-1}\partial_x\Phi_0, v_0 - 1, r_0^{n-1}\partial_x v_0, r_0^{2n-2}\partial_{xx}v_0, r_0^{3n-3}\partial_{xxx}v_0\right) \in L^2[0, +\infty);$$

$$\left(u_0, r_0^{n-1}\partial_x u_0, r_0^{2n-2}\partial_{xx}u_0, r_0^{3n-3}\partial_{xxx}u_0, r_0^{4n-4}\partial_{xxxx}u_0\right) \in L^2[0, +\infty). \tag{9}$$

Theorem (L-Luo-Zhong)

Let the initial data (v_0, u_0) be compatible with the boundary conditions (7) and satisfy conditions (9). Then the initial boundary value problem (5)-(8) admits a unique smooth solution (v, u, Φ_x) on $(x, t) \in [0, +\infty) \times [0, T]$ for every $T > 0$.

Difficulties

- Since the exterior domain is unbounded and the coefficients tend to infinity as $x \to +\infty$, some difficulties arise, for example, when $n > 1$, from the a priori estimates we could get only $u(x,t) = o(x^{-\frac{1}{2}+\frac{1}{2n}})$, but this is not sufficient to guarantee integration by parts where $u(x,t) = o(x^{-1+\frac{1}{n}})$ is required.
- For compreesible Navier-Stokes-Poisson equations involving Φ_x, the boundedness of v is subtle.

Lemma 1 (*the basic energy estimate*)

There is a positive constant E_0 independent of t, such that

$$\int_\Omega \left(\frac{1}{2}\rho|u|^2 + \tilde{A}(\rho) + \frac{1}{2}|\nabla\Phi|^2\right)(y,t)dy + \int_0^t \int_\Omega \left\{\mu|\nabla u|^2 + (\mu+\lambda)(\mathrm{div}u)^2\right\} dyds$$

$$= \int_\Omega \left(\frac{1}{2}\rho|u|^2 + \tilde{A}(\rho) + \frac{1}{2}|\nabla\Phi|^2\right)(y,0)dy,$$

where

$$\tilde{A}(\rho) = \begin{cases} \rho\ln\rho - \rho + 1, & \text{if } \gamma = 1, \\ \frac{\rho^\gamma}{\gamma-1} - \frac{\gamma}{\gamma-1}\rho + 1, & \text{if } \gamma > 1, \end{cases}$$

or equivalently,

$$\int_0^{+\infty}\left(\frac{1}{2}u^2+A(v)+\frac{1}{2}\frac{r^{2n-2}\Phi_x^2}{v}\right)(x,t)dx+\beta\int_0^t\int_0^{+\infty}\frac{\left((r^{n-1}u)_x\right)^2}{v}dxds$$
$$=\int_0^{+\infty}\left(\frac{1}{2}u^2+A(v)+\frac{1}{2}\frac{r^{2n-2}\Phi_x^2}{v}\right)(x,0)dx\equiv E_0,$$

where $A(v)$ is defined as

$$A(v)=\begin{cases} v-\ln v-1, & \text{if } \gamma=1,\\ \frac{v^{-\gamma+1}}{\gamma-1}+v-\frac{\gamma}{\gamma-1}, & \text{if } \gamma>1.\end{cases}$$

Remark

For $n=1$, the conservation of energy can also express as

$$\int_0^{+\infty}\left(\frac{1}{2}u^2+A(v)+\frac{1}{2}\left(\frac{\Phi_x}{v}\right)^2\right)(x,t)dx+\beta\int_0^t\int_0^{+\infty}\frac{u_x^2}{v}dxdt$$
$$=\int_0^{+\infty}\left(\frac{1}{2}u^2+A(v)+\frac{1}{2}\left(\frac{\Phi_x}{v}\right)^2\right)(x,0)dx.$$

Lemma 2 (*local bounds of the specific volume v*)

For each $t \geq 0$ and for any $i \in \mathbb{N}$ there exists a point $a_i(t) \in [i, i+1]$ such that

$$\alpha_1 \leq v(a_i(t), t) \leq \alpha_2, \quad \text{for } t \geq 0,\ i \in \mathbb{N}, \tag{10}$$

and

$$\alpha_1 \leq \int_i^{i+1} v(x,t)dx \leq \alpha_2, \quad \text{for } t \geq 0,\ i \in \mathbb{N}, \tag{11}$$

where $0 < \alpha_1 < 1$ and $1 < \alpha_2$ are two positive roots of the algebraic equation $A(y) = E_0$.

