The constant rank Theorem

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A technique to study the convexity of solutions to PDEs.

Elliptic equations: $F(D^2u, Du, u, x) = 0$, $\{F^{\alpha\beta}\} := \{\frac{\partial F}{\partial u_{\alpha\beta}}\} > 0$
Parabolic equations: $F(D^2u, Du, u, x, t) = u_t$

The classical solution u is (strictly) convex
$\Leftrightarrow P(x,y) =: u(\frac{x+y}{2}) - \frac{u(x)+u(y)}{2} \leq (<)0$
$\Leftrightarrow D^2u := \{\frac{\partial^2 u}{\partial x_i \partial x_j}\} \geq (>)0.$

The level set $\Sigma^c := \{x \in \Omega : u(x) = c\}$ is (strictly) convex
$\Leftrightarrow Q(x,y) =: u(\frac{x+y}{2}) - \min\{u(x), u(y)\} \geq (>)0$
$\Leftrightarrow II_{\Sigma^c} \geq (>)0.$

Introduction: "The constant rank Theorem"

The idea is from Caffarelli-Friedman [Duke Math. J., 1985] and Singer-Wong-Yau-Yau [Ann. Scuola Norm. Sup. Pisa,1985].

Consider a semipositive definite matrix $W = (W_{ij})_{N\times N}$, where $W_{ij} = W_{ij}(D^2u, Du, u, x)$ (W can also depend on t) satisfies the following conditions
(i) (W_{ij}) is diagonal or partial diagonal under a suitable coordinate,
(ii) $W_{ij} \in C^2$.
Suppose $l = \min\limits_{x\in\Omega} rank(W(x))$ is attained at some point $x_o \in \Omega$. We want to prove that W is of rank l at any $x \in \Omega$.

Introduction: "The constant rank Theorem"

So we consider a suitable test function

$$\phi = \sigma_{l+1}(W) \quad \text{or} \quad \phi = \sigma_{l+1}(W) + \frac{\sigma_{l+2}(W)}{\sigma_{l+1}(W)}, \tag{0.1}$$

where $\sigma_k(\lambda) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le N} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$, and prove a differential inequality

$$\begin{cases} \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} \phi_{\alpha\beta} \leqslant C_1 \phi + C_2 |\nabla \phi| & \text{in} \quad \mathcal{N}_{x_0}, \\ \phi(x_0) = 0, & \\ \phi \geqslant 0 & \text{in} \quad \mathcal{N}_{x_0}. \end{cases} \tag{0.2}$$

Utilizing the strong maximum principle and the continuous method, we can obtain

$$\sigma_{l+1}(W)(x) \equiv 0 \text{ in } \Omega. \tag{0.3}$$

Hence we can get *W is of constant rank l in Ω.*

The idea of Constant rank theorem

For the parabolic equations, assume $l = \min_{(x,t)\in\Omega\times(0,T]} rank(W(x,t))$ is attained at some point $(x_0, t_0) \in \Omega \times (0, T]$, and try to prove

$$\begin{cases} \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta}\phi_{\alpha\beta} - \phi_t \leqslant C_1\phi + C_2 |\nabla\phi| & \text{in} \quad \mathcal{N}_{x_0} \times (t_0 - \delta, t_0], \\ \phi(x_0, t_0) = 0, & \\ \phi \geqslant 0 & \text{in} \quad \mathcal{N}_{x_0} \times (t_0 - \delta, t_0], \end{cases} \tag{0.4}$$

Utilizing the strong maximum principle and the continuous method, we can obtain

$$\sigma_{l+1}(W)(x,t) \equiv 0 \text{ in } \Omega \times (0, t_0]. \tag{0.5}$$

Hence W is of constant rank l in $\Omega \times (0, t_0]$. Moreover, let $l(t)$ be the minimal rank of W in Ω, then $l(s) \leqslant l(t)$ for all $0 < s \leqslant t \leqslant T$.

Example 1:

Theorem (Makar-Limanov: Math. Notes Acad. Sci. USSR.1971)

Assume Ω is a C^2 convex domain, and u is the classical solution of

$$\begin{cases} \Delta u = 1, \ \textit{in} \ \ \Omega \subset \mathbb{R}^n \\ u = 0, \quad \ \ \textit{on} \ \ \partial\Omega \end{cases}$$

Then $-(-u)^{\frac{1}{2}}$ are strictly convex.

Theorem (Brascamp-Lieb: JFA 1976)

Assume Ω is a C^2 convex domain, and u is the classical solution of eigenvalue problem

$$\begin{cases} \Delta u = \lambda_1(-u), \ \textit{in} \ \ \Omega \subset \mathbb{R}^n \\ u = 0, \quad \ \ \textit{on} \ \ \partial\Omega \end{cases}$$

Then $-\log(-u)$ are strictly convex.

