Global Vanishing Viscosity Limit for Two Dimensional Incompressible Viscoelasticity

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Contents

Incompressible elastodynamics

For isotropic, hyperelastic and homogeneous incompressible materials, the motion can be described by the following elastodynamic system (in $\mathbb{R}^n$)

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = \nabla \cdot (\frac{\partial W(F)}{\partial F} F^\top), \\ \nabla \cdot v = 0. \end{cases} \tag{1}$$

We focus on Hookean elasticity $W(F) = \frac{1}{2}|F|^2$ for simplicity. The general case differs by the cubic and higher order nonlinear terms which won't make much difference.

Flow Map

Define the flow map:

$$\begin{cases} \frac{dx(t,y)}{dt} = v(t, x(t,y)), \\ x(0) = y. \end{cases}$$

The deformation tensor is defined through the flow map:

$$\tilde{F}(t,y) = \frac{\partial x(t,y)}{\partial y}.$$

Compatibility for deformation tensor

The incompressible condition is equivalent to $\nabla \cdot F^\top = 0$. In addition, one can deduce that

$$\begin{cases} \partial_t F + v \cdot \nabla F = \nabla v F, \\ F_{mj} \nabla_m F_{ik} = F_{lk} \nabla_l F_{ij}, \quad i, j, m, k, l \in \{1, 2, \cdots, n\}^1. \end{cases} \tag{2}$$

The above equations are basically the compatibility conditions for the velocity field and the flow map:

$$\begin{cases} D_t D_y x = D_y D_t x, \\ D_j D_k x_i = D_k D_j x_i, \quad i, j, k \in \{1, 2, \cdots, n\}. \end{cases}$$

Here D_t,D_i are derivatives with respect to Lagrangian coordinates.

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[1] Lei, Z., Liu, C., and Zhou, Y. *Global solutions for incompressible viscoelastic fluids*. Arch. Ration. Mech. Anal. 188 (2008), no. 3, 371–398

Full elastodynamics

Then the elastodynamics (1) can be equivalently rewritten as follows:

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = \nabla \cdot (\frac{\partial W(F)}{\partial F} F^\top), \\ \partial_t F + v \cdot \nabla F = \nabla v F, \\ \nabla \cdot v = 0, \quad \nabla \cdot F^\top = 0, \end{cases} \tag{3}$$

with the constraint

$$F_{mj} \nabla_m F_{ik} = F_{lk} \nabla_l F_{ij}, \quad i, j, m, k, l \in \{1, 2, \cdots, n\}. \tag{4}$$

Viscoelasticity

Taking into account of viscosity, we have the Oldroyd-B system of viscoelasticity:

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = \mu \Delta v + \nabla \cdot \left(\frac{\partial W(F)}{\partial F} F^\top\right), \\ \partial_t F + v \cdot \nabla F = \nabla v F, \\ \nabla \cdot v = 0, \quad \nabla \cdot F^\top = 0, \end{cases} \tag{5}$$

with the constraint

$$F_{mj} \nabla_m F_{ik} = F_{lk} \nabla_l F_{ij}, \quad i, j, m, k, l \in \{1, 2, \cdots, n\}. \tag{6}$$

Quasilinear wave equations

Let $W(F) = \frac{1}{2}|F|^2$, $F = I + G$. The incompressible elastodynamics (3) and (4) can be rewritten as

$$\begin{cases} \partial_t v - \nabla \cdot G = -\nabla p - v \cdot \nabla v + \nabla \cdot (GG^\top), \\ \partial_t G - \nabla v = -v \cdot \nabla G + \nabla v G, \\ \nabla \cdot v = 0, \quad \nabla \cdot G^\top = 0, \end{cases}$$

with the constraint

$$\partial_j G_{ik} - \partial_k G_{ij} = G_{lk} \partial_l G_{ij} - G_{lj} \partial_l G_{ik}, \quad i, j, k, l \in \{1, 2, \cdots, n\}.$$

Quasilinear wave equations:

$$(\partial_t^2 - \Delta) v = f_1, \ (\partial_t^2 - \Delta) G = f_2.$$

The tools for the long time existence

Vector field method:

- Generalized derivatives: $\partial = (\partial_t, \nabla)$, $\Omega_{ij} = -x_i\partial_j + x_j\partial_i$, $S = t\partial_t + r\partial_r$, $L_i = t\partial_i + x_i\partial_t$, $Z = (\partial, \Omega, S, L)$
- Klainerman-Sobolev inequalities (Klainerman 1985):

$$\langle t-r\rangle^{\frac{1}{2}}\langle t+r\rangle^{\frac{n-1}{2}}|u| \lesssim \sum_{|a|\le[\frac{n}{2}]+1} \|Z^a u\|_{L^2}.$$