Lemma (Key lemma: pointwise bounds on v)

There are two positive constants $\underline{v}$ and $\bar{v}$ such that

$$\underline{v} \leq v(x,t) \leq \bar{v}, \quad \textit{for } t \in [0,T].$$

Sketch of the proof of the key lemma

With the help of

$$(\ln v)_{xt} = \left[\frac{(r^{n-1}u)_x}{v}\right]_x,$$

Then the momentum equation becomes

$$\frac{1}{\beta}r^{1-n}u_t = -\frac{1}{\beta}p_x + [\ln v]_{xt} + \frac{1}{\beta}\frac{\Phi_x}{v}.$$

$$\frac{1}{v(x,t)}\exp\left\{\frac{1}{\beta}\int_0^t p(v)(x,s)ds\right\} = \frac{1}{v(a_i(t),t)}Y_i(t)B_i(x,t), \quad t \in [0,T]$$

where

$$B_i(x,t) := \frac{v(a_i(t),0)}{v(x,0)}\exp\left\{-\frac{1}{\beta}\int_{a_i(t)}^x \int_0^t \left(r^{1-n}u_t - \frac{\Phi_x}{v}\right)dsdy\right\} \geq 0,$$

and

$$Y_i(t) := \exp\left\{\frac{1}{\beta}\int_0^t p(v)(a_i(t),s)ds\right\} \geq 1.$$

Goal: estimate each term of $B_i(x,t)$.

$$\left|\int_{a_i(t)}^{x}\int_0^t r^{1-n}u_t dsdy\right| = \left|\int_{a_i(t)}^{x}\int_0^t \frac{\partial(r^{1-n}u)}{\partial t} + (n-1)r^{-n}r_t udsdy\right|$$
$$\leq C(T).$$

Now, we estimate the term $\int_{a_i(t)}^{x}\int_0^t \frac{\Phi_x}{v}dsdy$.

For case $n=1$. Notice that Remark 1 implies $\left\|\frac{\Phi_x}{v}\right\| \leq E_0$. Hence, by using Cauchy inequality, one has

$$\int_{a_i(t)}^{x}\int_0^t \frac{\Phi_x}{v}dsdy \leq \int_i^{i+1}\int_0^t \left|\frac{\Phi_x}{v}\right| dsdy \leq \int_0^t \left\|\frac{\Phi_x}{v}\right\| ds \leq C(T).$$

For case $n \geq 2$. By using the third equation in (5) and the boundary condition (7), we get

$$\frac{\Phi_x}{v} = -r^{2-2n} \int_0^x (v-1)(y,t)dy. \tag{12}$$

Let

$$w(x,t) := \int_0^x (v-1)(y,t)dy.$$

Since w is equivalent to

$$w = \int_0^1 (v-1)dy + \int_1^2 (v-1)dy + \cdots + \int_{[x]}^x (v-1)dy,$$

which implies

$$\alpha_1[x] - x \leq w \leq \alpha_2([x]+1) - x,$$

by (11). Hence

$$|w| \leq C_1(x+1).$$

$$v=\left(\frac{r^n}{n}\right)_x \Rightarrow \frac{r^n(x,t)}{n}-\frac{a^n}{n}=w+x,$$

which gives us another expression for r

$$r=\left(n(w+x)+a^n\right)^{\frac{1}{n}}\geq C_2(x+1)^{\frac{1}{n}}.$$

Therefore, for $n\geq 2$, one has

$$\left|r^{2-2n}w\right|\leq C_1C_2^{2-2n}(x+1)^{\frac{2}{n}-1}.$$

Combining the above with (12) to obtain

$$\begin{aligned}\left|\int_{a_i(t)}^{x}\int_0^t\frac{\Phi_x}{v}dsdy\right| &= \left|\int_{a_i(t)}^{x}\int_0^t r^{2-2n}wdsdy\right|\\ &\leq \left|\int_i^{i+1}\int_0^t C_1C_2^{2-2n}(1+x)^{\frac{2}{n}-1}dsdy\right|\\ &\leq C(T).\end{aligned}$$

