# Qualitative analysis on positive solutions of nonlinear elliptic equations 

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(1) Qualitative analysis on positive solutions of Lane-Emden equation
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Part 1 Lane-Emden problem

$$
\begin{cases}-\Delta u=u^{p} & \text { in } \Omega,  \tag{1}\\ u>0 & \text { in } \Omega, \\ u=0 & \text { on } \partial \Omega,\end{cases}
$$

where $\Omega \subset \mathbb{R}^{2}$ is a smooth bounded domain and $p>1$ is sufficiently large.

Lane-Emden equation (1) is an extremely simple looking semilinear elliptic equation with a power focusing nonlinearity, nevertheless it has a very rich structure in terms of the dependence of the solutions on the exponent $p$ of the power nonlinearity and on both the geometry and the topology of the domain.

In any smooth bounded domain $\Omega$, problem (1) admits at least one solution for any $p>1$, which can be obtained by standard variational methods, for example minimizing the associated energy functional on the Nehari manifold.

An interest subject is the qualitative properties of positive solutions when $p$ is large.

Ren-Wei, TAMS,1994: The existence and asymptotic behavior of the least energy solution on the convex region when $p$ is large.

Adimurthi-Grossi, 2003 PAMS: Some exact asymptotic estimates of the least energy solution by using the technique of blow-up analysis.

Grossi-Grumiau-Pacella, AIHPANL, 2013: Asymptotic behavior of low energy nodal solutions.

De Marchis-lanni-Pacella [JEMS 2015; MA 2017; AIM 2017]: Asymptotic analysis and sign changing bubble towers, Morse index formula for radial solutions, Exact Morse index for nodal radial solutions.

De Marchis-Grossi-lanni-Pacella[JMPA 2019] calculated the Morse index of the single spike solutions concentrating at a non-degenerate critical point of the Robin function and hence proved the non-degeneracy of these solutions.

A direct consequence of this non-degeneracy result and the results in convex domains, the positive solution of (1) satisfying

$$
\begin{equation*}
\sup _{p} p\left\|\nabla u_{p}\right\|_{2}^{2}<\infty \tag{2}
\end{equation*}
$$

is unique.

## Problem:

(1) Whether the positive solutions of (1) with multiple spikes are non-degenerate.
(2) The uniqueness of positive solutions of (1) in general domains.
(3) Morse index of the positive solutions of (1) with multiple spikes.

Let Green's function $G(x, \cdot)$ be the solution of

$$
\begin{cases}-\Delta G(x, \cdot)=\delta_{x} & \text { in } \Omega \\ G(x, \cdot)=0 & \text { on } \partial \Omega\end{cases}
$$

where $\delta_{x}$ is the Dirac function. For $G(x, y)$, we have the following form

$$
\begin{equation*}
G(x, y)=S(x, y)-H(x, y) \text { for }(x, y) \in \Omega \times \Omega \tag{3}
\end{equation*}
$$

where $S(x, y)=-\frac{1}{2 \pi} \ln |x-y|$ and $H(x, y)$ is the regular part of $G(x, y)$. For any $x \in \Omega$, we set $R(x):=H(x, x)$, which is called the Robin function. Let $a^{k}=\left(a_{1}, \cdots, a_{k}\right)$ with $a_{j} \in \Omega$ for $j=1, \cdots, k$ and $\Psi_{k}: \Omega^{k} \rightarrow \mathbb{R}$ be defined by

$$
\begin{equation*}
\Psi_{k}\left(a^{k}\right)=\sum_{j=1}^{k} \Psi_{k, j}\left(a^{k}\right), \text { with } \Psi_{k, j}\left(a^{k}\right)=R\left(a_{j}\right)-\sum_{m=1, m \neq j}^{k} G\left(a_{j}, a_{m}\right) \text {. } \tag{4}
\end{equation*}
$$

Also it is well known that

$$
\begin{equation*}
U(x)=-2 \ln \left(1+\frac{|x|^{2}}{8}\right) \tag{5}
\end{equation*}
$$

is a positive solution of the Liouville equation

$$
\left\{\begin{array}{l}
-\Delta U=e^{U} \text { in } \mathbb{R}^{2}, \\
\int_{\mathbb{R}^{2}} e^{U} d x=8 \pi
\end{array}\right.
$$

Now we state the following basic asymptotic behavior on the positive solutions of Lane-Emden problem (1).