- Generalized energy:

$$E_k(t) = \sum_{|a|\le k-1} \|\partial Z^a u(t,\cdot)\|_{L^2}^2.$$

2D wave equations are supercritical

Formally, generalized energy gives (for quadratic nonlinearity):

$$\frac{d}{dt}E_k(t) \lesssim \|\partial Z^{[(k-1)/2]}u\|_{L^\infty} E_k(t),$$

By the Klainerman-Sobolev inequality, the above inequality becomes

$$\frac{d}{dt}E_k(t) \lesssim \langle t\rangle^{-\frac{n-1}{2}} E_k^{\frac{3}{2}}(t).$$

This suggests:

- $n \geq 4$, integrable, small data global, subcritical!
- $n = 3$, small data almost global, critical!
- $n = 2$, far from integrability, supercritical!

Quasilinear wave equations

3-D case:

- finite time blow up, F. John (CPAM, 1981)
- almost global, John-Klainerman (CPAM, 1984)
- global under null conditions: Klainerman (Lect. in Appl. Math., 1986); Christodoulou (CPAM, 1986)

2-D case:

- finite time blow up: Alinhac (Ann. of Math., Acta Math., 1999)
- global under null conditions: Alinhac (Invent. Math., 2001)

Elastodynamics in 3D

Blow-up for compressible elastodynamics in 3D:

- John (CPAM, 1984), Tahvildar-Zadeh (Ann. Inst. H. Poincare-Phys. Theor, 1998)

Long time existence in 3D:

- almost global for compressible elastodynamics: John (CPAM, 1988), Klainerman, Sideris (CPAM, 1996),
- global for compressible elastodynamics: Sideris (Invent. Math., 1996, Ann. Math.,2000), Agemi (Invent. Math., 2000)
- global for incompressible elastodynamics: Sideris, Thomases (CPAM 2005, 2007)

Elastodynamics in 2D

Incompressible elastodynamics in 2D:

- Almost Global: Lei-Sideris-Zhou (Trans. AMS, 2015)
- Global: Lei (CPAM, 2016). Inherent strong null condition
- An alternative proof of global solution: Wang (Ann. Henri Poincaré, 2017). Space-time resonance, normal form, Z norm
- Uniform bound of the highest-order energy: Cai (arXiv:2010.08718, 2020)

Two dimensional compressible elastodynamics : open problem!

Related Results

Global for viscoelasticity:

Contributions of many authors:

- Lin-Liu-Zhang/Lei-Zhou, 2D,
- Lei-Liu-Zhou/Chen-Zhang, 3D
- Liu, Fang-Zhang, Hu-Wang, Qian-Zhang, Qian, Deng, Han....

Vanishing viscosity for viscoelasticity in 3D:

- Kessenich, P. (arXiv:0903.2824, 2009)

Main result

Theorem (Cai-Lei-Lin-Masmoudi, CPAM, 2019)

For sufficiently small initial displacement, the 2D incompressible viscoelasticity is globally well-posed uniformly for all $t \geq 0$ and all $\mu \geq 0$.

Equations near equilibrium

Let $W(F) = \frac{1}{2}|F|^2$. The deformation tensor perturbs around its equilibrium, $F = I + G$. Then the 2D incompressible viscoelastic system (5) and (6) can be rewritten as follows:

$$\begin{cases} \partial_t v - \mu\Delta v - \nabla \cdot G = -\nabla p - v \cdot \nabla v + \nabla \cdot (GG^\top), \\ \partial_t G - \nabla v = -v \cdot \nabla G + \nabla v G, \\ \nabla \cdot v = 0, \quad \nabla \cdot G^\top = 0, \end{cases} \tag{7}$$

with the constraint

$$(\nabla^\perp \cdot G)_i = G_{l2}\nabla_l G_{i1} - G_{l1}\nabla_l G_{i2}. \tag{8}$$

Inviscid case

Vector fields and generalized energy[2]:

- Generalized derivatives: $Z = (\partial, \Omega, S)$, $\partial = (\partial_t, \nabla)$, $\Omega = \partial_\theta = x^\perp \cdot \nabla$, $S = t\partial_t + r\partial_r$,
- Generalized energy:

$$E_\kappa(t) = \sum_{|a|\le\kappa} \|Z^a v\|_{L^2}^2 + \|Z^a G\|_{L^2}^2.$$

- Weighted L^2 norm:

$$X_\kappa(t) = \sum_{|a|\le\kappa-1} \|\langle t-r\rangle \nabla Z^a v\|_{L^2}^2 + \sum_{|a|\le\kappa-1} \|\langle t-r\rangle \nabla Z^a G\|_{L^2}^2.$$

[2] Lei, Zhen; Sideris, Thomas C.; Zhou, Yi. Almost global existence for 2-D incompressible isotropic elastodynamics. Trans. Amer. Math. Soc. 367 (2015), no. 11, 8175–8197.