Lemma 4 (*Sobolev-norm estimates of derivatives for u, v*)

$$\int_0^t \int_0^{+\infty} \left(v_t^2 + (r^{n-1}u)_x^2 + r^{2n-2}u_x^2 \right) dxds \leq C(T),$$

$$\int_0^{+\infty} r^{2n-2} v_x^2 dx + \int_0^t \int_0^{+\infty} r^{2n-2} v_x^2 dxds \leq C(T).$$

$$\int_0^{+\infty} [v_t^2 + (r^{n-1}u)_x^2 + r^{2n-2}u_x^2] dx + \int_0^t \int_0^{+\infty} u_t^2 dxds \leq C(T),$$

$$\|u\|_{L^\infty([0,+\infty))} \leq C(T).$$

Lemma 5(high derivatives of u, v)

$$\int_0^t\int_0^{+\infty}\Big(r^{4n-4}u_{xx}^2+r^{2n-2}v_{xt}^2\Big)dxds\le C(T).$$

$$\|r^{n-1}\Phi_x\|_{L^\infty([0,+\infty))}+\int_0^{+\infty}\left\{\left(\frac{r^{n-1}\Phi_x}{v}\right)_t^2+r^{4n-4}\Phi_{xx}^2+r^{2n-2}\Phi_{xt}^2\right\}dx\le C(T).$$

$$\int_\Omega\rho|u_t|^2dy+\int_0^t\int_\Omega\left(\mu|\nabla u_t|^2+(\mu+\lambda)|\mathrm{div}u_t|^2\right)dyds\le C(T),$$

or equivalently,

$$\int_0^{+\infty}u_t^2dx+\int_0^t\int_0^{+\infty}\Big(v_{tt}^2+(r^{n-1}u)_{xt}^2+r^{2n-2}u_{xt}^2\Big)dxds\le C(T).$$

Lemma 6 (*high derivatives of u, v*)

$$\int_0^{+\infty}\Big(r^{2n-2}(r^{n-1}u)_{xx}^2+r^{4n-4}u_{xx}^2+r^{2n-2}v_{xt}^2\Big)dx\le C(T).$$

$$\int_0^{+\infty}\Big(r^{4n-4}v_{xx}^2+r^{6n-6}\Phi_{xxx}^2\Big)dx+\int_0^t\int_0^{+\infty}r^{4n-4}v_{xx}^2dxds\le C(T).$$

$$\int_\Omega\Big(|u_{tt}|^2+\rho_{tt}^2\Big)dy+\int_0^t\int_\Omega\Big(|\nabla u_{tt}|^2+|\mathrm{div}u_{tt}|^2\Big)dyds\le C(T).$$

or equivalently,

$$\int_0^{+\infty}\Big(v_{tt}^2+u_{tt}^2+r^{2n-2}u_{xt}^2\Big)dx\le C(T).$$

$$\int_0^{+\infty}r^{6n-6}u_{xxx}^2dx\le C(T).$$

The total estimates

$$\int_0^{+\infty}\Big\{u^2+(v-1)^2+r^{2n-2}\Phi_x^2+u_t^2+v_t^2+r^{2n-2}u_x^2+r^{2n-2}v_x^2$$
$$+r^{2n-2}u_{xt}^2+r^{2n-2}v_{xt}^2+r^{4n-4}u_{xx}^2+r^{4n-4}v_{xx}^2$$
$$+r^{6n-6}u_{xxx}^2+r^{4n-4}\Phi_{xx}^2+r^{6n-6}\Phi_{xxx}^2\Big\}dx\le C(T).$$