Theorem A. De Marchis-lanni-Pacella [JEMS 2015] Let $\left(u_{p}\right)$ be a family of solutions to (1) and (2). Then there exist a finite number of $k$ of distinct points $x_{\infty, j} \in \Omega, j=1, \cdots, k$ and a subsequence of $p$ (still denoted by $p$ ) such that setting $\mathcal{S}:=\left\{x_{\infty, 1}, \cdots, x_{\infty, k}\right\}$, one has

$$
\begin{equation*}
\lim _{p \rightarrow+\infty} p u_{p}=8 \pi \sqrt{e} \sum_{j=1}^{k} G\left(x, x_{\infty, j}\right) \text { in } C_{l o c}^{2}(\Omega \backslash \mathcal{S}) \tag{6}
\end{equation*}
$$

the energy satisfies

$$
\begin{equation*}
\lim _{p \rightarrow+\infty} p \int_{\Omega}\left|\nabla u_{p}(x)\right|^{2} d x=8 \pi e \cdot k \tag{7}
\end{equation*}
$$

and the concentrated points $x_{\infty, j}, j=1, \cdots, k$ fulfill the system

$$
\nabla_{x} \Psi_{k}\left(x_{\infty, 1}, \cdots, x_{\infty, k}\right)=0
$$

Moreover, for some small fixed $r>0$, let $x_{p, j} \in \overline{B_{2 r}\left(x_{\infty, j}\right)}$ be the sequence defined as

$$
u_{p}\left(x_{p, j}\right)=\frac{\max }{B_{2 r}\left(x_{\infty}, j\right)} u_{p}(x)
$$

then for any $j=1, \cdots, k$, it holds

$$
\lim _{p \rightarrow+\infty} x_{p, j}=x_{\infty, j}, \lim _{p \rightarrow+\infty} u_{p}\left(x_{p, j}\right)=\sqrt{e}, \lim _{p \rightarrow+\infty} \varepsilon_{p, j}=0
$$

where $\varepsilon_{p, j}=\left(p\left(u_{p}\left(x_{p, j}\right)\right)^{p-1}\right)^{-1 / 2}$. And setting

$$
\begin{equation*}
w_{p, j}(y):=\frac{p}{u_{p}\left(x_{p, j}\right)}\left(u_{p}\left(x_{p, j}+\varepsilon_{p, j} y\right)-u_{p}\left(x_{p, j}\right)\right), y \in \Omega_{p, j}:=\frac{\Omega-x_{p, j}}{\varepsilon_{p, j}}, \tag{8}
\end{equation*}
$$

one has

$$
\begin{equation*}
\lim _{p \rightarrow+\infty} w_{p, j}=U \text { in } C_{l o c}^{2}\left(\mathbb{R}^{2}\right) \tag{9}
\end{equation*}
$$

To prove the non-degeneracy of the multi-spike solutions and the local uniqueness of the solutions, a complete understanding of the properties of such solutions is crucial. In particular, we need to obtain a sharp estimate for the concentration rate $\varepsilon_{p, j}$ at $x_{p, j}$ than those stated in Theorem A.

## Theorem 1 (Grossi-lanni-L-Yan,2020)

Let $\left(u_{p}\right)$ be a family of solutions to (1) and (2), $x_{p, j}, \varepsilon_{p, j}$ and $w_{p, j}$ with $j=1, \cdots, k$ be defined in Theorem A above, then it holds

$$
\begin{equation*}
u_{p}\left(x_{p, j}\right)=\sqrt{e}\left(1-\frac{\ln p}{p-1}+\frac{1}{p}\left(4 \pi \Psi_{k, j}\left(x_{\infty}\right)+4 \sqrt{2}+2\right)\right)+O\left(\frac{1}{p^{2-\delta}}\right) \tag{10}
\end{equation*}
$$

where $\delta$ is a small fixed constant, $\Psi_{k, j}$ is the function in (4), $x_{\infty}:=\left(x_{\infty, 1}, \cdots, x_{\infty, k}\right)$. Consequently,

$$
\begin{equation*}
\varepsilon_{p, j}=\frac{1}{p} e^{-\frac{p-1}{4}}\left(e^{-\left(2 \pi \Psi_{k, j}\left(x_{\infty}\right)+2 \sqrt{2}+1\right)}+O\left(\frac{1}{p^{1-\delta}}\right)\right), \tag{11}
\end{equation*}
$$

## Theorem 1, Continue.

 and$$
\begin{equation*}
\frac{\varepsilon_{p, j}}{\varepsilon_{p, s}}=e^{2 \pi\left(\Psi_{k, s}\left(x_{\infty}\right)-\Psi_{k, j}\left(x_{\infty}\right)\right)}+O\left(\frac{1}{p^{1-\delta}}\right) \text { for } 1 \leq j, s \leq k \tag{12}
\end{equation*}
$$

Moreover, it holds

$$
\begin{equation*}
\lim _{p \rightarrow+\infty} p\left(w_{p, j}-U\right)=w_{0} \text { in } C_{l o c}^{2}\left(\mathbb{R}^{2}\right) \tag{13}
\end{equation*}
$$

where $w_{0}$ solves the non-homogeneous linear equation

$$
\begin{equation*}
-\Delta u-e^{U(x)} u=-\frac{U^{2}(x)}{2} e^{U(x)} \text { in } \mathbb{R}^{2} \tag{14}
\end{equation*}
$$

Now we define the linearized operator of Lane-Emden problem (1) by

$$
\mathcal{L}_{p}(\xi):=-\Delta \xi-p u_{p}^{p-1} \xi
$$

where $u_{p}$ is a positive solution of (1) satisfying (2). With the sharp estimates in Theorem 1, we can prove the following non-degeneracy result for a multi-spike positive solutions of (1).

