Decay and Vanishing of some D-Solutions of the Navier–Stokes equations

Na Zhao
(Institute of Applied Physics and Computational Mathematics)

Joint with Carrillo Bryan, Xinghong Pan
and Qi S. Zhang

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Overview

Introduction

Main result and proof

D-solutions

- Solutions to the steady Navier–Stokes equations:

$$\begin{cases} (u\cdot\nabla)u+\nabla p-\Delta u=f, \text{ in } D\subset\mathbb{R}^3, \\ \nabla\cdot u=0, \end{cases} \tag{1}$$

  with finite Dirichlet integral:

$$\int_D |\nabla u(x)|^2 dx < +\infty$$

  and various boundary conditions, with also the requirement that u vanishes at infinity; here $D\subset\mathbb{R}^3$ is a noncompact or compact, connected domain.

- Homogeneous D-solutions: $f=0$, $u=0$ on ∂D.
- Basic noncompact domains: $\mathbb{R}^3$(whole space), $\mathbb{R}^2\times[0,1]$(slab).

Previous work

- Leray, 1933: variational method.
- Liouville type property:

  Given a noncompact domain, is a homogeneous D-solution equal to 0 ?
- $n = 2$: $\mathbb{R}^2$, Gilbarg–Weinberger, [Ann. Scuola Norm. Sup. Pisa Cl. Sci, 1978].
- $n \geq 4$: $\mathbb{R}^n$, Galdi's book, 2011.
- $n = 3$: $\mathbb{R}^3$,
  - axially symmetric case without swirl ($u_\theta = 0$): Koch–Nadirashvili–Seregin–Sverak, [Acta Math., 2009]; Korobkov–Pileckas–Russo, [JMFM, 2015].
  - other cases: unknown!

Previous work: $\mathbb{R}^3$

Theorem (Galdi, 2011)

Let u be a D-solutions to steady Navier–Stokes equations, P is the associated pressure. Then, there exist a constant $P_0 \in \mathbb{R}$ such that

$$\lim_{|x|\to\infty} |D^\alpha u(x)| + \lim_{|x|\to\infty} |D^\alpha (P(x) - P_0)| = 0,$$

for any $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in [\mathbb{N} \cup 0]^3$.

A priori estimate:

- Sobolev embedding: $\nabla u \in L^2(\mathbb{R}^3) \Rightarrow u \in L^6(\mathbb{R}^3)$.
- Galdi's result $\Rightarrow u \in L^\infty(\mathbb{R}^3)$.
- $u \in L^p(\mathbb{R}^3)$, for any $p \in [6, \infty]$.

Previous work: $\mathbb{R}^3$

Extra integral or decay assumption:

- Galdi, 2011: $u \in L^{\frac{9}{2}}(\mathbb{R}^3) \Rightarrow u \equiv 0$.
- Chae–Wolf, [JDE, 2016]: improved Galdi's result by a log factor.
- Chae, [CMP, 2014]: $\Delta u \in L^{\frac{6}{5}}(\mathbb{R}^3) \Rightarrow u \equiv 0$.

  Notice: $\|\Delta u\|_{L^{\frac{6}{5}}}$ scales the same way as $\|\nabla u\|_{L^2}$.
- Seregin, [Nonlinearity, 2016]: $u \in L^6(\mathbb{R}^3) \cap BMO^{-1} \Rightarrow u \equiv 0$.
- Kozono–Terasawa–Wakasugi, [JFA, 2017]:

  $\nabla \times u = o(|x|^{-\frac{5}{3}})$ or $u = o(|x|^{-\frac{2}{3}}) \Rightarrow u \equiv 0$.

Main difficulty:

- Decay of u at infinity is not fast enough.
- No a priori decay rate is known in $\mathbb{R}^3$.