Example 1: Idea of proof

Step 1: Constant rank Theorem: $D^2 v \geq 0 \Rightarrow Rank D^2 v = constant$.

From the regularity theory, $u \in C^\infty(\Omega) \cap C^2(\overline{\Omega})$.
Let $v = -(-u)^{\frac{1}{2}}$, then v satisfies $2(-v)\Delta v - 2|\nabla v|^2 = 1$.
Assume the minimum rank l of $D^2 v$ is attained at $x_0 \in \Omega$, and $l \leqslant n-1$. For a small neighborhood $\mathcal{N}_{x_0}$ and any fixed point $x \in \mathcal{N}_{x_0}$, we can rotate the coordinates such that

$$D^2 v \text{ is diagonal.} \tag{0.6}$$

Also, we can assume $v_{11} \geq v_{22} \geq \cdots \geq v_{nn}$. Denote $G = \{v_{11}, \cdots, v_{ll}\}$, $\quad B = \{v_{l+1 l+1}, \cdots, v_{nn}\}$, and $G = \{1, \cdots, l\}$, $B = \{l+1, \cdots, n\}$. Then $v_{ii} \geq \delta > 0$ for $i \in G$ if $\mathcal{N}_{x_0}$ small enough.
Let

$$\phi(x) = \sigma_{l+1}(D^2 v), \tag{0.7}$$

Example 1: Idea of proof

If $h(x) \leq C(\phi + |\nabla\phi|)$, $\forall x \in \mathcal{N}_{x_0}$, we say $h \sim 0$.
Also $h \sim 0 \quad$ if $\quad h \lesssim 0$, and $\ -h \lesssim 0$.
Since $\phi = \sigma_{l+1}(D^2v) \geq \sigma_l(G) \sum\limits_{i\in B} v_{ii}$, then

$$v_{ii} \sim 0, \ \ \forall i \in B.$$

Taking the first derivatives of ϕ, we get

$$\begin{aligned} \phi_\alpha &= \sum_{i=1}^n \frac{\partial\sigma_{l+1}(D^2v)}{\partial v_{ii}} v_{ii\alpha} \\ &= \sum_{i\in G} \sigma_l(D^2v|i) v_{ii\alpha} + \sum_{i\in B} \sigma_l(D^2v|i) v_{ii\alpha} \\ &\sim \sigma_l(G) \sum_{i\in B} v_{ii\alpha}, \end{aligned}$$

Then

$$\sum_{i\in B} v_{ii\alpha} \sim 0.$$

Example 1: Idea of proof

$$
\begin{aligned}
2(-v)\Delta\phi =& 2(-v)\Big[\sum_{i=1}^{n}\frac{\partial\sigma_{l+1}(D^2v)}{\partial v_{ii}}v_{ii\alpha\alpha}+\sum_{i,j,k,l=1}^{n-1}\frac{\partial^2\sigma_{l+1}(D^2v)}{\partial v_{ij}\partial v_{kl}}v_{ij\alpha}v_{kl\alpha}\Big]\\
\sim& 2(-v)\sigma_l(G)\sum_{j\in B}\left[v_{jj\alpha\alpha}-2\sum_{i\in G}\frac{{v_{ij\alpha}}^2}{v_{ii}}\right]\\
\lesssim& \sigma_l(G)\sum_{j\in B}\left[4v_j\Delta v_j-4(-v)\sum_{i\in G}\frac{{v_{iij}}^2}{v_{ii}}\right]\\
\lesssim& \sigma_l(G)\sum_{j\in B}\left[4\frac{v_j^2}{-v}\Delta v-4(-v)\frac{(\sum_{i\in G}v_{iij})^2}{\sum_{i\in G}v_{ii}}\right]\\
\sim& 0.
\end{aligned}
\tag{0.8}
$$

Example 1: Idea of proof

Applying the strong maximum principle, and we obtain

$$\phi(x) = \sigma_{l+1}(D^2 v) \equiv 0, \quad x \in \mathcal{N}_{x_0}. \tag{0.9}$$

By the continuity method, $Rank\{D^2 v\} = l$ in Ω.

Step 2: Full rank Theorem: $RankD^2 v = const. \Rightarrow D^2 v > 0.$

By a result of Caffarelli-Spruck [CPDE, 1982] or Korevaar [Indiana, 1983], v is strictly convex near $\partial\Omega$.

Hence $l = n$ and v is strictly convex in Ω.

Example 1: Idea of proof

Step 3: Deformation Theorem: $\Rightarrow D^2 v > 0$.