## Theorem 2 (Grossi-lanni-L-Yan,2020)

Let $u_{p}$ be a positive solution of (1) satisfying (2), $k$ be the number of the bubbles of $u_{p}$ and $\xi_{p} \in H_{0}^{1}(\Omega)$ be a solution of $\mathcal{L}_{p}\left(\xi_{p}\right)=0$. Suppose that $x_{\infty}:=\left(x_{\infty, 1}, \cdots, x_{\infty, k}\right)$ is a nondegenerate critical point of $\Psi_{k}(x)$, then $\xi_{p}=0$ for large $p$.

For the local uniqueness of solutions to (1), we have the following result.

## Theorem 3 (Grossi-lanni-L-Yan,2020)

Let $u_{p}^{(1)}$ and $u_{p}^{(2)}$ be two positive solutions to (1) with

$$
\begin{equation*}
\lim _{p \rightarrow+\infty} p \int_{\Omega}\left|\nabla u_{p}^{(I)}(x)\right|^{2} d x=8 \pi e \quad \text { for } I=1,2 \tag{15}
\end{equation*}
$$

which concentrate at the same non-degenerate critical point of Robin function $R(x)$. Then $u_{p}^{(1)} \equiv u_{p}^{(2)}$ for large $p$.

Remarks on local uniqueness:

If $\Omega \subset \mathbb{R}^{2}$ is a smooth bounded convex domain, then from [Caffarelli-Friedman, DMJ, 1985], we know that the Robin function $R(x)$ is strictly convex and so it has a unique critical point which is a strict minimum. Moreover the corresponding Hessian matrix $D^{2} R(x)$ at this point is positive definite.

On the other hand, we know from [Kamburov-Sirakov, CVPDE, 2018] that $\sup _{p} p\left\|\nabla u_{p}\right\|_{2}^{2}<\infty$ holds when $\Omega$ is convex. By [Grossi-Takahashi, JFA, 2010], problem (1) possesses no solutions with $k$ spike for $k \geq 2$ when $\Omega$ is convex.

Hence from Theorem 3, problem (1) admits a unique solution for large $p$ and convex domain. This gives another proof of the uniqueness result in a convex domain obtained in [De Marchis-Grossi-lanni-Pacella, JMPA 2019].

Let us point out that the uniqueness result in [Caffarelli-Friedman, DMJ, 1985] holds only if $\Omega$ is convex. Its proof uses a uniqueness result for (1) when $p$ is close to one, and the uniqueness of Morse index one solution proved by [Lin, 1994]. If $\Omega$ is convex, the solution has one spike and its Morse index is one.

But a solution with one spike concentrating at a non-degenerate critical point of the Robin function $R(x)$ has Morse index larger than one if this critical point is not a local minimum point of $R(x)$.

Theorem 3 tells us that each non-degenerate critical point of the Robin function can generate exactly one single spike solution for (1), and if all the critical points of $R(x)$ are non-degenerate, then the number of one spike solutions equals the number of the critical points of $R(x)$.

Ideas:

## Proposition 4

For any fixed small $d>0$, it holds

$$
\begin{equation*}
u_{p}(x)=\sum_{j=1}^{k} C_{p, j} G\left(x_{p, j}, x\right)+o\left(\sum_{j=1}^{k} \frac{\varepsilon_{p, j}}{p}\right) \text { in } C^{1}\left(\Omega \backslash \bigcup_{j=1}^{k} B_{2 d}\left(x_{p, j}\right)\right), \tag{16}
\end{equation*}
$$

where

$$
\begin{equation*}
C_{p, j}:=\int_{B_{d}\left(x_{\rho, j}\right)} u_{p}^{p}(y) d y=\frac{1}{p}(8 \pi \sqrt{e}+o(1)) \tag{17}
\end{equation*}
$$

By potential theory.

Now we define

$$
v_{p, j}(x):=p\left(w_{p, j}(x)-U(x)\right)
$$

## Proposition 5

For any small fixed $d_{0}>0$ and fixed $\tau \in(0,1)$, there exists $C>0$ such that

$$
\begin{equation*}
\left|v_{p, j}\right| \leq C(1+|x|)^{\tau} \text { in } B_{\frac{d_{0}}{\varepsilon_{j, p}}}(0) . \tag{18}
\end{equation*}
$$

Inspired by [Chen-Lin,2002,CPAM, Lin-Yan, 2018, AIM].