Weighted $L^\infty - L^2$ estimate

Lemma (Lei-Sideris-Zhou, Trans. AMS, 2015)

For all $f \in H^2(\mathbb{R}^2)$, there holds

$$r^{\frac{1}{2}}|f(x)| \lesssim \sum_{a=0,1} [\|\partial_r \Omega^a f\|_{L^2}^2 + \|\Omega^a f\|_{L^2}^2]^{\frac{1}{2}},$$

$$r^{\frac{1}{2}}\langle t-r\rangle|f(x)| \lesssim \sum_{a=0,1} [\|\langle t-r\rangle\partial_r \Omega^a f\|_{L^2}^2 + \|\langle t-r\rangle\Omega^a f\|_{L^2}^2]^{\frac{1}{2}},$$

$$\langle t\rangle\|f\|_{L^\infty(r\le\langle t\rangle/2)} \lesssim \sum_{|a|\le 2} \|\langle t-r\rangle\partial^a f\|_{L^2},$$

provided the right-hand side is finite.

Temporal Decay

Lemma (Lei-Sideris-Zhou, Trans. AMS, 2015)

Assume $E_\kappa \ll 1$, there holds

$$
\begin{aligned}
&X_\kappa \lesssim E_\kappa, \\
&\langle t\rangle^{\frac{1}{2}}|Z^{\kappa-2}v|, \langle t\rangle^{\frac{1}{2}}|Z^{\kappa-2}G| \lesssim E_\kappa^{\frac{1}{2}}, \\
&\|r(\partial_r Z^{\kappa-1}v + \partial_r Z^{\kappa-1}G\omega)\|_{L^2} + \|r\partial_r G\omega^\perp\|_{L^2} \lesssim E_\kappa^{\frac{1}{2}}.
\end{aligned}
$$

Good unknowns: $g = v + G\omega,\ G\omega^\perp$.

Null condition

For the following quasilinear wave equations:

$$\partial_t^2 u - \Delta u = Q(\partial u, \partial^2 u),$$

Definition (null condition)

We say Q satisfies the null condition if Q can be written in the following form:

$$Q(\partial u, \partial^2 u) = Q_1(\partial u, g(\partial u)) + Q_2(g, \partial^2 u) + \mathcal{R},$$

where $\mathcal{R}$ satisfies

$$|\mathcal{R}| \lesssim \frac{|\Gamma u||\partial^2 u| + |\partial u||\partial \Gamma u|}{1+t}, \quad r \geq \frac{t+1}{2}.$$

Here g is good unknowns of the following form:

$$g(u) = \omega \partial_t u + \nabla u.$$

Systems satisfy null condition

The nonlinear terms in the momentum equation can be organized as follows:

$$
\begin{aligned}
& v \cdot \nabla v - \nabla \cdot (GG^T) \\
&= (v + G\omega) \cdot \nabla v - (G\omega^{\perp})_j (\nabla_j v + \nabla_j G\omega) - (G\omega^{\perp})_j \nabla_j G\omega^{\perp} \\
&= Q_1(g, \nabla v) + Q_2(G\omega, g(\partial u)) + Q_2(g, g(\nabla u)).
\end{aligned}
$$

We observe a similar fact for the G-equation in (7)-(8).

Ghost weight

Heuristic calculation ($\mu = 0$). Consider the energy estimate with weight $e^q = e^{\arctan(r-t)}$ (S. Alinhac, Invent. Math., 2001):

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}(|Z^\kappa v|^2 + |Z^\kappa G|^2)e^q dx \\
&+\frac{1}{2}\int_{\mathbb{R}^2}\frac{|Z^\kappa v + Z^\kappa G\omega|^2 + |Z^\kappa G\omega^\perp|^2}{1+(t-r)^2}e^q dx \\
&= \text{nonlinear terms.}
\end{aligned}$$

Bad commutators when $\mu > 0$

"Bad commutators" will appear when ghost weight is applied.

A heuristic calculation:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}(|Z^\kappa v|^2+|Z^\kappa G|^2)e^q dx+\ \mu\int_{\mathbb{R}^2}|\nabla Z^\kappa v|^2 e^q dx\\&+\frac{1}{2}\int_{\mathbb{R}^2}\frac{|Z^\kappa v+Z^\kappa G\omega|^2+|Z^\kappa G\omega^\perp|^2}{1+(t-r)^2}e^q dx\\&=\frac{1}{2}\mu\int_{\mathbb{R}^2}|Z^\kappa v|^2\Delta e^q dx+\cdots\end{aligned}$$

Difficulties

Main Difficulties

- The use of ghost weight is the key step to obtain desired estimate. However, the appearance of viscosity seems preventing us using this weight. Exactly, the commutator due to ghost weight and viscosity is very serious
- In dimension two, the time decay rate is $\frac{1}{\sqrt{1+t}}$, which is supercritical. Even within null conditions, the temporal decay is still critical

- 2D Soblev inequality is critical!
- 2D Hardy's inequality is not correct!
- No compact support!

