Boussinesq System With Measure Forcing

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CAMIS, South China Normal University, Nov. 28, 2020

Boussinesq System

Boussinesq system

$$\begin{cases} \partial_t \theta + v \cdot \nabla \theta - \Delta \theta = \mu, \\ \partial_t v + v \cdot \nabla v - \Delta v + \nabla p = \theta\, e_d, \\ \operatorname{div} v = 0, \end{cases} \tag{1}$$

where $d = 2, 3$, $e_d = (0, \cdots, 0, 1)$, v the velocity and θ the temperature.

- System (1) is widely used in geophysics, atmosphere.
- System (1) can be rigorously derived from compressible Navier-Stokes -Fourier equations by taking a low Mach number limit (see Feireisl, Novotny [FN09]).
- For system (1) with $\mu = 0$:

  the well-posedness was studied in Cannon, DiBenedetto [CD80], Guo [Guo89] etc.

  Brandolese, Schonbek [BS12] proved that the total energy $\|u(t)\|_{L^2}^2$ may grow in time.

Boussinesq System With Measure Forcing

We consider system (1) in which μ is given by a heat source transported by the flow:

$$\begin{cases} \partial_t \mu + v \cdot \nabla \mu = 0, \\ \partial_t \theta + v \cdot \nabla \theta - \Delta \theta = \mu, \\ \partial_t v + v \cdot \nabla v - \Delta v + \nabla p = \theta \, e_d, \\ \operatorname{div} v = 0, \\ (\mu, \theta, v)|_{t=0}(x) = (\mu_0, \theta_0, v_0)(x), \end{cases} \tag{2}$$

where $x \in \mathbb{R}^d$.

- One can think of $\mu(x,t) = \sum_{i=1}^{m} \lambda_i \delta(x - x_i(t))$, $\lambda_i \in \mathbb{R}$.
- From the perspective of physical modeling, it would describe the movement of water after putting chemical material, like Sodium (Na), rapidly reacting with water.
- We assume total transfer of energy by μ is constant in time.
- We also neglect all other chemical or thermodynamical effects.

Related works

3D Navier-Stokes system with singular forcing

$$\begin{cases} \partial_t v + v \cdot \nabla v - \Delta v + \nabla p = F, \\ \mathrm{div}\, v = 0, \quad v|_{t=0} = v_0, \quad x \in \mathbb{R}^3. \end{cases} \tag{3}$$

- "Landau solutions" are a family of explicit formulas of (v, p) which solves system (3) (with $F = 0$) under the conditions they are *steady, symmetric about x_1-axis, homogeneous of degree -1, regular except origin*. See Landau [LL59], Tian, Xin [TX98], Sverak [Sve11].

  "Landau solutions" indeed are distributional solution to system (3) with singular force $F = (b\delta_0, 0, 0)$. See Cannone, Karch [CK04].
- If $v_0 \in \mathcal{PM}^2$ and $F \in C_w([0,\infty), \mathcal{PM}^0)$ are sufficiently small [1], then system (3) can be solved globally and uniquely. cf. Cannone, Karch [CK04].

[1] Pseudomeasure space $\mathcal{PM}^a$, $a \geq 0$ is the space of $f \in \mathcal{S}'(\mathbb{R}^d)$ such that $\|f\|_{\mathcal{PM}^a} = \operatorname{ess\,sup}_{\mathbb{R}^d} |\xi|^a |\hat{f}(\xi)| < \infty$.

Question: If $\mu_0 \in \mathcal{M}(\mathbb{R}^d)$ with[2] $\operatorname{supp} \mu_0 \subset B_{R_0}(0)$ for some $R_0 > 0$, can we solve the system (2) uniquely and globally?

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[2] Denote $\mathcal{M} = \mathcal{M}(\mathbb{R}^d)$ as the space of finite Radon measures defined on $\mathbb{R}^d$ with *total variation* topology, i.e., for any μ Radon measure,

$$\|\mu\|_{\mathcal{M}(\mathbb{R}^d)} = |\mu|(\mathbb{R}^d) := \sup \left\{ \left| \int_{\mathbb{R}^d} f \mathrm{d}\mu \right| : \|f\|_{L^\infty} \leq 1 \right\}.$$

From Riesz representation theorem, $\mathcal{M}(\mathbb{R}^d)$ is the dual space of $C_0(\mathbb{R}^d)$.