As a consequence of Proposition 5, we can prove

## Proposition 6

It holds

$$
\lim _{p \rightarrow+\infty} v_{p, j}=w_{0} \text { in } C_{l o c}^{2}\left(\mathbb{R}^{2}\right),
$$

where $w_{0}$ solves the non-homogeneous linear equation

$$
\begin{equation*}
-\Delta u-e^{U(x)} u=-\frac{U^{2}(x)}{2} e^{U(x)} \text { in } \mathbb{R}^{2} \tag{19}
\end{equation*}
$$

Next to prove Theorem 1, we use the idea in [De Marchis-lanni-Pacella, JEMS 2015].

For non-degeneracy and local uniqueness. Our ideas are local Pohozeav identities based on blow-up analysis, which is established by Deng-Lin-Yan in 2015, JMPA. This method has been widely used to consider some related problem:

Cao, Guo, Lin, Peng, Wei, Yan.......

Cao-Peng-Yan's book: Singularly Perturbed Methods for Nonlinear Elliptic Problems, 2020.

Cao-Li-L, CVPDE 2015. L-Peng-Wang, CVPDE 2020.
Cao-L-Peng, TAMS, 2020: The number of positive solutions to the Brezis-Nirenberg problem.

The Brezis-Nirenberg problem:

$$
\begin{cases}-\Delta u=u^{\frac{N+2}{N-2}}+\varepsilon u, u>0, & \text { in } \Omega,  \tag{2}\\ u=0, & \text { on } \partial \Omega,\end{cases}
$$

where $\varepsilon$ is a small positive parameter, $\Omega \subset \subset \mathbb{R}^{N}$ is smooth, $N \geq 3$.

Assumption A: The problem

$$
\begin{cases}-\Delta u=u^{N+2}, u>0, & \text { in } \Omega,  \tag{21}\\ u=0, & \text { on } \partial \Omega\end{cases}
$$

has no solutions.

## Theorem 7 (Cao-L-Peng,TAMS,2020)

Let $N \geq 6$, for any $k \in\left[1, k_{0}\right]$, if the the critical points of $\Psi_{k}$ are nondegenerate and $M_{k}(x)$ is a positive matrix at these points, Assumption $A$ holds, then

$$
\text { the number of the positive solution to (20) is } \sum_{i=1}^{k_{0}} l_{i} \text {, }
$$

where $k_{0}$ is the largest number of the blow-up points of the positive solution to the equation (20) and $l_{i}$ is the number of the critical point of $\Psi_{i}$.

## Morse index:

Furthermore, we calculate the morse index of the above positive solutions by local Pohozeav identities based on blow-up analysis.

For more results on morse index, we can refer to Massimo Grossi and Filomena Pacella's paper or the book
[Lucio Damascelli, Filomena Pacella: Morse index of solutions of nonlinear elliptic equation,2019].

Part 2. We consider the following nonlinear elliptic equation

$$
\begin{cases}-\Delta u=f(u) & \text { in } \Omega,  \tag{22}\\ u>0 & \text { in } \Omega, \\ u=0 & \text { on } \partial \Omega\end{cases}
$$

Here $\Omega \subset \mathbb{R}^{N}$ with $N \geq 2$ is a smooth bounded domain.
$f$ is a smooth nonlinearity.

We focus on the number of the critical points of positive solutions.

The above problem is a generalization of the elastic torsion problem $(f(u) \equiv 1)$, a classical topic in PDEs, with references dating back to St. Venant(1856).

From then, many techniques and important results to address this problem were developed in the literature (Morse theory, degree theory, complex analysis, etc.).

Here an open problem is
How many critical points does the positive solution possess.

The calculation of the number of critical points of a function is strictly related to the topological properties of the domain.

This link is clearly highlighted in the following Poincaré-Hopf Theorem.
Theorem A (Poincaré-Hopf Theorem). Let $\Omega \subset \mathbb{R}^{N}, N \geq 2$, be a smooth bounded domain. Let $v$ be a vector field on $\Omega$ with isolated zeroes $x_{1}, . ., x_{k}$ and such that $v(x) \cdot \nu(x)<0$ for any $x \in \partial \Omega$ (here $\nu$ is the outward normal vector to $\partial \Omega$ ). Then we have the formula

$$
\begin{equation*}
\sum_{i=1}^{k} \operatorname{index}_{x_{i}}(v)=(-1)^{N} \chi(\Omega) \tag{23}
\end{equation*}
$$

where index $_{x}(v)=\operatorname{deg}(v, B(x, \delta), 0)$ with small fixed $\delta>0$ and $\chi(\Omega)$ is the Euler characteristic of $\Omega$.

Choosing $v=\nabla u$ in Theorem A, we get a link between an analytic problem (to look for critical points of $u$ ) and a topological invariant (the Euler characteristic of $\Omega$ ).