- $\Omega = B_1(0)$, $u = \frac{1}{2n}|x|^2 - \frac{1}{2n}$, and then $v = -\sqrt{\frac{1}{2n} - \frac{1}{2n}|x|^2}$ and

$$D^2 v = \sqrt{\frac{1}{2n}}\left(\frac{1}{2n} - \frac{1}{2n}|x|^2\right)^{-\frac{3}{2}}[(1-|x|^2)\delta_{ij} + x_i x_j] > 0$$

- $\Omega_t = (1-t)B_1(0) + t\Omega$, u^t is the solution, $v^t = -(-u^t)^{\frac{1}{2}}$.
- $t = 0$, holds. If $t \to t_0 \in (0,1]$, such that

$$D^2 v^{t_0} \geq 0, \text{ but not } > 0,$$

it is a contradiction by Full rank Theorem. Hence $t = 1$, holds.

Example 2: level sets of harmonic function in convex rings

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega = \Omega_0 \backslash \overline{\Omega_1}, \\ u = 0 & \text{on} \quad \partial\Omega_0, \\ u = 1 & \text{in} \quad \partial\Omega_1, \end{cases} \tag{0.10}$$

where $\Omega = \Omega_0 \setminus \overline{\Omega}_1$ is a C^2 convex ring in $\mathbb{R}^n$ ($n \geq 2$), i.e. Ω_0 and Ω_1 are C^2 bounded convex domains, and $\overline{\Omega}_1 \subset \Omega_0$.

Theorem (Gabriel: JLMS 1957; Lewis: ARMA 1977)

For any $c \in (0, 1)$, the level sets $\Sigma^c = \{x \in \Omega : u(x) = c\}$ are strictly convex.

Gabriel: JLMS 1957: Concavity function method

$$Q(x, y) = u(\frac{x+y}{2}) - \min\{u(x), u(y)\}, \qquad (x, y) \in \Omega \times \Omega$$

Lewis: ARMA 1977: p-harmonic function, strict convexity

Example 2: Idea of proof

Step 1: Constant rank Theorem: $II_{\Sigma^c} \geq 0 \Rightarrow Rank\ II_{\Sigma^c} = constant$.

From the regularity theory, $u \in C^\infty(\Omega) \cap C^2(\overline{\Omega})$.
From Kawohl [book, 1985], $|\nabla u| \neq 0$ in Ω.
Suppose $a(x) = \{a_{ij}(x)\}_{n-1\times n-1}$ be the Weingarten curvature matrix of level set Σ^c, and $a \geq 0$. By rotating the coordinates, such that $u_n \neq 0$, then

$$a_{ij} = -\frac{|u_n|}{|\nabla u|{u_n}^3} A_{ij}, \quad 1 \leq i, j \leq n-1, \tag{0.11}$$

where

$$\begin{aligned} A_{ij} =& h_{ij} - \frac{u_i u_l h_{jl}}{W(1+W)u_n^2} - \frac{u_j u_l h_{il}}{W(1+W)u_n^2} + \frac{u_i u_j u_k u_l h_{kl}}{W^2(1+W)^2 u_n^4}, \\ h_{ij} =& u_n^2 u_{ij} + u_{nn} u_i u_j - u_n u_j u_{in} - u_n u_i u_{jn}, \\ W =& \frac{|\nabla u|}{|u_n|}. \end{aligned}$$

Example 2: Idea of proof

Assume the minimum rank l of $a(x)$ is attained at $x_0 \in \Omega$, and $l \leqslant n-2$. For a small neighborhood $\mathcal{N}_{x_0}$ and any fixed point $x \in \mathcal{N}_{x_0}$, we can rotate the coordinates such that

$$|\nabla u(x)| = u_n(x) > 0, \quad \{u_{ij}\}_{1\le i,j\le n-1} \text{is diagonal.} \tag{0.12}$$

Then $a_{ij} = -\frac{1}{{u_n}^3} h_{ij} = -\frac{1}{u_n} u_{ij}$ is diagonal. Also, we can assume $u_{11} \le u_{22} \le \cdots \le u_{n-1n-1}$. Denote $G = \{a_{11}, \cdots, a_{ll}\}$, $B = \{a_{l+1,l+1}, \cdots, a_{n-1,n-1}\}$, and $G = \{1, \cdots, l\}$, $B = \{l+1, \cdots, n-1\}$. Then $a_{ii} \ge \delta > 0$ for $i \in G$ if $\mathcal{N}_{x_0}$ small enough. Let

$$\phi(x) = \sigma_{l+1}(a_{ij}), \tag{0.13}$$

If $h(x) \le C_1\phi + C_2|\nabla\phi|$, $\forall x \in \mathcal{N}_{x_0}$, we say $h \lesssim 0$.
Also $h \sim 0 \quad \text{if} \quad h \lesssim 0, \text{ and } -h \lesssim 0$.