Previous work: $\mathbb{R}^3$

Axially symmetric D-solution:

A priori decay estimate:

- Choe–Jin, [JMFM, 2009]: $|u^r| + |u^z| \lesssim \left(\frac{\ln r}{r}\right)^{\frac{1}{2}}$, $|u^\theta| \leq \frac{(\ln r)^{\frac{1}{8}}}{r^{\frac{3}{8}}}$.
- Weng, [JMFM, 2017]: $|u(x)| \leq C\left(\frac{\ln r}{r}\right)^{\frac{1}{2}}, |w^\theta(x)| \leq Cr^{-(\frac{19}{16})^-}$,
$$|w^r(x)| + |w^z(x)| \leq Cr^{-(\frac{67}{64})^-}.$$
- Carrillo–Pan–Zhang, [JFA, 2020]:
  - Brezis–Gallouet inequality: $|u(x)| \leq C\left(\frac{\ln r}{r}\right)^{\frac{1}{2}}$;
  - $|w^\theta(x)| \leq C\frac{(\ln r)^{\frac{3}{4}}}{r^{\frac{5}{4}}}$, $|w^r(x)| + |w^z(x)| \leq C\frac{(\ln r)^{\frac{11}{8}}}{r^{\frac{9}{8}}}$.

Extra decay assumption:

- Wang, [JDE, 2019] or Z., [NA, 2019]:
$w = o(r^{-\frac{5}{3}})$ or $u = o(r^{-\frac{2}{3}}) \Rightarrow u = 0$.

Previous work: $\mathbb{R}^2 \times [0, 1]$

- Carrillo–Pan–Zhang, [JFA, 2020]: Axially symmetric D-solutions with periodic boundary conditions:

$$\int_0^1 u^r dz = \int_0^1 u^z dz = 0 \Rightarrow u = 0.$$

- Other cases:

Case 1 D-solutions with Dirichlet boundary?

Case 2 D-solutions with periodic boundary without extra assumption?

Main result 1: Dirichlet boundary condition

Theorem (Carrillo–Pan–Zhang–Z., ARMA, 2020)

Let u be a smooth, bounded solution to the problem

$$\begin{cases} (u \cdot \nabla)u + \nabla p - \Delta u = 0, \quad \mathit{in} \quad \mathbb{R}^2 \times [0,1], \\ \nabla \cdot u = 0, \\ u(x)|_{x_3=0} = u(x)|_{x_3=1} = 0, \end{cases} \tag{2}$$

such that the Dirichlet integral satisfies the condition:

$$\int_0^1 \int_{\mathbb{R}^2} |\nabla u(x)|^2 dx < \infty. \tag{3}$$

Then, $u \equiv 0$.

Main result 1: Dirichlet boundary condition

Remark:

- If the Dirichlet integral is infinite, then 0 may not be the unique solution.
  Example: $u = (x_3 - x_3^2, 0, 0), p = -2x_1$.
- Poincare inequality: $\nabla u \in L^2 \Rightarrow u \in L^2$. u decays like $\frac{1}{|x|}$ in the integral sense.
- Due to the boundary, the difficulty is to deal with the pressure!

Main result 1: Pressure estimate

Lemma (Pressure estimate)

Let u, p be the solution, then we have

$$\|p - p_R\|_{L^2(\Omega_R)} \leq C_0 R, \tag{4}$$

where $C_0 = C(\|u\|_{L^\infty}, \|u\|_{L^2}, \|\nabla u\|_{L^2})$ *and* $p_R := \frac{1}{|\Omega_R|}\int_{\Omega_R} p dx$ *is the average of* p *on* Ω_R *with* $\Omega_R = \left\{x' \in \mathbb{R}^2 \middle| |x'| \leq R\right\} \times [0,1]$.

Proof: By using Bogovskii estimate and scaling technique, for any $f \in L^2(\Omega_R)$ with $\int_{\Omega_R} f = 0$, there exists at least one $V : \Omega_R \to \mathbb{R}^3$ such that

$$\nabla \cdot V = f, \quad V \in W_0^{1,2}(\Omega_R), \quad \|\nabla V\|_{L^2(\Omega_R)} \leq CR\|f\|_{L^2(\Omega_R)}$$

Let $f = p - p_R$, one has

$$\int_{\Omega_R} \nabla(p - p_R) \cdot V dx = \int_{\Omega_R} (\Delta u - u \cdot \nabla u) \cdot V dx. \tag{5}$$