The first case studied in the literature is when $\Omega$ is a (strictly) convex domain. In this case $\chi(\Omega)=1$ and so (23) becomes

$$
\begin{equation*}
\sum_{i=1}^{k} \operatorname{index}_{x_{i}}(\nabla u)=(-1)^{N} \tag{24}
\end{equation*}
$$

Of course since $u$ is a solution to (22), we always have a maximum point for $u$ whose index is $(-1)^{N}$. The question is now when does the sum in (24) reduce to a singleton?

Here we list some results that give an affirmative answer to the question (25).

- $f(s)=1$ and $\Omega \subset \mathbb{R}^{2}$ is a convex bounded domain (Makar-Limanov,1971, MZ).
- $f(s)=\lambda_{1} s$ where $\lambda_{1}$ is the first eigenvalue of the Laplace operator and $\Omega \subset \mathbb{R}^{N}$ is a strictly convex bounded domain (Brascamp and Lieb,1976, JFA; Korevaar, 1983,IUMJ).
- $f$ locally lipschitz and $\Omega \subset \mathbb{R}^{N}$ is a symmetric bounded domain convex in any direction (Gidas, Ni and Nirenberg,1979,CMP).
- $f \geq 0, \Omega \subset \mathbb{R}^{2}$ is a bounded domain with positive curvature and $u$ is a semi-stable solution to (22) (Cabré and Chanillo,1998, Selecta Math.).

Here we consider the domain

$$
\Omega_{\varepsilon}=\Omega \backslash B(P, \varepsilon) \text { with } P \in \Omega \text { and } \varepsilon \text { small, }
$$

and a solution $u_{\varepsilon}$ of

$$
\begin{cases}-\Delta u=f(u) & \text { in } \Omega_{\varepsilon}  \tag{26}\\ u>0 & \text { in } \Omega_{\varepsilon} \\ u=0 & \text { on } \partial \Omega_{\varepsilon}\end{cases}
$$

We suppose

$$
\begin{equation*}
\left|u_{\varepsilon}\right| \leq C \text { in } \Omega_{\varepsilon} \text { with } C \text { independent of } \varepsilon . \tag{27}
\end{equation*}
$$

By the standard regularity theory, an immediate consequence of (27) is that there exist a sequence $\varepsilon_{n} \rightarrow 0$ and $u_{0} \in C^{2}(\Omega)$, the solution to (22), such that

$$
\left\{\begin{array}{l}
u_{\varepsilon_{n}} \rightharpoonup u_{0} \text { weakly in } H_{0}^{1}(\Omega) \text { (here we extend } u_{\varepsilon_{n}} \text { to } 0 \text { in } B(P, \varepsilon) \text { ), }  \tag{28}\\
u_{\varepsilon_{n}} \rightarrow u_{0} \text { in } C^{2}(K) \text { for any compact set } K \subset \Omega \backslash P .
\end{array}\right.
$$

If we write (23) for $v=\nabla u_{\varepsilon}$ (again assuming that the number of critical point of $u_{\varepsilon}$ is finite) and denote by
$\mathcal{C}=\left\{\right.$ critical points of $u_{0}$ in $\left.\Omega\right\}$ and $\mathcal{C}_{1}=\left\{\nabla u_{\varepsilon}(x)=0\right\} \cap\{\operatorname{dist}(x, \mathcal{C})>\delta\}$, and observing that $\chi\left(\Omega_{\varepsilon}\right)=\chi(\Omega)+(-1)^{N-1}$, we get that if $P \notin \mathcal{C}$ we have

$$
\begin{equation*}
\sum_{x_{i} \in \mathcal{C}_{1}} i n d e x_{x_{i}}\left(\nabla u_{\varepsilon}\right)=-1 . \tag{29}
\end{equation*}
$$

Hence we have that the solution $u_{\varepsilon}$ has at least one additional critical point which is away from $\mathcal{C}$. As before a natural question arises

> when does the sum in (29) reduce to a singleton?

General speaking, we consider the variety of the number of critical points of the positive solution after digging a small ball in $\Omega$.

## Theorem 8 (Grossi-L,2020)

Suppose that $u_{\varepsilon}$ is a solution to (26) which verifies (27) and $u_{0}$ its weak limit. We have that if

$$
P \text { is not a critical point of } u_{0},
$$

then for $\varepsilon$ small enough there is exactly one critical point for $u_{\varepsilon}$ in $B(P, d) \backslash B(P, \varepsilon)$ (here $B(P, d) \subset \Omega$ is chosen not containing any critical point of $u_{0}$ ).
Moreover the critical point $x_{\varepsilon} \in B(P, d)$ of $u_{\varepsilon}$ is a saddle point of index -1 which verifies

$$
u_{\varepsilon}\left(x_{\varepsilon}\right) \rightarrow u_{0}(P) .
$$

Which answer the question (30).