The First Key Ingredient

Strategy: we introduce a new type energy named "modified generalized energy" which has one more traditional derivative than the generalized energy.

Higher order energy estimate:

- generalized energy estimate : no ghost weight.
- Modified generalized energy estimate: with ghost weight.

Now we can absorb the commutator!

The realization of the first key idea

Modified higher-order energy estimate (with ghost weight):

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}(|\nabla Z^\kappa v|^2+|\nabla Z^\kappa G|^2)e^q dx+\ \mu\int_{\mathbb{R}^2}|\nabla^2 Z^\kappa v|^2 e^q dx$$
$$+\frac{1}{2}\int_{\mathbb{R}^2}\frac{|\nabla Z^\kappa v+\nabla Z^\kappa G\omega|^2+|\nabla Z^\kappa G\omega^\perp|^2}{1+(t-r)^2}e^q dx$$
$$=\frac{1}{2}\mu\int_{\mathbb{R}^2}|\nabla Z^\kappa v|^2\Delta e^q dx+\text{nonlinear terms}$$

Higher-order energy estimate (without ghost weight):

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}(|Z^\kappa v|^2+|Z^\kappa G|^2)dx+\ \mu\int_{\mathbb{R}^2}|\nabla Z^\kappa v|^2 dx$$
$$=\text{nonlinear terms}$$

Strong Null Condition

We start with the following scalar quasilinear wave equations

$$\partial_t^2 u - \Delta u = Q(\partial u, \partial^2 u).$$

Here Q is a bilinear form.

Definition (Strong Null Condition)

We say Q satisfies the strong null condition if

$$Q(\partial u, \partial^2 u) = Q_1(\partial u, g(\partial u)) + \mathcal{R},$$

where ther reminder $\mathcal{R}$ satisfies

$$|\mathcal{R}| \lesssim \frac{|\partial u||\partial Z u|}{1+t}, \quad r \geq \frac{t+1}{2}.$$

Here g is good known in the sense of Alinhac: $g(u) = \omega \partial_t u + \nabla u$.

Original system satisfy null condition but doesn't satisfy the strong null condition

Following Lei-Sideris-Zhou (Trans AMS, 2015), we call $v + G\omega$ and $G\omega^{\perp}$ good unknowns. One writes the nonlinear terms in the momentum equation as

$$\begin{aligned}
& v \cdot \nabla v - \nabla \cdot (GG^T) \\
&= (v + G\omega) \cdot \nabla v - (G\omega)_j (\nabla_j v + \nabla_j G\omega) - (G\omega^{\perp})_j \nabla_j G\omega^{\perp} \\
&= Q_1(g, \nabla v) + Q_2(G\omega, g(\partial u)) + Q_2(g, g(\nabla u)).
\end{aligned}$$

Obviously, Q_1 must present and thus system doesn't explicitly exhibit the strong null structure (note that the system now is of first-order. One can observe a similar fact for the G-equation in (7)-(8).

Second main ingredient:

Our second key idea is to transform the viscoelastic system to a "fully nonlinear" one, together with a transformed fully nonlinear constraint. It turns out that the transformed systems satisfy the "strong null condition".

A simple example

Inspired by Lei (CPAM, 2016), consider the following quasilinear wave equations

$$\begin{cases} \Box v = \partial_t(|\partial_t v|^2 - |\nabla v|^2), \\ v(0,\cdot) = v_0, \partial_t v(0,\cdot) = v_1. \end{cases} \tag{9}$$

It's easy to verify that it doesn't satisfy the strong null condition. However, let

$$v = \partial_t u.$$

Choosing appropriate data, then u satisfies

$$\Box u = |\partial_t^2 u|^2 - |\nabla \partial_t u|^2. \tag{10}$$

Now the strong null condition is satisfied!

Problems

Consider the following two dimensional wave equations:

$$\begin{cases} \Box u = N_{\alpha\beta\mu\nu}\partial_\alpha\partial_\beta u \partial_\mu\partial_\nu u, \\ u(0,\cdot) = \varphi, \partial_t u(0,\cdot) = \psi, \end{cases} \tag{11}$$

Here $\partial := (\partial_t, \partial_1, \partial_2)$, $\Box$ is d' Alembertian operator, $\varphi, \psi \in H^k_\Lambda$, $u = u(t, x_1, x_2)$ is the unknown. Our goal is to prove the global well-posedness for (11) without compact support.