Main result 1: Pressure estimate

Integration by parts indicate that

$$
\begin{aligned}
& \int_{\Omega_R} (p - p_R)^2 dx \\
= & \int_{\Omega_R} (p - p_R) \nabla \cdot V dx \\
= & \int_{\Omega_R} \sum_{i,j=1}^{3} \partial_i u^j \partial_i V^j + \nabla \cdot (u \otimes u) \cdot V dx \\
= & \int_{\Omega_R} \sum_{i,j=1}^{3} (\partial_i u^j - u^i u^j) \partial_i V^j dx \\
\leq & \|\nabla V\|_{L^2(\Omega_R)} \big(\|\nabla u\|_{L^2(\Omega_R)} + \|u\|_{L^\infty(\Omega_R)} \|u\|_{L^2(\Omega_R)} \big) \\
\leq & \frac{\varepsilon}{R^2} \|\nabla V\|_{L^2(\Omega_R)}^2 + C_\varepsilon R^2 \big(\|\nabla u\|_{L^2(\Omega_R)} + \|u\|_{L^\infty(\Omega_R)} \|u\|_{L^2(\Omega_R)} \big)^2 \\
\leq & C\varepsilon \|p - p_R\|_{L^2(\Omega_R)}^2 + C_\varepsilon R^2 \big(\|\nabla u\|_{L^2(\Omega_R)} + \|u\|_{L^\infty(\Omega_R)} \|u\|_{L^2(\Omega_R)} \big)^2 .
\end{aligned}
$$

By choosing ε is small enough, we can get the pressure estimate.

Main result 1: Vanishing of u

Proof of main result 1: Let $\phi(s)$ be a smooth cut-off function satisfying

$$\phi(s) = \begin{cases} 1 & s \in [0, 1/2], \\ 0 & s \geq 1, \end{cases} \tag{6}$$

with the usual property that ϕ, ϕ' and ϕ'' are bounded. Set $\phi_R(y') = \phi(\frac{|y'|}{R})$ where R is a large positive number. For convenience of notation, we denote $I = [0, 1]$. Now testing the Navier-Stokes equation

$$u \cdot \nabla u + \nabla p = \Delta u$$

with $u\phi_R$, we obtain

$$\int_{\mathbb{R}^2 \times I} -\Delta u \cdot (u\phi_R) dx = \int_{\mathbb{R}^2 \times I} -(u \cdot \nabla u + \nabla(p - p_R)) \cdot (u\phi_R) dx.$$

Main result 1: Vanishing of u

Integration by parts indicates that

$$\int_{\mathbb{R}^2\times I} |\nabla u|^2 \phi_R dx = \underbrace{\frac{1}{2}\int_{\mathbb{R}^2\times I} |u|^2 \Delta\phi_R dx}_{I_1} + \underbrace{\frac{1}{2}\int_{\mathbb{R}^2\times I} |u|^2 u\cdot\nabla\phi_R dx}_{I_2} + \underbrace{\int_{\mathbb{R}^2\times I} (p-p_R) u\cdot\nabla\phi_R dx}_{I_3}.$$

Denote $\bar{B}_{R/\frac{1}{2}R} := \{x' | 1/2R \le |x'| \le R\}$ i.e. the dyadic annulus. Then we have, since ϕ_R depends only on r, that

Main result 1: Vanishing of u

$$I_1 \lesssim \frac{1}{R^2}\|u\|^2_{L^2(\bar{B}_{R/\frac{1}{2}R}\times I)}; \qquad I_2 \lesssim \frac{\|u^r\|_\infty}{R}\|u\|^2_{L^2(\bar{B}_{R/\frac{1}{2}R}\times I)};$$

$$I_3 \lesssim \frac{1}{R}\|u^r\|_{L^2(\bar{B}_{R/\frac{1}{2}R}\times I)}\|p-p_R\|_{L^2(\bar{B}_{R/\frac{1}{2}R}\times I)} \lesssim C_0\|u\|_{L^2(\bar{B}_{R/\frac{1}{2}R}\times I)}.$$