Example 2: Idea of proof

Since $\phi = \sigma_{l+1}(a_{ij}) \geq \sigma_l(G) \sum_{i\in B} a_{ii}$, then

$$a_{ii} \sim 0, h_{ii} \sim 0, u_{ii} \sim 0, \quad \forall i \in B.$$

Taking the first derivatives of ϕ, we get

$$\begin{aligned}
\phi_\alpha &= \sum_{ij=1}^{n-1} \frac{\partial \sigma_{l+1}(a)}{\partial a_{ij}} a_{ij,\alpha} = \sum_{i\in G} \sigma_l(a|i) a_{ii,\alpha} + \sum_{i\in B} \sigma_l(a|i) a_{ii,\alpha} \\
&\sim \sigma_l(G) \sum_{i\in B} a_{ii,\alpha} \sim -u_n^{-3} \sigma_l(G) \sum_{i\in B} h_{ii,\alpha} \\
&\sim -u_n^{-3} \sigma_l(G) \sum_{i\in B} [u_n^2 u_{ii\alpha} - 2u_n u_{in} u_{i\alpha}],
\end{aligned}$$

hence

$$\sum_{i\in B} a_{ii,\alpha} \sim 0, \quad \sum_{i\in B} h_{ii,\alpha} \sim 0, \quad \sum_{i\in B} [u_n^2 u_{ii\alpha} - 2u_n u_{in} u_{i\alpha}] \sim 0. \quad (0.14)$$

Example 2: Idea of proof

$$\begin{aligned}
\Delta\phi &= \sum_{i,j=1}^{n-1}\frac{\partial\sigma_{l+1}(a)}{\partial a_{ij}}a_{ij,\alpha\alpha}+\sum_{i,j,k,l=1}^{n-1}\frac{\partial^2\sigma_{l+1}(a)}{\partial a_{ij}\partial a_{kl}}a_{ij,\alpha}a_{kl,\alpha}\\
&\sim \sigma_l(G)\sum_{j\in B}\left[a_{jj,\alpha\alpha}-2\sum_{i\in G}\frac{a_{ij,\alpha}a_{ij,\alpha}}{a_{ii}}\right]\\
&\sim \sigma_l(G)\frac{2}{u_n^3}\sum_{j\in B,i\in G}\sum_{\alpha=1}^{n}\frac{[u_nu_{ij\alpha}-2u_{i\alpha}u_{jn}]^2}{u_{ii}}\\
&\leq C(\phi+|\nabla\phi|). \qquad (0.15)
\end{aligned}$$

Applying the strong maximum principle, and the continuity method, $Rank\{a_{ij}\}=l$ in Ω.

Example 2: Idea of proof

Step 2: Full rank Theorem: $II_{\Sigma^c} \geq 0 \Rightarrow II_{\Sigma^c} > 0$.

Σ^c are $(n-1)$ dimensional closed hypersurface, so $l = n-1$ (level sets are strictly convex).

Step 3: Deformation Theorem: $\Rightarrow II_{\Sigma^c} > 0$.

- $\Omega = B_2(0) \setminus \overline{B_1(0)}$, $u = -\frac{2^{n-1}}{2^{n-1}+1}\frac{1}{|x|^{n-1}} + \frac{1}{2^{n-1}+1}$, and then $\{x : u(x) = c\}$ is a sphere.
- $\Omega_t = (1-t)B_2(0) \setminus \overline{B_1(0)} + t\Omega$, u^t is the solution.
- $t = 0$, holds. If $t \to t_0 \in (0, 1]$, such that

$$II_{u^{t_0}} \geq 0, \text{ but not } > 0,$$

it is a contradiction by Full rank Theorem. Hence $t = 1$, holds.