❸ Global existence and exponential stability of the NSP system on exterior domains

Main theorem

Theorem (L-Luo-Zhong)

Let Ω be an exterior domain in $\mathbb{R}^3$ with compact smooth boundary. Assume that $\rho_0 - 1, u_0 \in H^3(\Omega)$ and

$$\int_\Omega (\rho_0 - 1)dx = 0. \tag{13}$$

For $\alpha > 0$, there exists a constant $\delta > 0$ such that if

$$\|(\rho_0 - 1, u_0)\|_3 \le \delta,$$

then there exists a unique smooth global-in-time solution $(\rho, u, \nabla\Phi)$ to IBV problem (1)- (3). Moreover, there are positive constants C and σ such that

$$\left\{ \left\|(\rho - 1, u, \nabla\Phi)\right\|_3^2 + \|\rho_t\|_2^2 + \|u_t\|_1^2 \right\}(t) \le Ce^{-\sigma t}\left\{ \left\|(\rho - 1, u, \nabla\Phi)\right\|_3^2 + \|\rho_t\|_2^2 + \|u_t\|_1^2 \right\}(0).$$

The linearized problem

Let $q = \rho - 1$, then we have

$$\begin{cases} L^0 \equiv q_t + \mathrm{div} u + u \cdot \nabla q = -q \mathrm{div} u \equiv f^0, \\ L \equiv u_t + \gamma \nabla q - \nabla \Phi - \mu \Delta u - (\mu + \lambda) \nabla(\mathrm{div} u) + \alpha u = f, \\ \triangle \Phi = q, \end{cases} \tag{14}$$

where the nonlinear term is

$$f = \mu \frac{q}{q+1} \Delta u + (\mu + \lambda) \frac{q}{q+1} \nabla(\mathrm{div} u) - \gamma(\rho^{\gamma-1} - 1) \nabla q - u \cdot \nabla u.$$

(14) is enclosed with the initial data

$$(q, u)(\cdot, 0) = (q_0, u_0), \tag{15}$$

and the boundary condition

$$u|_{\partial\Omega} = 0, \quad \nabla \Phi \cdot \nu|_{\partial\Omega} = 0. \tag{16}$$

For clarity, we introduce

$$\mathcal{E}(t) = \|(q, u, \nabla\Phi)\|_3 + \|q_t\|_2 + \|u_t\|_1,$$

and

$$\mathcal{D}(t) = \|q\|_3 + \|\nabla\Phi\|_4 + \|u\|_4 + \|q_t\|_2 + \|u_t\|_2.$$

Proposition (*a priori estimates*)

Let (q, u, Φ) be a solution of the initial boundary value problem (14)-(16) in time interval $t \in [0, T]$. Then there exists positive constants δ and σ which are independent of t, such that if

$$\sup_{0\le t\le T} \mathcal{E}(t) \le \delta,$$

then there holds, for any $t \in [0, T]$,

$$\mathcal{E}(t) \le C\mathcal{E}(0)e^{-\sigma t}.$$

$$
\begin{aligned}
&\mathrm{div}u = -\frac{dq}{dt} - q\mathrm{div}u \equiv -\frac{dq}{dt} + f^0\\
&-\mu\Delta u + (\gamma\nabla q - \nabla\Phi) = -u_t + (\mu+\lambda)\nabla(\mathrm{div}u) - \alpha u + f,\\
&u|_{\partial\Omega} = u|_{\infty} = 0.
\end{aligned} \tag{17}
$$

Lemma 1

Let Ω be any exterior domain. Then for $k = 2, 3, 4$

$$
\|\nabla^k u\|^2 + \|\nabla^{k-1}(\gamma q - \Phi)\|^2 \le C\left\{\left\|\frac{dq}{dt}\right\|_{k-1}^2 + \|f^0\|_{k-1}^2 + \|u_t\|_{k-2}^2 + \|u\|_{k-2}^2 + \|f\|_{k-2}^2 + \|\nabla u\|^2\right\},
$$

where the last term on the right-hand side is necessary in the case of exterior domain.