Null condition

Denote

$$N(u,v) = N_{\alpha\beta\mu\nu}\partial_\alpha\partial_\beta u\partial_\mu\partial_\nu u.$$

We impose the null condition for $N(u,v)$:

$$N_{\alpha\beta\mu\nu}X_\alpha X_\beta X_\mu X_\nu = 0, \tag{12}$$

for all $X \in \Sigma$ where

$$\Sigma = \{X \in \mathbb{R}^3 : X_0^2 = X_1^2 + X_2^2\}.$$

Main result

Theorem(Cai-Lei-Masmoudi, JMPA, 2018)

Let $M > 0$, $0 < \gamma < \frac{1}{8}$ be two given constants and $(\varphi, \psi) \in H^k_\Lambda$, with $k \geq 8$. Suppose that the nonlinearities satisfy the null condition (12), and

$$\|(\varphi, \psi)\|_{H^k_\Lambda} < M, \quad \|(\varphi, \psi)\|_{H^{k-1}_\Lambda} < \epsilon. \tag{13}$$

There exists a positive constant $\epsilon_0 < e^{-M}$ which depend on $M,\ k,\ \gamma$ such that, if $\epsilon \leq \epsilon_0$, the fully nonlinear wave equation (11) with initial data $(u(0), \partial_t u(0)) = (\varphi, \psi)$ has a unique global solution which satisfies $E_k(t) \leq C_0 M^2 \langle t \rangle^\gamma$ and $E_{k-1}(t) \leq C_0 \epsilon^2 e^{C_0 M}$ for some $C_0 > 1$ uniformly for $0 \leq t < \infty$.

Application

Consider the following quasilinear wave equations

$$\begin{cases} \Box v = A_l \partial_l (N_{\mu\delta} \partial_\mu v \partial_\delta v), \\ v(0,x) = v_0(x), \partial_t v(0,x) = v_1(x). \end{cases} \tag{14}$$

Theorem (Cai-Lei-Masmoudi, JMPA, 2018)

For quasilinear wave equations (14), assume that for all $X \in \Sigma$, there holds the null condition

$$N_{\mu\delta} X_\mu X_\delta = 0 \tag{15}$$

For suitable small initial data, (14) has unique global classical solutions.

Equivalent new formulation of the 2d viselasticity

Lemma

Let $v = \nabla^\perp V, \quad G^\top = \nabla^\perp H$. For classical solutions, the system (7) is equivalent to :

$$\begin{cases} \partial_t V - \mu\Delta V - \nabla \cdot H \\ \qquad = \nabla^\perp \cdot \nabla \cdot \Delta^{-1}\big(-\nabla^\perp V \otimes \nabla^\perp V + \nabla^\perp H \otimes \nabla^\perp H\big), \\ \partial_t H - \nabla V = \nabla^\perp H \nabla V, \end{cases} \tag{16}$$

and (8) is equivalent to :

$$\nabla^\perp \cdot H = \nabla^\perp H_2 \cdot \nabla H_1. \tag{17}$$

The new system satisfies the strong null condition

For the new systems, the good quantities are $V + H \cdot \omega$ and $H \cdot \omega^{\perp}$. We can calculate that

$$
\begin{aligned}
&\nabla_i^{\perp} V \nabla_j^{\perp} V - \nabla_i^{\perp} H \cdot \nabla_j^{\perp} H \\
&= (\nabla_i^{\perp} V + \nabla_i^{\perp} H \ \cdot \omega) \nabla_j^{\perp} V - \nabla_i^{\perp} H \ \cdot \omega (\nabla_j^{\perp} V + \nabla_j^{\perp} H \ \cdot \omega) \\
&\quad - \nabla_i^{\perp} H \ \cdot \omega^{\perp} \nabla_j^{\perp} H \ \cdot \omega^{\perp}.
\end{aligned}
$$

Now the strong null condition is satisfied. One can show the similar phenomenon for $H-$equation and the constraint.

Application of rotational operators

For two-dimensional case, rotation operator is:

$$\Omega = x^{\perp} \cdot \nabla = \partial_\theta.$$

Applying rotation operator onto (16) and (17) yielding

$$\begin{cases} \partial_t \widetilde{\Omega} V - \nabla \cdot \widetilde{\Omega} H \\ \quad = \nabla^{\perp} \cdot \nabla \cdot \Delta^{-1}\big(- \nabla^{\perp} \widetilde{\Omega} V \otimes \nabla^{\perp} V + \nabla^{\perp} \widetilde{\Omega} H \otimes \nabla^{\perp} H \big) \\ \qquad + \nabla^{\perp} \cdot \nabla \cdot \Delta^{-1}\big(- \nabla^{\perp} V \otimes \nabla^{\perp} \widetilde{\Omega} V + \nabla^{\perp} H \otimes \nabla^{\perp} \widetilde{\Omega} H \big), \\ \partial_t \widetilde{\Omega} H - \nabla \widetilde{\Omega} V = \nabla^{\perp} \widetilde{\Omega} H \nabla V + \nabla^{\perp} H \nabla \widetilde{\Omega} V, \end{cases} \tag{18}$$

and

$$\nabla^{\perp} \cdot \widetilde{\Omega} H = \nabla^{\perp} \widetilde{\Omega} H_2 \cdot \nabla H_1 + \nabla^{\perp} H_2 \cdot \nabla \widetilde{\Omega} H_1, \tag{19}$$

here

$$\begin{cases} \widetilde{\Omega} V = \Omega V, \\ \widetilde{\Omega} H = \Omega H - H^{\perp}. \end{cases}$$