Background

Convexity of solutions is an important geometric property, and there is a vast literature devoted to it. There are two important methods

- **macroscopic methods**: weak maximum principle, including concavity function method, and convex envelope method.

$$P(x,y) = u(\frac{x+y}{2}) - \frac{u(x)+u(y)}{2}$$

$$Q(x,y) = u(\frac{x+y}{2}) - \min\{u(x), u(y)\}$$

- **microscopic methods**: strong maximum principle, mainly is the constant rank theorem.

$$Rank\{W_{ij}\} = constant$$

Background

For the elliptic equations and parabolic equations, the convexity of the solution

- (spatial) convex solutions $\dashleftarrow\dashrightarrow$ $D_x^2 u \geq 0$
  - Semilinear or quasilinear elliptic equations: Korevaar[Indiana, 1983], Kennington[Indiana, 1985], Kawohl[Math. Meth. Appl, 1986]; Caffarelli-Friedman[Duke, 1985], Korevaar-Lewis[ARMA, 1987]
  - Eigenvalue problem: Brascamp-Lieb[JFA, 1976], Caffarelli-Spruck[CPDE, 1982], Singer-Wong-Yau-Yau[Ann. Scuola Norm. Sup. Pisa, 1985], Liu-Ma-Xu [Adv.Math, 2010],
  - Christoffel-Minkowski problems: Guan-Ma[Invent. Math, 2003], Guan-Lin-Ma[Chin. Ann. Math., 2006], Guan-Ma-Zhou[CPAM, 2006]
  - Fully nonlinear elliptic and parabolic equations: Alvarez-Lasry-Lions[J.M.P.A, 1997], Caffarelli-Guan-Ma[CPAM, 2007], Bian-Guan[Invent. Math, 2009], Ma-Xu[JFA, 2008]

Background

- Convexity of (spatial) level sets $\longleftrightarrow$ $II_x \geq 0$
  - Elliptic equations:
    Shiffman[Ann. Math, 1956], Gabriel[J. London Math. Soc., 1957], Lewis[Arch. Rat. Mech. Anal., 1977], Caffarelli-Spruck[CPDE, 1982], Bianchini-Longinetti-Salani [Indiana, 2009], Korevaar[CPDE, 1990], Bian-Guan-Ma-Xu [Indiana, 2011], Guan-Xu[J. Reine Angew. Math, 2013]
  - Parabolic equations:
    C.-Shi[Sci. China Math., 2011], C. [D.C.D.S, 2014]

Background

For the **parabolic equations**, the **spacetime convexity** is also a basic geometric property.

- Spacetime convex solutions $\longleftrightarrow$ $D^2_{x,t}u \geq 0$
  - Borell[AIHP Probab. Statist,1996; Potential Anal, 2000]: Brownian motion
  - Hu-Ma[Manu. Math, 2013]: heat equation
  - C.-Hu[Acta Math. Sin., 2013]: fully nonlinear parabolic equation
- Convexity of spacetime level sets $\longleftrightarrow$ $II_{x,t} \geq 0$
  - ? ? ? ? ?

Consider the convexity of spacetime level sets of the heat equation in the convex rings.

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u & \text{in} \quad \Omega \times (0, +\infty), \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \\ u(x,t) = 0 & \text{on} \quad \partial\Omega_0 \times (0, +\infty), \\ u(x,t) = 1 & \text{on} \quad \partial\Omega_1 \times (0, +\infty), \end{cases} \tag{0.16}$$

where $\Omega = \Omega_0 \backslash \overline{\Omega_1}$, Ω_0 and Ω_1 are bounded convex $C^{2,\alpha}$ domain in $\mathbb{R}^n$ with $\overline{\Omega_1} \subset \Omega_0$.

Basic Porblems: The (strict) convexity of
(1) spatial level sets $\Sigma_x^{c,t} := \{x \in \Omega : u(x,t) = c\}$.
(2) spacetime level sets $\Sigma_{x,t}^{c} := \{(x,t) \in \Omega \times (0, +\infty) : u(x,t) = c\}$.

This work is joint with Xi-Nan Ma and Paolo Salani.

Difficulties

The strict convexity of spacetime level sets is very difficult to study.

- For heat equation in convex rings, there is no result about the strict convexity of spatial level sets (until our work).
- We cannot prove the constant rank Theorem of $II_{\Sigma_x^{c,t}}$ under the condition $II_{\Sigma_x^{c,t}} \geq 0$.
- The characterization of spacetime level sets is complicated.
- The calculations are very hard when we prove the constant rank Theorem of $II_{\Sigma_{x,t}^c}$ under the condition $II_{\Sigma_{x,t}^c} \geq 0$.

Theorem (C.-Ma-Salani, Memoirs AMS, 2019)

If u is a space-time quasiconcave solution of (0.16), with $u_t > 0$. Then
(1) the spatial level sets $\Sigma_x^{c,t}$ of u are strictly convex for every $c \in (0,1)$ and $t \in (0,+\infty)$.
(2) the spacetime level sets $\Sigma_{x,t}^{c}$ of u are strictly convex for every $c \in (0,1)$.