Firstly, notice that differentiation of the system (14) with respect to t will keep the boundary conditions (16). Compute the integral

$$\int_\Omega \Big\{\partial_t^l (L^0 - f^0)\gamma \partial_t^l q + \partial_t^l (L - f)\cdot \partial_t^l u\Big\} dx = 0, \quad l = 0, 1,$$

we obtain

Lemma 2 (*estimates of* $(q, u, \nabla\Phi)$ *and* $(q_t, u_t, \nabla\Phi_t)$)

Under the conditions in Proposition, then for $l = 0, 1$, there exists a constant $C > 0$ independent of t such that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \Big\{\gamma|\partial_t^l q|^2 + |\partial_t^l u|^2 + |\partial_t^l \nabla\Phi|^2\Big\} dx + C\int_\Omega \Big\{|\partial_t^l \nabla u|^2 + \Big|\partial_t^l \frac{dq}{dt}\Big|^2 + |\partial_t^l u|^2\Big\} dx$$

$$\leq C\delta\mathcal{D}^2(t).$$

Lemma 3

Assume that the conditions in Proposition hold, then there exists a constant $C > 0$ independent of t such that for $k = 0, 1$,

$$\frac{d}{dt}\left\{\frac{1}{2}\int_\Omega\Big(\mu|\partial_t^k\nabla u|^2 + (\mu+\lambda)|\partial_t^k \mathrm{div} u|^2 + \alpha|\partial_t^k u|^2\Big)dx - \int_\Omega \gamma\partial_t^k q\partial_t^k \mathrm{div} u dx\right.$$
$$\left. - \int_\Omega \partial_t^k(\nabla\Phi)\cdot\partial_t^k u dx\right\} + C\int_\Omega\Big(\gamma|\partial_t^k q_t|^2 + |\partial_t^k u_t|^2\Big)dx \tag{18}$$
$$\leq C\int_\Omega\Big(|\partial_t^k\nabla u|^2 + |\partial_t^k u|^2\Big)dx + C\delta\mathcal{D}^2(t).$$

Lemma 4

Under the conditions in Proposition, then it holds that

$$\gamma\|q\|^2 + \|\nabla\Phi\|^2 + \|\nabla q\|^2 \leq C\Big(\|u\|^2 + \|u_t\|^2 + \|\Delta u\|^2 + \|\nabla \mathrm{div} u\|^2\Big) + C\delta\mathcal{D}^2(t).$$

We shall separate the estimates into that away from the boundary and that near the boundary. Let $\chi_0(x)$ be any fixed $C^\infty(\Omega)$ cut-off function such that *support*$\chi_0 \subset \Omega$ and $\chi_0 \equiv 1$ outside of a bounded region $O \subset \Omega$.

Lemma 5 (*the estimates on the region away from the boundary*)

Assume that the conditions in Proposition hold, for $k = 1, 2, 3$, it has

$$\frac{d}{dt}\left\{\frac{\gamma}{2}\int_\Omega \chi_0^2|\nabla^k q|^2 dx + \frac{1}{2}\int_\Omega \chi_0^2|\nabla^k u|^2 dx + \frac{1}{2\gamma}\int_\Omega \chi_0^2|\nabla^k \Phi|^2 dx - \int_\Omega \chi_0^2 \nabla^k q \cdot \nabla^k \Phi dx\right\}$$

$$+\int_\Omega \left\{\chi_0^2|\nabla^{k+1} u|^2 + \chi_0^2\left|\nabla^k \frac{dq}{dt}\right|^2 + \alpha\chi_0^2|\nabla^k u|^2\right\}dx + \frac{1}{8}\int_\Omega \chi_0^2|\gamma\nabla^k q - \nabla^k \Phi|^2 dx$$

$$\leq C\int_\Omega \left\{|\nabla^{k-1} u|^2 + |\nabla^{k-1} u_t|^2 + |\nabla^k u|^2\right\}dx + C\delta\mathcal{D}^2(t). \tag{19}$$