## A Remark

## The condition that $P$ is not a critical point of $u_{0}$ cannot be removed.

An easy counterexample can be constructed when $\Omega=B(0,1)$ and $u_{0}$ is the first eigenfunction of $-\Delta$ with zero Dirichlet boundary condition. Then

$$
\nabla u_{0}(0)=0
$$

Let $\Omega_{\varepsilon}=B(0,1) \backslash B(0, \varepsilon)$ and $u_{\varepsilon}$ be the first radial eigenfunction in the annulus $\Omega_{\varepsilon}$. Of course $u_{\varepsilon}$ has infinitely many critical points in $B(0, d) \backslash B(0, \varepsilon)$ for any $\varepsilon>0$ small and $d \in(0,1)$.

## Theorem 9 (Grossi-L,2020)

Suppose that $u_{\varepsilon}$ is a solution to (26) which verifies (27). Denoting by $u_{0}$ its weak limit we get that if
$P$ is not a critical point of $u_{0}$ and all critical points of $u_{0}$ are nondegenerate, then
$\sharp\left\{\right.$ critical points of $u_{\varepsilon}$ in $\left.\Omega_{\varepsilon}\right\}=\sharp\left\{\right.$ critical points of $u_{0}$ in $\left.\Omega\right\}+1$.

Next we consider the case $\nabla u_{0}(P)=0$. As noted in above Remark it is even possible to have infinitely many critical points. Moreover, formula (29) becomes

$$
\begin{equation*}
\sum_{x_{i} \in B(P, d) \backslash B(P, \varepsilon)} \text { index }_{x_{x_{i}}}\left(\nabla u_{\varepsilon}\right)=\operatorname{index}_{P}\left(\nabla u_{0}\right)-1, \tag{31}
\end{equation*}
$$

where $B(P, d) \subset \Omega$ is chosen not containing any critical point of $u_{0}$.
Hence the number of critical points of $u_{\varepsilon}$ in a neighborhood of $P$ is strongly depending of the index of $\nabla u_{0}$ at $P$. In particular, if $P$ is a maximum point for $u_{0}$ then (31) becomes

$$
\sum_{x_{i} \in B(P, d) \backslash B(P, \varepsilon)} \text { index }_{x_{i}}\left(\nabla u_{\varepsilon}\right)=(-1)^{N}-1 .
$$

We need an additional technical assumption.
Suppose that $u_{\varepsilon}$ and $u_{0}$ verify

$$
\begin{align*}
& \int_{\Omega_{\varepsilon}}\left(f\left(u_{\varepsilon}(y)\right)-f\left(u_{0}(y)\right)\right) \frac{\partial G(x, y)}{\partial x_{i}} d y \\
& \quad=o(|x-P|)+ \begin{cases}o\left(\frac{\varepsilon^{N-2}}{|x-P|^{N-1}}\right) & \text { for } N \geq 3, \\
o\left(\frac{1}{|x-P| \cdot|\log \varepsilon|}\right) & \text { for } N=2\end{cases} \tag{32}
\end{align*}
$$

Some cases where it is verified are the following.

- $f(s) \equiv 1$.
- $\Omega$ convex and symmetric with respect to $P$ as in the Gidas, Ni and Nirenberg Theorem.

Denote by $\mathbf{H}(P)$ the Hessian matrix of $u_{0}$ at $P$.

## Theorem 10 (Grossi-L,2020)

Suppose that $u_{\varepsilon}$ is a solution to (26) which verifies (27). Denoting by $u_{0}$ its weak limit we get that if (32) holds and all critical points of $u_{0}$ are nondegenerate, for small $\varepsilon$ we have that
$\sharp\left\{\right.$ critical points of $u_{\varepsilon}$ in $\left.\Omega_{\varepsilon}\right\} \geq \sharp\left\{\right.$ critical points of $u_{0}$ in $\left.\Omega\right\}+2$ index $_{P}\left(\nabla u_{0}\right)-1$.
Furthermore, if the negative eigenvalues of $\boldsymbol{H}(P)$ are simple, then
$\sharp\left\{\right.$ critical points of $u_{\varepsilon}$ in $\left.\Omega_{\varepsilon}\right\}=\sharp\left\{\right.$ critical points of $u_{0}$ in $\left.\Omega\right\}+2$ index $_{P}\left(\nabla u_{0}\right)-1$.

## Examples and Applications

Example 1: Assume that $\Omega_{\varepsilon}=B(0,1) \backslash B(P, \varepsilon)$ and $\phi_{1, \varepsilon}$ is the first eigenfunction of $-\Delta$ in $B(0,1) \backslash B(P, \varepsilon)$. Then we have that

$$
\sharp\left\{\text { critical points of } \phi_{1, \varepsilon} \text { in } \Omega_{\varepsilon}\right\}= \begin{cases}\infty & \text { if } P=0, \\ 2 & \text { if } P \neq 0,\end{cases}
$$

for $\varepsilon$ small enough.