Now let $R \to +\infty$, using $u \in L^2(\mathbb{R}^2 \times I)$, we arrive at

$$\int_{\mathbb{R}^2\times I} |\nabla u|^2 dx = 0,$$

which shows that $u \equiv c$. Besides, recall $u = 0$ at the boundary, then at last we deduce

$$u \equiv 0.$$

Main result 2: Periodic boundary condition

Theorem (Carrillo–Pan–Zhang–Z., ARMA, 2020)

Let u be a smooth axially symmetric solution to the problem

$$\begin{cases} (u\cdot\nabla)u + \nabla p - \Delta u = 0, \quad \textit{in} \quad \mathbb{R}^2 \times S^1 = \mathbb{R}^2 \times [-\pi, \pi], \\ \nabla \cdot u = 0, \\ u(x_1, x_2, z) = u(x_1, x_2, z + 2\pi), \\ \lim\limits_{|x|\to\infty} u = 0, \end{cases} \tag{7}$$

with finite Dirichlet integral:

$$\int_{-\pi}^{\pi}\int_{\mathbb{R}^2} |\nabla u(x)|^2 dx < +\infty.$$

Then $u = 0$.

Main result 2: Periodic boundary condition

Ideas:

- In this case, we can't get $u \in L^2$ by Poincaré inequality! However, we have $u^r \in L^2$ since

$$\int_{-\pi}^{\pi} u^r(r,z)dz = 0, \quad \forall r \geq 0.$$

This can be seen from integrating the divergence free condition in the z direction:

$$\partial_r u^r + \frac{u^r}{r} + \partial_z u^z = 0, \Rightarrow r\partial_r \int_{-\pi}^{\pi} u^r dz + \int_{-\pi}^{\pi} u^r dz = 0.$$

Therefore

$$\left(r \int_{-\pi}^{\pi} u^r(r,z)dz \right)' = 0, \Rightarrow \int_{-\pi}^{\pi} u^r(r,z)dz = 0 \times \int_{-\pi}^{\pi} u^r(0,z)dz = 0.$$

Main result 2: Periodic boundary condition

- $u^r \in L^2$ is enough to deal with I_2 since the cut-off function ϕ_R depends only on r!

$$
\begin{aligned}
I_2 =& \frac{1}{2}\int_{\mathbb{R}^2\times I} |u|^2 u\cdot\nabla\phi_R dx = \frac{1}{2}\int_{\mathbb{R}^2\times I} |u|^2 u^r \partial_r\phi_R dx \\
\lesssim& \|u\|^2_{L^\infty(\bar{B}_{R/\frac{1}{2}R}\times I)} \|u^r\|_{L^2(\bar{B}_{R/\frac{1}{2}R}\times I)} \|\partial_r\phi_R\|_{L^2(\bar{B}_{R/\frac{1}{2}R}\times I)} \\
\lesssim& \|u\|^2_{L^\infty(\bar{B}_{R/\frac{1}{2}R}\times I)} \|u^r\|_{L^2(\bar{B}_{R/\frac{1}{2}R}\times I)}.
\end{aligned}
$$

- If we can prove $p \in L^\infty$, then one can deal with I_3 the same way as I_2. However, we can't get $p \in L^\infty$ in general. Thus the main difficulty is to deal with the pressure.
- Our idea is to prove the oscillation of p in dyadic annulus is bounded even though p is not bounded.

Main result 2: Pressure estimate

Boundedness of the oscillation of p in dyadic annulus:

- For $R > 1$ and any $R < r < 2R$, $z \in [-\pi, \pi]$, the oscillation of p has the following estimate:

$$|p(r,z) - p(R,0)| \lesssim 1.$$

Key idea:

1. For any $R > 1$, $R < r < 2R$, $\left| \int_{-\pi}^{\pi} (p(r,z) - p(R,z)) dz \right| \lesssim 1$.

$$(u^r \partial_r + u^z \partial_z) u^r - \frac{(u^\theta)^2}{r} + \partial_r p = (\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2 - \frac{1}{r^2}) u^r$$