Key: “Twice” constant rank Theorem technique

- Constant Rank Theorem of $II_{\Sigma_x^{c,t}}$ $\Rightarrow$ constant rank properties
- Constant Rank Theorem of $II_{\Sigma_{x,t}^{c}}$.

the second fundamental form of the spatial level sets

- Spatial level set: $\Sigma_x^{c,t} = \{x \in \Omega | u(x,t) = c\}$
- If $u_n \neq 0$, the Weingarten curvature tensor is

$$a_{ij} = -\frac{|u_n|}{|\nabla u| {u_n}^3} A_{ij} \geq 0, \quad 1 \leq i,j \leq n-1, \tag{0.17}$$

where $\nabla u = (u_1, u_2, \cdots, u_{n-1}, u_n)$ is the spatial gradient of u,

$$A_{ij} = h_{ij} - \frac{u_i u_l h_{jl}}{W(1+W)u_n^2} - \frac{u_j u_l h_{il}}{W(1+W)u_n^2} + \frac{u_i u_j u_k u_l h_{kl}}{W^2(1+W)^2 u_n^4},$$

$$W = \frac{|\nabla u|}{|u_n|},$$

$$h_{ij} = u_n^2 u_{ij} + u_{nn} u_i u_j - u_n u_j u_{in} - u_n u_i u_{jn}, \quad 1 \leq i,j \leq n-1.$$

the spacetime level sets and the spacetime second fundamental form

- Spacetime level set: $\Sigma^c_{x,t} = \{(x,t) | u(x,t) = c\}$
- If $u_t \neq 0$, the Weingarten curvature tensor is

$$\hat{a}_{\alpha\beta} = -\frac{|u_t|}{|Du|{u_t}^3}\hat{A}_{\alpha\beta}, \quad 1 \leq \alpha, \beta \leq n, \tag{0.18}$$

where $Du = (u_1, \cdots, u_n, u_t)$ is the spacetime gradient of u,

$$\hat{A}_{\alpha\beta} = \hat{h}_{\alpha\beta} - \frac{u_\alpha u_\gamma \hat{h}_{\beta\gamma}}{\hat{W}(1+\hat{W})u_t^2} - \frac{u_\beta u_\gamma \hat{h}_{\alpha\gamma}}{\hat{W}(1+\hat{W})u_t^2} + \frac{u_\alpha u_\beta u_\gamma u_\eta \hat{h}_{\gamma\eta}}{\hat{W}^2(1+\hat{W})^2 u_t^4},$$

$$\hat{W} = \frac{|Du|}{|u_t|},$$

$$\hat{h}_{\alpha\beta} = u_t^2 u_{\alpha\beta} + u_{tt}u_\alpha u_\beta - u_t u_\beta u_{\alpha t} - u_t u_\alpha u_{\beta t}, \quad 1 \leq \alpha, \beta \leq n.$$

Constant rank theorem of spatial level sets

Assume $l = \min\limits_{(x,t)\in\Omega\times(0,T]} rank\{a_{ij}(x,t)\}$ attained at some point $(x_0, t_0) \in \Omega \times (0, T]$. We just need to prove a differential inequality

$$\begin{cases} \Delta\phi - \phi_t \leqslant C_1\phi + C_2\,|\nabla\phi| & \text{in} \quad \mathcal{O}\times(t_0-\delta, t_0], \\ \phi(x_0, t_0) = 0, & \\ \phi \geqslant 0 & \text{in} \quad \mathcal{O}\times(t_0-\delta, t_0], \end{cases} \tag{0.19}$$

where

$$\phi = \sigma_{l+1}(\{a_{ij}(x,t)\}). \tag{0.20}$$

Then use the strong maximum principle and the continuous method.

Constant rank theorem of spatial level sets

Similarly to Bian-Guan-Ma-Xu [Indiana Univ. Math. J, 2011], we can get

$$\begin{aligned}\Delta\phi(x,t)-\phi_t &= -u_n^{-3}\sigma_l(G)\sum_{j\in B}[-4u_nu_{nj}u_{tj}+6u_{nj}^2u_t]\\ &\quad+2u_n^{-3}\sigma_l(G)\sum_{j\in B,i\in G}\sum_{\alpha=1}^{n}\frac{[u_nu_{ij\alpha}-2u_{i\alpha}u_{jn}]^2}{u_{ii}}\\ &\quad+O(\phi+|\nabla\phi|).\end{aligned}$$

The solution is **space-time quasiconcave**, then for $j\in B$ we get

$$\hat{a}_{jn}^2\le\hat{a}_{jj}\hat{a}_{nn}=O(\phi),\quad\text{and}\quad\hat{h}_{jn}^2=O(\phi),\tag{0.21}$$

hence

$$-4u_nu_{nj}u_{tj}+6u_{nj}^2u_t=\frac{2}{u_t}[u_tu_{nj}+\frac{\hat{h}_{jn}}{u_t}]^2+O(\phi).\tag{0.22}$$