Application of scaling operators

Define the modified scaling operator:

$$\widetilde{S} = S - 1.$$

Applying modified scaling operator $\tilde{S}$ onto (16) and (17) gives

$$\begin{cases} \partial_t \widetilde{S}V - \mu\Delta(\widetilde{S}-1)V - \nabla\cdot\widetilde{S}H \\ \quad = \nabla^\perp\cdot\nabla\cdot\Delta^{-1}\big(-\nabla^\perp\widetilde{S}V\otimes\nabla^\perp V + \nabla^\perp\widetilde{S}H\otimes\nabla^\perp H\big) \\ \qquad +\nabla^\perp\cdot\nabla\cdot\Delta^{-1}\big(-\nabla^\perp V\otimes\nabla^\perp\widetilde{S}V + \nabla^\perp H\otimes\nabla^\perp\widetilde{S}H\big), \\ \partial_t\widetilde{S}H - \nabla\widetilde{S}V = \nabla^\perp\widetilde{S}H\nabla V + \nabla^\perp H\nabla\widetilde{S}V, \end{cases} \tag{20}$$

and

$$\nabla^\perp\cdot\widetilde{S}H = \nabla^\perp\widetilde{S}H_2\cdot\nabla H_1 + \nabla^\perp H_2\cdot\nabla\widetilde{S}H_1. \tag{21}$$

Viscoelasticity has no scaling invariance. When $\mu > 0$, there is an extra term coming from the commutation.

Application of Generalized Operators

Let

$$\Gamma \in \{\partial_t, \partial_1, \partial_2, \widetilde{\Omega}\}.$$

Repeatedly using (20), (21) and (18), (19) gives

$$\begin{cases} \partial_t \widetilde{S}^\alpha \Gamma^a V - \mu \Delta \sum_{l=0}^{\alpha} C_\alpha^l (-1)^{\alpha-l} \widetilde{S}^l \Gamma^a V - \nabla \cdot \widetilde{S}^\alpha \Gamma^a H = f_{\alpha a}^1, \\ \partial_t \widetilde{S}^\alpha \Gamma^a H - \nabla \widetilde{S}^\alpha \Gamma^a V = f_{\alpha a}^2, \end{cases} \tag{22}$$

and

$$\nabla^\perp \cdot \widetilde{S}^\alpha \Gamma^a H = f_{\alpha a}^3,$$

The nonlinearities are

$$\begin{cases} f^1_{\alpha a} = \sum\limits_{\substack{b+c=a \\ \beta+\gamma=\alpha}} C^\beta_\alpha C^b_a \nabla^\perp \cdot \nabla \cdot \Delta^{-1}\big(- \nabla^\perp \widetilde{S}^\beta \Gamma^b V \otimes \nabla^\perp \widetilde{S}^\gamma \Gamma^c V \\ \qquad\qquad\qquad + \nabla^\perp \widetilde{S}^\beta \Gamma^b H \otimes \nabla^\perp \widetilde{S}^\gamma \Gamma^c H\big), \\ f^2_{\alpha a} = \sum\limits_{\substack{b+c=a \\ \beta+\gamma=\alpha}} C^\beta_\alpha C^b_a (\nabla^\perp \widetilde{S}^\beta \Gamma^b H \nabla \widetilde{S}^\gamma \Gamma^c V), \\ f^3_{\alpha a} = \sum\limits_{\substack{b+c=a \\ \beta+\gamma=\alpha}} C^\beta_\alpha C^b_a (\nabla^\perp \widetilde{S}^\beta \Gamma^b H_2 \cdot \nabla \widetilde{S}^\gamma \Gamma^c H_1). \end{cases}$$

Several norms

Denote $V^{(\alpha,a)} = \widetilde{S}^\alpha \Gamma^a V$, $H^{(\alpha,a)} = \widetilde{S}^\alpha \Gamma^a H$, $U = (V, H)$.
Generalized energy is defined, for $\kappa \geq 1$ by

$$E_\kappa(t) = \sum_{|\alpha|+|a| \leq \kappa} \|U^{(\alpha,a)}(t,\cdot)\|_{L^2}^2,$$