Our next goal is to establish the estimates near the boundary $\partial\Omega$. For this purpose, we choose a finite number of bounded open sets $\{O_j\}_{j=1}^N$ in $\mathbb{R}^3$ such that

$$\partial\Omega \subset \bigcup_{j=1}^{N} O_j,$$

The local geodesic coordinates (ξ, ζ, r) will be set up in each set O_j as follows:
(1) The boundary $O_j \cap \Omega$ is the image of smooth functions $z = z^i(\xi, \zeta)$ satisfying

$$|z_\xi| = 1, \quad z_\xi z_\zeta = 0, \quad |z_\zeta| \geq \widetilde{\tau} > 0,$$

where $\widetilde{\tau}$ is some positive constant independent of $j = 1, 2, \cdots, N$.
(2) Any x in O_j is represented by

$$x^i = x^i(\xi, \zeta, r) = rn^i(\xi, \zeta) + z^i(\xi, \zeta), \tag{20}$$

where $n^i(\xi, \zeta)$ is the external unit normal vector at the point of the boundary coordinate (ξ, ζ).

Lemma 6 (the tangential derivatives $\partial = (\partial_\xi, \partial_\zeta)$)

For any positive ε and $k = 1, 2, 3$, it holds that

$$\frac{d}{dt}\left\{\frac{\gamma}{2}\|\chi_j\partial^k q\|^2 + \frac{1}{2}\|\chi_j\partial^k u\|^2 + \frac{1}{2\gamma}\|\chi_j\partial^k \Phi\|^2 - \int_\Omega \chi_j^2\partial^k q\cdot\partial^k\Phi dx\right\}$$

$$+ C\left\{\|\chi_j\nabla\partial^k u\|^2 + \|\chi_j\partial^k\frac{dq}{dt}\|^2 + \alpha\|\chi_j\partial^k u\|^2\right\}$$

$$\leq C\left\{\varepsilon\|\gamma\nabla q - \nabla\Phi\|_{k-1}^2 + (1+\varepsilon^{-1})\|\nabla u\|_{k-1}^2 + \delta\mathcal{D}^2(t)\right\}.$$

Lemma 7(*the mixed derivatives*)

Under the conditions in Proposition, then for $k+l=0,1,2$, there exists a constant $C>0$ independent of t such that

$$\frac{d}{dt}\left\{\frac{\gamma}{2}\|\chi_j\partial^k\partial_r^l q_r\|^2+\frac{1}{2}\|\chi_j\partial^k\partial_r^l\Phi_r\|^2-\int_\Omega\chi_j^2\partial^k\partial_r^l q_r\partial^k\partial_r^l\Phi_r dx\right\}$$

$$+\frac{1}{2}\|\chi_j\partial^k\partial_r^l(\gamma q_r-\Phi_r)\|^2+\frac{2\mu+\lambda}{2}\left\|\chi_j\partial^k\partial_r^l\left(\frac{dq}{dt}\right)_r\right\|^2$$

$$\leq C\left\{\|\partial^k\partial_r^l u_t\|^2+\|\chi_j\partial^k\partial_r^l\nabla\partial u\|^2+\|\partial^k\partial_r^l u\|^2+\delta\mathcal{D}^2(t)\right\}.$$

Taking $\chi_j \partial^k (k=1,2)$ to Stokes equation (17), we have

$$
\begin{aligned}
&\operatorname{div}(\chi_j \partial^k u) = \chi_j \partial^k f^0 - \chi_j \partial^k \left(\frac{dq}{dt}\right) + \nabla \chi_j \cdot \partial^k u, \\
&-\mu \Delta(\chi_j \partial^k u) + \nabla\Big(\chi_j \partial^k (\gamma q - \Phi)\Big) = -\chi_j \partial^k u_t - \alpha \chi_j \partial^k u + (\gamma \partial^k q - \partial^k \Phi) \nabla \chi_j \\
&-\mu \Delta \chi_j \partial^k u - 2\mu \nabla \chi_j \partial^k \nabla u + \chi_j \partial^k f - (\mu+\lambda) \chi_j \partial^k \left(\nabla \frac{dq}{dt}\right) + (\mu+\lambda) \chi_j \partial^k \nabla f^0 \\
&\chi_j \partial^k u|_{\partial\Omega} = 0.
\end{aligned}
\tag{21}
$$