Example 2: Let $\Omega \subset \mathbb{R}^{N}$ be symmetric and convex with respect $x_{1}, . ., x_{N}$ with $N \geq 2, P \neq 0$ and $u_{\varepsilon}$ solution of

$$
\begin{cases}-\Delta u=u^{p} & \text { in } \Omega_{\varepsilon}  \tag{33}\\ u>0 & \text { in } \Omega_{\varepsilon} \\ u=0 & \text { on } \partial \Omega_{\varepsilon}\end{cases}
$$

with $1<p<\frac{N+2}{N-2}$ for $N \geq 3$ and $p>1$ if $N=2$. Moreover assume that

$$
\begin{equation*}
\int_{\Omega_{\varepsilon}}\left|\nabla u_{\varepsilon}\right|^{2} \leq C, \quad C \text { independent of } \varepsilon \tag{34}
\end{equation*}
$$

Then $u_{\varepsilon}$ admits exactly two critical points.
Here the key point is to obtain a priori estimate $\left\|u_{\varepsilon}\right\|_{L^{\infty}\left(\Omega_{\varepsilon}\right)} \leq C$ by blow-up analysis in Gidas-Spruck's paper(1981,CPDE).

Example 3: Let $\Omega \subset \mathbb{R}^{N}, N \geq 3$ be convex and $u_{\varepsilon}$ solution of (33) satisfying (34) with $\nabla u_{0}(P) \neq 0$. Then for $p$ sufficiently close to $\frac{N+2}{N-2}$ we have that $u_{\varepsilon}$ admits exactly two critical points for $\varepsilon$ small enough.

Example 4: Let $\Omega \subset \mathbb{R}^{2}$ be a smooth bounded domain whose boundary has positive curvature and $u_{\varepsilon}$ semi-stable positive solution to

$$
\begin{cases}-\Delta u=\lambda f(u) & \text { in } \Omega_{\varepsilon} \\ u=0 & \text { on } \partial \Omega_{\varepsilon}\end{cases}
$$

where $f>0$ is an increasing function and $\nabla u_{0}(P) \neq 0$. Then for any $0<\lambda<\lambda^{*}$ and $\varepsilon$ small enough we have that $u_{\varepsilon}$ admits exactly two critical points.

Example 5: Assume that $\Omega$ and $f$ are like in Gidas, Ni and Nirenberg Theorem, $P=0$. Then if all the eigenvalues of $H(P)$ are simple then for $\varepsilon$ small enough we have that
$\sharp\left\{\right.$ critical points of $u_{\varepsilon}$ in $\left.\Omega_{\varepsilon}\right\}=2 N$.

Example 6: Let $\Omega_{\varepsilon}$ be the annulus $B(0,1) \backslash B(0, \varepsilon), u_{\varepsilon}$ a radial solution to (26) and $u_{0}$ its weak limit. Assume that $f(s)>0$ for $s>0$ and set $r=|x|$ and $u_{\varepsilon}=u_{\varepsilon}(r)$. We have that for $\varepsilon>0$ small enough $u_{\varepsilon}(r)$ has a unique critical point $r=r_{\varepsilon}$ given by

$$
r_{\varepsilon}= \begin{cases}{\left[\left(\frac{N(N-2) u_{0}(0)}{f\left(u_{0}(0)\right)}\right)^{\frac{1}{N}}+o(1)\right] \varepsilon^{\frac{N-2}{N}}} & \text { if } N \geq 3 \\ \left(\sqrt{\frac{u_{0}(0)}{2 f\left(u_{0}(0)\right)}}+o(1)\right) \frac{1}{\sqrt{|\log \varepsilon|}} & \text { if } N=2\end{cases}
$$

## An application on the spectral gap

Theorem A. [Steinerberger, 2018, JFA] Let $\Omega \subset \mathbb{R}^{2}$ be a bounded, convex domain and assume the solution of

$$
\begin{cases}-\Delta u=1 & \text { in } \Omega \\ u=0 & \text { on } \partial \Omega\end{cases}
$$

assumes its maximum $x_{0} \in \Omega$. There are universal constants $c_{1}, c_{2}>0$ such that

$$
\begin{equation*}
\lambda_{\max }\left(D^{2} u\left(x_{0}\right)\right) \leq-c_{1} \exp \left(-c_{2} \frac{\operatorname{diam}(\Omega)}{\operatorname{inrad}(\Omega)}\right) . \tag{35}
\end{equation*}
$$

Steinerberger proposed the following open problem:
"Does Theorem A also hold true on domains that are not convex but merely simply connected or perhaps only bounded?"