Modified generalized energy:

$$\mathcal{E}_\kappa(t) = \sum_{|\alpha|+|a|+1 \leq \kappa} \|\nabla U^{(\alpha,a)}(t,\cdot)\|_{L^2}^2.$$

The weighted energy norm of Klainerman-Sideris:

$$X_\kappa(t) = \sum_{|\alpha|+|a|+1 \leq \kappa} \|\langle r-t \rangle \nabla U^{(\alpha,a)}\|_{L^2}^2,$$

Weighted energy to capture the temporal decay of good unknowns:

$$Y_\kappa(t) = \sum_{|\alpha|+|a|+1 \leq \kappa} \big(\|r(\partial_r V^{(\alpha,a)} + \partial_r H^{(\alpha,a)} \cdot \omega)\|_{L^2}^2 + \|r \partial_r H^{(\alpha,a)} \cdot \omega^\perp\|_{L^2}^2 \big).$$

Estimate of weighted L^2 norm

Lemma (estimate of weighted L^2 norm)

Suppose that $(V, H) \in H_\Gamma^{\kappa-1}$ solves (16) and (17) with $\kappa \geq 12$, and suppose $E_{\kappa-3} \ll 1$. Then there holds

$$X_{\kappa-4} + Y_{\kappa-4} \lesssim E_{\kappa-3}, \quad X_{\kappa-2} + Y_{\kappa-2} \lesssim E_{\kappa-1}.$$

Modified higher-order energy estimate

Let $\kappa \geq 12$, $|\alpha| + |a| \leq \kappa - 1$, $\sigma = r - t$, $q(\sigma) = \arctan \sigma$. Denote $e^q = e^{q(\sigma)}$. Applying ∇ onto (22). Then taking L^2 norm of the first resulting equations with $\nabla V^{(\alpha,a)} e^q$ and taking L^2 norm of the second resulting equations with $\nabla H^{(\alpha,a)} e^q$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}(|\nabla V^{(\alpha,a)}|^2 + |\nabla H^{(\alpha,a)}|^2)e^q dx \\
&-\int_{\mathbb{R}^2}\mu\nabla\Delta\sum_{l=0}^{\alpha}C_\alpha^l(-1)^{\alpha-l}V^{(l,a)}\cdot\nabla V^{(\alpha,a)}e^q dx \\
&+\frac{1}{2}\sum_{1\leq i\leq 2}\int_{\mathbb{R}^2}\frac{|\nabla_i V^{(\alpha,a)} + \nabla_i H^{(\alpha,a)}\cdot\omega|^2 + |\nabla_i H^{(\alpha,a)}\cdot\omega^\perp|^2}{\langle t-r\rangle^2}e^q dx \\
&=\int_{\mathbb{R}^2}(\nabla f^1_{\alpha a}\cdot\nabla V^{(\alpha,a)} + \nabla f^2_{\alpha a} : \nabla H^{(\alpha,a)})e^q dx \\
&= I_1 + I_2,
\end{aligned}
$$

where

$$I_1 = \int_{\mathbb{R}^2} \nabla\big[\nabla^\perp \cdot \nabla \cdot \Delta^{-1}\big(- \nabla^\perp V^{(\alpha,a)} \otimes \nabla^\perp V + \nabla^\perp H^{(\alpha,a)} \otimes \nabla^\perp H\big) + \nabla^\perp \cdot \nabla \cdot \Delta^{-1}\big(- \nabla^\perp V \otimes \nabla^\perp V^{(\alpha,a)} + \nabla^\perp H \otimes \nabla^\perp H^{(\alpha,a)}\big)\big] \cdot \nabla V^{(\alpha,a)} e^q dx + \int_{\mathbb{R}^2} \nabla(\nabla^\perp H^{(\alpha,a)} \nabla V + \nabla^\perp H \nabla V^{(\alpha,a)}) : \nabla H^{(\alpha,a)} e^q dx,$$

and

$$I_2 = \sum_{\substack{\beta+\gamma=\alpha,\, b+c=a \\ |\beta|+|b|,\, |\gamma|+|c| < |\alpha|+|a|}} \Big\{ C_\alpha^\beta C_a^b \int_{\mathbb{R}^2} \nabla\big[\nabla^\perp \cdot \nabla \cdot \Delta^{-1}\big(- \nabla^\perp V^{(\beta,b)} \otimes \nabla^\perp V^{(\gamma,c)} + \nabla^\perp H^{(\beta,b)} \otimes \nabla^\perp H^{(\gamma,c)}\big)\big] \cdot \nabla V^{(\alpha,a)} e^q dx + C_\alpha^\beta C_a^b \int_{\mathbb{R}^2} \nabla(\nabla^\perp H^{(\beta,b)} \nabla V^{(\gamma,c)}) : \nabla H^{(\alpha,a)} e^q dx \Big\}.$$

Problem: I_1 contains the highest-order derivatives, it may lose derivative!