Main result 2: Pressure estimate

Integrating the above equation on z from $-\pi$ to π, we can get

$$\begin{aligned}\partial_r \int_{-\pi}^{\pi} p dz &= \int_{-\pi}^{\pi} \left[-(u^r \partial_r + u^z \partial_z) u^r + \frac{(u^\theta)^2}{r} + (\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2 - \frac{1}{r^2}) u^r \right] dz \\
&= -\int_{-\pi}^{\pi} \frac{1}{2} \partial_r (u^r)^2 dz - \int_{-\pi}^{\pi} u^z \partial_z u^r dz + \int_{-\pi}^{\pi} \frac{(u^\theta)^2}{r} dz \\
&+ \partial_r^2 \int_{-\pi}^{\pi} u^r dz + \frac{1}{r}\partial_r \int_{-\pi}^{\pi} u^r dz - \frac{1}{r^2} \int_{-\pi}^{\pi} u^r dz.\end{aligned}$$

Picking any $r_0 \in [R, 2R]$, integrating the above on r from R to r_0, we find

$$\begin{aligned}&\left| \int_{-\pi}^{\pi} p(r_0, z) dz - \int_{-\pi}^{\pi} p(R, z) dz \right| \lesssim \left| \int_R^{r_0} \int_{-\pi}^{\pi} \partial_r (u^r)^2 dz dr \right| \\
&+ \left| \int_R^{r_0} \int_{-\pi}^{\pi} u^z \partial_z u^r dz dr \right| + \left| \int_R^{r_0} \int_{-\pi}^{\pi} \frac{(u^\theta)^2}{r} dz dr \right| + \left| \int_R^{r_0} \partial_r^2 \int_{-\pi}^{\pi} u^r dz dr \right| \\
&+ \left| \int_R^{r_0} \frac{1}{r} \partial_r \int_{-\pi}^{\pi} u^r dz dr \right| + \left| \int_R^{r_0} \frac{1}{r^2} \int_{-\pi}^{\pi} u^r dz dr \right| \lesssim 1.\end{aligned}$$

Main result 2: Pressure estimate

2. By mean value theorem, for fixed $R > 1$, for any $R < r < 2R$, $\exists z(r)$ such that
$$\big|p(r, z(r)) - p(R, z(r))\big| \lesssim 1.$$
3. Boundedness of $u, \nabla u, \nabla^2 u \Rightarrow |\partial_z p| \lesssim 1$ uniformly in R.
$$(u^r \partial_r + u^z \partial_z) u^z + \partial_z p = (\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2) u^z.$$
4. Combination of the above and uniform boundedness of $\partial_z p$, we can get for $R > 1$, any $R < r < 2R$, and $z \in [-\pi, \pi]$, the following bound on oscillation of p:
$$\begin{aligned}
&|p(r,z) - p(R,0)| \\
&= |(p(r,z) - p(r,z(r))) + (p(r,z(r)) - p(R,z(r))) + (p(R,z(r)) - p(R,0))| \\
&\leq (|\partial_z p(r, z_1)| + |\partial_z p(R, z_2)|)(|z - z(r)| + |z(r)|) + |p(r,z(r)) - p(R,z(r))| \\
&\lesssim 1,
\end{aligned}$$
where $z_1, z_2 \in [-\pi, \pi]$ and we have used the mean value theorem.

Main result 2: Vanishing of u

Proof of main result 2:

For I_1:

$$
\begin{aligned}
I_1 &= \frac{1}{2}\int_{\mathbb{R}^2\times I} |u|^2 \Delta\phi_R dx \\
&\lesssim \|u\|^2_{L^\infty(\bar{B}_{2R/R}\times[-\pi,\pi])}\|\Delta\phi_R\|_{L^1(\bar{B}_{2R/R}\times[-\pi,\pi])} \\
&\lesssim \|u\|^2_{L^\infty(\bar{B}_{2R/R}\times[-\pi,\pi])}.
\end{aligned}
$$

For I_3:

$$
\begin{aligned}
I_3 &= \int_{\mathbb{R}^2\times I} (p - p(R,0))u^r \partial_r\phi_R dx \\
&\lesssim \sup_{R<r<2R, z\in[-\pi,\pi]} \big|(p(r,z) - p(R,0))\big| \|u^r\|_{L^2(\bar{B}_{2R/R}\times[-\pi,\pi])} \\
&\quad \|\partial_r\phi_R\|_{L^2(\bar{B}_{2R/R}\times[-\pi,\pi])} \\
&\lesssim \|u^r\|_{L^2(\bar{B}_{2R/R}\times[-\pi,\pi])}.
\end{aligned}
$$