Constant rank properties

Corollary

$$u_{ii} = 0, \ \textit{for } i \in B, \tag{0.23}$$
$$u_{ni} = 0, \ \textit{for } i \in B, \tag{0.24}$$
$$u_{ij\alpha} = 0, \ \textit{for } i \in B, j \in G, \alpha = 1, \cdots, n. \tag{0.25}$$

Under above assumptions, we can get

$$\begin{aligned}
&\hat{a}_{ij}(x,t) \equiv 0, \quad \text{i or j } \in B,\\
&\hat{a}_{in}(x,t) = \hat{a}_{ni}(x,t) \equiv 0, \quad i \in B,\\
&D\hat{a}_{ij}(x,t) = (\nabla \hat{a}_{ij}, \hat{a}_{ij,t})(x,t) \equiv 0, \quad \text{i or j } \in B,\\
&D\hat{a}_{in}(x,t) = D\hat{a}_{ni}(x,t) \equiv 0, \quad i \in B.
\end{aligned}$$

Constant rank theorem of spacetime level sets

Assume $l = \min_{(x,t)\in\Omega\times(0,T]} rank\{\hat{a}_{\alpha\beta}\}$ attained at some point $(x_0, t_0) \in \Omega \times (0, T]$. We just need to prove a differential inequality

$$\begin{cases} \Delta\varphi - \varphi_t \leqslant C_1\varphi + C_2\,|\nabla\varphi|, & \text{in} \quad \mathcal{O} \times (t_0 - \delta, t_0], \\ \varphi(x_0, t_0) = 0, & \\ \varphi \geqslant 0 & \text{in} \quad \mathcal{O} \times (t_0 - \delta, t_0], \end{cases} \tag{0.26}$$

where $\mathcal{O} \times (t_0 - \delta, t_0]$ is a small parabolic neighborhood of (x_0, t_0).

$$\varphi = \sigma_{l+1}(\{\hat{a}_{\alpha\beta}\}), \tag{0.27}$$

Then use the strong maximum principle and the continuous method.

Proof of (spacetime) Constant Rank Theorem

By hard calculations, we get

$$\begin{aligned}
\Delta\varphi - \varphi_t \sim & \sigma_l(G)\left(1 + \sum_{i\in G}\frac{\hat{a}_{in}^2}{\hat{a}_{ii}^2}\right)\sum_{i\in B}\Delta\hat{a}_{ii} \\
& + \sigma_l(G)\Bigg[\left(\Delta\hat{a}_{nn} - \hat{a}_{nn,t}\right) \\
& \qquad - 2\sum_{i\in G}\frac{\hat{a}_{in}}{\hat{a}_{ii}}\left(\Delta\hat{a}_{in} - \hat{a}_{in,t}\right) \\
& \qquad + \sum_{ij\in G}\frac{\hat{a}_{in}}{\hat{a}_{ii}}\frac{\hat{a}_{jn}}{\hat{a}_{jj}}\left(\Delta\hat{a}_{ij} - \hat{a}_{ij,t}\right)\Bigg] \\
& - 2\sigma_l(G)\sum_{i\in G}\frac{1}{\hat{a}_{ii}}\sum_{\alpha=1}^{n}\left[\hat{a}_{in,\alpha} - \sum_{j\in G}\frac{\hat{a}_{jn}}{\hat{a}_{jj}}\hat{a}_{ij,\alpha}\right]^2
\end{aligned}$$

Proof of (spacetime) Constant Rank Theorem

$$
\begin{aligned}
&\sim \sigma_l(G)\left(1+\sum_{i\in G}\frac{\hat{a}_{in}^2}{\hat{a}_{ii}^2}\right)\left(-\frac{|u_t|}{|Du|{u_t}^3}\right)\sum_{i\in B}\Delta\hat{A}_{ii}\\
&+\sigma_l(G)\left(-\frac{|u_t|}{|Du|{u_t}^3}\right)\Bigg[\left(\Delta\hat{A}_{nn}-\hat{A}_{nn,t}\right)\\
&\qquad\qquad -2\sum_{i\in G}\frac{\hat{A}_{in}}{\hat{A}_{ii}}\left(\Delta\hat{A}_{in}-\hat{A}_{in,t}\right)\\
&\qquad\qquad +\sum_{ij\in G}\frac{\hat{A}_{in}}{\hat{A}_{ii}}\frac{\hat{A}_{jn}}{\hat{A}_{jj}}\left(\Delta\hat{A}_{ij}-\hat{A}_{ij,t}\right)\Bigg]\\
&-2\sigma_l(G)\left(-\frac{|u_t|}{|Du|{u_t}^3}\right)\sum_{i\in G}\frac{1}{\hat{A}_{ii}}\sum_{\alpha=1}^{n}\left[\hat{A}_{in,\alpha}-\sum_{j\in G}\frac{\hat{A}_{jn}}{\hat{A}_{jj}}\hat{A}_{ij,\alpha}\right]^2
\end{aligned}
$$