## Theorem 11 (Chen-L, 2020)

Let $\Omega \subset \mathbb{R}^{2}$ be a bounded, convex domain, $\Omega_{\varepsilon}=\Omega \backslash B(P, \varepsilon)$ with $P \in \Omega$ and $B(P, \varepsilon)$ denote the ball centered at $P$ and radius $\varepsilon$. If $u_{\varepsilon}$ is the solution of

$$
\begin{cases}-\Delta u=1 & \text { in } \Omega_{\varepsilon}, \\ u=0 & \text { on } \partial \Omega_{\varepsilon}\end{cases}
$$

with its maximum $x_{\varepsilon} \in \Omega_{\varepsilon}$. Let $\lambda_{1}$ and $\lambda_{2}$ be two eigenvalues of $D^{2} u_{0}(x)$ at $P$, then

$$
\lim _{\varepsilon \rightarrow 0} \lambda_{\max }\left(D^{2} u_{\varepsilon}\left(x_{\varepsilon}\right)\right)= \begin{cases}\max \left\{\lambda_{1}, \lambda_{2}\right\} & \text { if } x_{0} \neq P \\ \max \left\{\lambda_{1}, \lambda_{2},-\left|\lambda_{2}-\lambda_{1}\right|\right\} & \text { if } x_{0}=P\end{cases}
$$

where $x_{0}$ is the point in Theorem $A$.

A remark: Here we take $\Omega \subset \mathbb{R}^{2}$ a bounded and convex domain, $\Omega_{\varepsilon}=\Omega \backslash B(P, \varepsilon)$ with $x_{0}=P$ and $\varepsilon$ small.

Now we suppose that (35) is true for $\Omega_{\varepsilon}$, then there exist two positive constants $c_{3}$ and $c_{4}$, which is independent with $\varepsilon$, such that

$$
\begin{equation*}
\lambda_{\max }\left(D^{2} u_{\varepsilon}\left(x_{\varepsilon}\right)\right) \leq-c_{1} \exp \left(-c_{2} \frac{\operatorname{diam}\left(\Omega_{\varepsilon}\right)}{\operatorname{inrad}\left(\Omega_{\varepsilon}\right)}\right) \leq-c_{3} \exp \left(-c_{4} \frac{\operatorname{diam}(\Omega)}{\operatorname{inrad}(\Omega)}\right) . \tag{36}
\end{equation*}
$$

On the other hand, moreover if we suppose $\lambda_{1}=\lambda_{2}$ (for example $\Omega=B(0,1)$ ), then Theorem 11 gives us

$$
\lim _{\varepsilon \rightarrow 0} \lambda_{\max }\left(D^{2} u_{\varepsilon}\left(X_{\varepsilon}\right)\right)=0
$$

which is a contradiction with (36).
Hence we deduce that (35) doesn't hold for above non-convex domain $\Omega_{\varepsilon}$, which gives a negative answer to above open problem.

## Key ideas.

The key step is to derive sharp $C^{2}$ expansions of the solution $u_{\varepsilon}$ which improve (28). For $N \geq 3$ our basic estimate near $\partial B(P, \varepsilon)$ is the following

$$
u_{\varepsilon}(x)=u_{0}(x)-\frac{u_{0}(P)+o(1)}{|x-P|^{N-2}} \varepsilon^{N-2}+o(1) .
$$

Note that near $\partial B(P, \varepsilon)$ there is an interaction between the weak limit $u_{0}$ and the fundamental solution of the Laplacian.

Another crucial result is to derive that $u_{\varepsilon}$ and $u_{0}(x)-\frac{u_{0}(P)}{|x-P|^{N-2}} \varepsilon^{N-2}$ are close in the $C^{2}$-topology in $B(P, d) \backslash B(P, \varepsilon)$.

Let us write down the equation satisfied by $u_{\varepsilon}-u_{0}$ where $u_{0}$ and $u_{\varepsilon}$ are solutions of (22) and (26) respectively,

$$
\begin{cases}-\Delta\left(u_{\varepsilon}-u_{0}\right)=f\left(u_{\varepsilon}\right)-f\left(u_{0}\right) & \text { in } \Omega_{\varepsilon} \\ u_{\varepsilon}-u_{0}=0 & \text { on } \partial \Omega \\ u_{\varepsilon}-u_{0}=-u_{0} & \text { on } \partial B(P, \varepsilon) .\end{cases}
$$

By Green's representation formula, we get

$$
\begin{align*}
u_{\varepsilon}(x)= & u_{0}(x)+\int_{\partial B(P, \varepsilon)} \frac{\partial G_{\varepsilon}(x, y)}{\partial \nu_{y}} u_{0}(y) d \sigma(y)  \tag{37}\\
& +\int_{\Omega_{\varepsilon}}\left(f\left(u_{\varepsilon}(y)\right)-f\left(u_{0}(y)\right)\right) G_{\varepsilon}(x, y) d y
\end{align*}
$$

A Remark: Above involving Green's function can be used to handle the critical point of Robin function and Kirchhoff-Routh function [Grossi-L-Yan,2020].

## Thank you!