Main result 3: D-solutions in $\mathbb{R}^3$

Theorem (Carrillo–Pan–Zhang–Z., ARMA, 2020)

Let $u_\rho = u_\rho(x)$ be the radial component of 3 dimensional D-solutions in spherical coordinates. If

$$u_\rho(x) \leq \frac{C}{|x|}, \qquad x \in \mathbb{R}^3, \tag{8}$$

for some positive constant C, then $u \equiv 0$.

Remark: We only impose the condition on the positive part of the radial component of the solution, there is no restriction on the other two components.

Main result 3: D-solutions in $\mathbb{R}^3$

Head pressure $Q := \frac{1}{2}|u|^2 + p - p_1$.

- Equation of Q is: $-\Delta Q + u \cdot \nabla Q = -|\mathrm{curl}\, u|^2$.
- Decay of u and p gives $\lim_{|x|\to\infty} Q = 0$.
- Maximum principle: $Q \leq 0$.
- If one can show $Q \equiv 0$, then the Liouville problem can be done!

Main result 3: D-solutions in $\mathbb{R}^3$

Proof of main result 3: Testing the Navier-Stokes equation $u \cdot \nabla u + \nabla p = \Delta u$ with $u\phi_R$, we can get

$$\int_{\mathbb{R}^3} -\Delta u(u\phi_R)dx = \int_{\mathbb{R}^3} -(u \cdot \nabla u + \nabla p)(u\phi_R)dx.$$

Integration by parts indicates that

$$\begin{aligned}
&\int_{\mathbb{R}^3} |\nabla u|^2 \phi_R dx - \frac{1}{2}\int_{\mathbb{R}^3} |u|^2 \Delta\phi_R dx \\
&= \frac{1}{2}\int_{\mathbb{R}^3} |u|^2 u \cdot \nabla\phi_R dx + \int_{\mathbb{R}^3} (p - p_1) u \cdot \nabla\phi_R dx \\
&= \int_{\mathbb{R}^3} Qu \cdot \nabla\phi_R dx = \int_{\mathbb{R}^3} Qu_\rho \partial_\rho \phi_R dx.
\end{aligned}$$

Main result 3: D-solutions in $\mathbb{R}^3$

- Then we get

$$\int_{\mathbb{R}^3} |\nabla u|^2 \phi_R dx \lesssim \int_{B_{R/\frac{1}{2}R}} |u|^2 \Delta \phi_R dx - \int_{\mathbb{R}^3} Q u_\rho^- \partial_\rho \phi_R dx + \int_{\mathbb{R}^3} Q u_\rho^+ \partial_\rho \phi_R dx,$$

where $u_\rho^- =: -\min\{0, u_\rho\}$ and $u_\rho^+ =: \max\{0, u_\rho\}$.

- Since $u \in L^6(\mathbb{R}^3)$, $p - p_1 \in L^3(\mathbb{R}^3)$, we have $Q \in L^3(\mathbb{R}^3)$. Choosing $\partial_\rho \phi_R \leq 0$, one has

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla u|^2 \phi_R dx \lesssim & \|u\|^2_{L^6(B_{R/\frac{1}{2}R})} \|\Delta \phi_R\|_{L^{\frac{3}{2}}(B_{R/\frac{1}{2}R})} - \int_{\mathbb{R}^3} Q u_\rho^- \partial_\rho \phi_R dx \\ & + \sup_{B_{R/\frac{1}{2}R}} u_\rho^+ \|Q\|_{L^3(B_{R/\frac{1}{2}R})} \|\nabla \phi_R\|_{L^{\frac{3}{2}}(B_{R/\frac{1}{2}R})} \\ \lesssim & \|u\|^2_{L^6(B_{R/\frac{1}{2}R})} + \|Q\|_{L^3(B_{R/\frac{1}{2}R})}.\end{aligned}$$

Thanks for your attention!