Lemma 8

For $l = 0, 1, 2$, it holds

$$\|\nabla^{2+l}u\|^2 + \|\nabla^{1+l}(\gamma q - \Phi)\|^2 \le C\left\{ \left\|\frac{dq}{dt}\right\|_{1+l}^2 + \|f^0\|_{1+l}^2 + \|u_t\|_l^2 + \alpha\|u\|_l^2 + \|f\|_l^2 + \|\nabla u\|^2 \right\}.$$

And for $k = 1, 2$, $l + k = 1, 2$, it holds

$$\|\chi_j\nabla^{2+l}\partial^k u\|^2 + \|\chi_j\nabla^{1+l}\partial^k(\gamma q - \Phi)\|^2$$

$$\le C\left\{ \left\|\chi_j\partial^k\frac{dq}{dt}\right\|_{1+l}^2 + \|f^0\|_{1+l+k}^2 + \|u_t\|_{l+k}^2 + \|u\|_{l+k}^2 + \|f\|_{l+k}^2 + \|\gamma\nabla q - \nabla\Phi\|_{l+k-1}^2 + \|\nabla u\|_{l+k}^2 \right\}.$$

Outlines of the proof of main theorem

Step 1. Combining the above lemmas, we arrive at

$$\frac{1}{2}\frac{d}{dt}\Big\{\sum_{l=0}^{1}\Big(\gamma\|\partial_t^l q\|^2+\|\partial_t^l u\|^2+\|\nabla\partial_t^l\Phi\|^2\Big)$$

$$+\frac{\eta}{2}\sum_{l=0}^{1}\Big(\mu\|\partial_t^l\nabla u\|^2+(\mu+\lambda)\|\partial_t^l \mathrm{div} u\|^2+\alpha\|\partial_t^l u\|^2-\int_\Omega\gamma\partial_t^l q\partial_t^l \mathrm{div} u dx-\int_\Omega\partial_t^l(\nabla\Phi)\cdot\partial_t^l u dx\Big)$$

$$+\frac{\eta^2}{2}\sum_{k=1}^{3}\Big(\gamma\|\chi_0\nabla^k q\|^2 dx+\frac{1}{\gamma}\|\chi_0\nabla^k\Phi\|^2+\|\chi_0\nabla^k u\|^2-2\int_\Omega\chi_0^2\nabla^k q\cdot\nabla^k\Phi dx\Big)$$

$$+\frac{\eta}{2}\sum_{k=1}^{3}\Big(\gamma\|\chi_j\partial^k q\|^2+\|\chi_j\partial^k u\|^2+\frac{1}{\gamma}\|\chi_j\partial^k\Phi\|^2-2\int_\Omega\chi_j^2\partial^k q\cdot\partial^k\Phi dx\Big)$$

$$+\frac{\eta^2}{2}\sum_{k+l=0}^{2}\Big(\gamma\|\chi_j\partial^k\partial_r^l q_r\|^2+\frac{1}{\gamma}\|\chi_j\partial^k\partial_r^l\Phi_r\|^2-2\int_\Omega\chi_j^2\partial^k\partial_r^l q_r\partial^k\partial_r^l\Phi_r dx\Big)\Big\}$$

$$+C\Big\{\|u\|_1^2+\|u_t\|_1^2+\Big\|\frac{dq}{dt}\Big\|_3^2+\|q_{tt}\|^2+\|u_{tt}\|^2\Big\}$$

$$\leq C\delta\mathcal{D}^2(t).$$

Step 2. Denote the left-hand side on the above inequality as $\frac{d}{dt}\widetilde{\mathcal{E}}^2(t) + C\widetilde{\mathcal{D}}^2(t)$, that is

$$\frac{d}{dt}\widetilde{\mathcal{E}}^2(t) + C\widetilde{\mathcal{D}}^2(t) \leq C\delta\mathcal{D}^2(t). \tag{22}$$