After integration by parts, I_1 gains one derivative. Moreover, it still satisfies the null condition!

$$\begin{aligned}
I_1 =& \int_{\mathbb{R}^2} \nabla_i^\perp \nabla_j \Delta^{-1}\big(\nabla_i^\perp H^{(\alpha,a)} \cdot \nabla_k \nabla_j^\perp H - \nabla_i^\perp V^{(\alpha,a)} \nabla_k \nabla_j^\perp V\big) \nabla_k V^{(\alpha,a)} e^q dx \\
&+ \int_{\mathbb{R}^2} \nabla_i^\perp \nabla_j \Delta^{-1}\big(\nabla_k V^{(\alpha,a)} \nabla_i^\perp \nabla_j^\perp V - \nabla_k H^{(\alpha,a)} \cdot \nabla_i^\perp \nabla_j^\perp H\big) \nabla_k V^{(\alpha,a)} e^q dx \\
&+ \int_{\mathbb{R}^2} \nabla_i^\perp \nabla_j \Delta^{-1}\big(\nabla_k \nabla_i^\perp H \cdot \nabla_j^\perp H^{(\alpha,a)} - \nabla_k \nabla_i^\perp V \nabla_j^\perp V^{(\alpha,a)}\big) \nabla_k V^{(\alpha,a)} e^q dx \\
&+ \int_{\mathbb{R}^2} \nabla_i^\perp \nabla_j \Delta^{-1}\big(\nabla_i^\perp \nabla_j^\perp V \nabla_k V^{(\alpha,a)} - \nabla_i^\perp \nabla_j^\perp H \cdot \nabla_k H^{(\alpha,a)}\big) \nabla_k V^{(\alpha,a)} e^q dx \\
&+ \frac{1}{2} \int_{\mathbb{R}^2} |\nabla V^{(\alpha,a)}|^2 \nabla_j (\nabla_j^\perp V e^q) dx - \frac{1}{2} \int_{\mathbb{R}^2} |\nabla H^{(\alpha,a)}|^2 \nabla_j^\perp (\nabla_j V e^q) dx \\
&- \int_{\mathbb{R}^2} \nabla_k H^{(\alpha,a)} \cdot \nabla_j^\perp H \nabla_j \big(\nabla_k V^{(\alpha,a)} e^q\big) dx + \int_{\mathbb{R}^2} \nabla_j^\perp H_i \nabla_k \nabla_j V^{(\alpha,a)} \nabla_k H_i^{(\alpha,a)} e^q dx \\
&+ \int_{\mathbb{R}^2} (\nabla_j^\perp H_i^{(\alpha,a)} \nabla_k \nabla_j V + \nabla_k \nabla_j^\perp H_i \nabla_j V^{(\alpha,a)}) \nabla_k H_i^{(\alpha,a)} e^q dx.
\end{aligned}$$

Other technical difficulties

- No ghost weight in the higher-order energy estimate. How to deal with the nonlinearities?
  $\Rightarrow$ Using the modified ghost weight energy in modified energy estimate!
- Derivative loss of $X_{\kappa-1}$ and $Y_{\kappa-1}$.
  $\Rightarrow X_{\kappa-1}$: using the "fully nonlinear" structure.
  $\Rightarrow Y_{\kappa-1}$: using the modified ghost weight energy in modified energy estimate!

Finally, we obtain the following *a priori* higher-order energy estimate

$$\mathcal{E}_\kappa(t) + E_{\kappa-1}(t) + \sum_{|\alpha|+|a|\le\kappa-1} \mu \int_0^t \int_{\mathbb{R}^2} |\Delta V^{(\alpha,a)}(\tau)|^2 + |\nabla V^{(\alpha,a)}(\tau)|^2 dxd\tau$$
$$\lesssim \int_0^t \langle\tau\rangle^{-1}(\mathcal{E}_\kappa(\tau) + E_{\kappa-1}(\tau))E_{\kappa-3}^{\frac{1}{2}}(\tau)d\tau + \mathcal{E}_\kappa(0) + E_{\kappa-1}(0),$$

and the lower-order energy estimate:

$$E_{\kappa-3}(t) + \sum_{|\alpha|+|a|\le\kappa-3} \mu \int_0^t \int_{\mathbb{R}^2} |\nabla V^{(\alpha,a)}(\tau)|^2 dxd\tau$$
$$\lesssim E_{\kappa-3}(0) + \int_0^t \langle\tau\rangle^{-\frac{3}{2}} E_{\kappa-3}(\tau)E_{\kappa-1}^{\frac{1}{2}}(\tau)d\tau.$$

Thank you!