Proof of (spacetime) Constant Rank Theorem

$$
\begin{aligned}
\sim\sigma_l(G)\Big(-\frac{|u_t|}{|Du|u_t^3}\Big)\frac{1}{\hat{W}^2}\Big[(\Delta\hat{h}_{nn}-\hat{h}_{nn,t})
&-2\sum_{i\in G}\frac{\hat{h}_{in}}{\hat{h}_{ii}}(\Delta\hat{h}_{in}-\hat{h}_{in,t})\\
&+\sum_{ij\in G}\frac{\hat{h}_{in}}{\hat{h}_{ii}}\frac{\hat{h}_{jn}}{\hat{h}_{jj}}(\Delta\hat{h}_{ij}-\hat{h}_{ij,t})\Big]\\
-2\sigma_l(G)\Big(-\frac{|u_t|}{|Du|u_t^3}\Big)\frac{1}{\hat{W}^2}\sum_{i\in G}\frac{1}{\hat{h}_{ii}}\sum_{\alpha=1}^{n}\Big[\hat{h}_{in,\alpha}-\sum_{j\in G}\frac{\hat{h}_{jn}}{\hat{h}_{jj}}\hat{h}_{ij,\alpha}\Big]^2
\end{aligned}
$$

Proof of (spacetime) Constant Rank Theorem

$$
\begin{aligned}
\sim & \sigma_l(G)\Big(-\frac{|u_t|}{|Du|{u_t}^3}\Big)\frac{2}{\hat{W}^2}\bullet \\
& \Big\{-\sum_{i\in G}\frac{1}{\hat{h}_{ii}}[\hat{h}_{in,n}-\sum_{j\in G}\frac{\hat{h}_{jn}}{\hat{h}_{jj}}\hat{h}_{ij,n}-\hat{h}_{in}(\frac{u_{nn}}{u_n}-\frac{u_{nt}}{u_t}-\sum_{i\in G}\frac{\hat{h}_{in}}{\hat{h}_{ii}}\frac{u_{in}}{u_n})]^2 \\
& -\sum_{i\in G}\frac{1}{\hat{h}_{ii}}[\hat{h}_{in,i}-\sum_{j\in G}\frac{\hat{h}_{jn}}{\hat{h}_{jj}}\hat{h}_{ij,i}-\hat{h}_{ii}(\frac{u_{nn}}{u_n}-\frac{u_{nt}}{u_t}-\sum_{i\in G}\frac{\hat{h}_{in}}{\hat{h}_{ii}}\frac{u_{in}}{u_n})]^2 \\
& -\sum_{i\in G}\frac{1}{\hat{h}_{ii}}\sum_{\alpha\in G,\alpha\neq i}[\hat{h}_{in,\alpha}-\sum_{j\in G}\frac{\hat{h}_{jn}}{\hat{h}_{jj}}\hat{h}_{ij,\alpha}]^2 \\
& +[\frac{u_{nn}}{u_n}-\frac{u_{nt}}{u_t}-\sum_{i\in G}\frac{\hat{h}_{in}}{\hat{h}_{ii}}\frac{u_{in}}{u_n}]^2u_t^3\Big\} \\
\leq & C(\varphi+|\nabla\varphi|)
\end{aligned}
$$

Remark

- Borell [C.M.P, 1982]

$$\begin{cases} \frac{\partial u^B}{\partial t} = \Delta u^B & \text{in} \quad \Omega \times (0, +\infty), \\ u^B(x, 0) = 0 & \text{in} \quad \Omega, \\ u^B(x, t) = 0 & \text{on} \quad \partial\Omega_0 \times (0, +\infty), \\ u^B(x, t) = 1 & \text{on} \quad \partial\Omega_1 \times (0, +\infty), \end{cases} \tag{0.28}$$

  Then the spacetime level sets $\{(x, t) | u^B(x, t) = c\}$ is convex.
- By the constant rank theorem, we can get the spacetime level sets of u^B is strictly convex for $t > 0$ and $c \in (0, 1)$.
- Deformating from $u^B(x, \epsilon)$ to $u_0(x)$, we can get the spacetime level sets of u is strictly convex for $t > 0$ and $c \in (0, 1)$.

Thank You !