Hölder's inequality and Lemma 1 yields

$$C^{-1}\mathcal{E}^2(t) \leq \widetilde{\mathcal{E}}^2(t) \leq C\mathcal{E}^2(t). \tag{23}$$

We need to show that $\mathcal{D}^2(t) \leq C\widetilde{\mathcal{D}}^2(t)$, where

$$\mathcal{D}^2(t) \equiv \|q\|_3^2 + \|\nabla\Phi\|_4^2 + \|u\|_4^2 + \|q_t\|_2^2 + \|u_t\|_2^2,$$

$$\widetilde{\mathcal{D}}^2(t) \equiv \|u\|_1^2 + \|u_t\|_1^2 + \left\|\frac{dq}{dt}\right\|_3^2 + \|q_{tt}\|^2 + \|u_{tt}\|^2.$$

Noting that $\|u\|_4^2$ and $\|u_t\|_2^2$ are bounded by $C\widetilde{\mathcal{D}}^2(t) + C\delta\mathcal{D}^2(t)$ due to Lemma 1 and Lemma 8.

Estimate the term $\|q\|_3$
Lemma 4 means that

$$\|q\|^2 + \|\nabla q\|^2 + \|\nabla\Phi\|^2 \le C\widetilde{\mathcal{D}}^2(t) + C\delta\mathcal{D}^2(t).$$

Noting Lemma 8 with $l = 1$ tells

$$\|\gamma\nabla^2 q - \nabla^2\Phi\|^2 \le C\widetilde{\mathcal{D}}^2(t) + C\delta\mathcal{D}^2(t),$$

which together with the elliptic estimate: $\|\nabla^2\Phi\|^2 \le C\Big(\|q\|^2 + \|\nabla\Phi\|^2\Big)$ yields

$$\begin{aligned}\|\nabla^2 q\|^2 &\le \frac{1}{\gamma}\|\gamma\nabla^2 q - \nabla^2\Phi\|^2 + \frac{1}{\gamma}\|\nabla^2\Phi\|^2 \\ &\le C\Big(\|\gamma\nabla^2 q - \nabla^2\Phi\|^2 + \|q\|^2 + \|\nabla\Phi\|^2\Big) \\ &\le C\widetilde{\mathcal{D}}^2(t) + C\delta\mathcal{D}^2(t).\end{aligned}$$

Similarly, by using : $\|\nabla^3\Phi\| \le C\Big(\|q\|_1^2 + \|\nabla\Phi\|^2\Big)$, it holds

$$\begin{aligned}\|\nabla^3 q\|^2 &\le C\Big(\|\gamma\nabla^3 q - \nabla^3\Phi\|^2 + \|q\|_1^2 + \|\nabla\Phi\|^2\Big)\\ &\le C\widetilde{\mathcal{D}}^2(t) + C\delta\mathcal{D}^2(t).\end{aligned}$$

Furthermore, the elliptic estimates implies $\|\nabla^4\Phi\|^2 \le \|q\|_2^2 + \|\nabla\Phi\|^2$, so it arrives at

$$\mathcal{D}^2(t) \le C\widetilde{\mathcal{D}}^2(t) + C\delta\mathcal{D}^2(t),$$

which implies

$$\mathcal{D}^2(t) \le C\widetilde{\mathcal{D}}^2(t). \tag{24}$$

Then, putting (24) and (23) into (22), we obtain

$$\frac{d}{dt}\widetilde{\mathcal{E}}^2(t) + \sigma\widetilde{\mathcal{E}}^2(t) \le 0,$$

where $\sigma > 0$ is a constant independent of t, which gives

$$\widetilde{\mathcal{E}}(t) \le e^{-\sigma t}\widetilde{\mathcal{E}}(0).$$

Then

$$\mathcal{E}(t) \le Ce^{-\sigma t}\mathcal{E}(0).$$

Thank you very much for your attention!