Main Result

Theorem 1 (P. B. Mucha & L. Xue, 2020)

Let $\mu_0 \in \mathcal{M}_+(\mathbb{R}^2)$ *with* $\operatorname{supp}\mu_0 \subset B_{R_0}(0)$ *for some* $R_0 > 0$.
For each $\sigma \in (0,2)$, *let* $\theta_0 \in L^1 \cap B^{2-\sigma}_{\frac{2}{2-\sigma},\infty}(\mathbb{R}^2)$ *with* $\theta_0 \geq 0$, *and* $v_0 \in H^1(\mathbb{R}^2)$ *be a divergence-free vector field with vorticity* $\omega_0 = \partial_1 v_{2,0} - \partial_2 v_{1,0} \in B^{3-\sigma}_{\frac{2}{2-\sigma},\infty}(\mathbb{R}^2)$.
Then system (2) *admits a global in time unique solution* (μ, θ, v) *such that for any* $T > 0$,

$$\mu \in L^\infty(0,T;\mathcal{M}_+(\mathbb{R}^2)), \quad \textit{with} \quad \operatorname{supp}\mu \subset B_{R_0+C_T}, \tag{4}$$

$$\theta \in L^\infty(0,T;L^1 \cap B^{2-\sigma}_{\frac{2}{2-\sigma},\infty}(\mathbb{R}^2)), \quad \textit{with} \quad \theta \geq 0 \ \text{ on } \ [0,T]\times\mathbb{R}^2,$$

$$v \in L^\infty(0,T;H^1 \cap W^{1,\infty}) \cap L^2(0,T;H^2), \quad \nabla v \in L^\infty(0,T;B^{3-\sigma}_{\frac{2}{2-\sigma},\infty}).$$

Some Remarks

- Difficulty: The heat source is a measure which is not vanishing in time.
  - Solvability of $(2)_1$ requires high regularity of the velocity (Lipschitz continuity) to guarantee existence and uniqueness.

    One can not expect too high regularity of solutions since they are generated by a measure.

    Our solutions are regular enough and $(2)_1$ can be solved in terms of characteristics.
  - In obtaining a priori estimates, the source μ given as a measure does not allow to use the standard bounds by energy norms.
  - We prove uniqueness in Lagrangian coordinates: the regularity is high enough to define the Lagrangian coordinates. After the transformation, μ becomes fixed in time.

Some remarks

- It is an application of inhomogeneous Besov spaces $L^\infty(0,T;B^s_{p,\infty}(\mathbb{R}^2))$ to address the regularity of solutions.
  - Properties of these spaces allow to consider regularity of (μ,θ,v) in the L^∞-norm in time, which is required by the basic bound of μ (4).
  - By embedding $B^\sigma_{2/\sigma,1}(\mathbb{R}^2) \subset C_0(\mathbb{R}^2)$ for $\sigma \in (0,2)$, we have

    $$\mathcal{M}(\mathbb{R}^2) \subset B^{-\sigma}_{\frac{2}{2-\sigma},\infty}(\mathbb{R}^2),$$

    thus μ belongs to $L^\infty(0,T;B^{-\sigma}_{\frac{2}{2-\sigma},\infty}(\mathbb{R}^2))$.

    This framework fits to the regularity properties of the right-hand side of equation $(2)_2$.
  - Besov spaces admit the theory of maximal regularity for the heat and Stokes equations, which allows to maintain the full information about the solutions.

A priori estimates: estimation of source μ

$$\partial_t \mu + v \cdot \nabla \mu = 0, \quad \mu|_{t=0} = \mu_0 \in \mathcal{M}_+(\mathbb{R}^2).$$

Proposition 1

Let $\mu_0 \in \mathcal{M}_+(\mathbb{R}^2)$ *satisfy that* $\operatorname{supp} \mu_0 \subset B_{R_0}(0)$ *for some* $R_0 > 0$. *Let* $T > 0$ *be any given, and* (μ, θ, v) *be smooth functions on* $\mathbb{R}^2 \times [0, T]$ *solving system* (2).

Then for every $t \in [0, T]$, *we have* $\mu(t, x) = \mu_t(x) \in \mathcal{M}_+(\mathbb{R}^2)$ *with*

$$\|\mu_t\|_{\mathcal{M}(\mathbb{R}^2)} \leq \|\mu_0\|_{\mathcal{M}(\mathbb{R}^2)}, \quad \forall t \in [0, T],$$

and also $\operatorname{supp} \mu_t \subset B_{R_0 + C}(0)$ *with* $C = \|v\|_{L^1_T(L^\infty)}$.

A priori estimates: L^1-Estimation of temperature θ

$$\partial_t \theta + v \cdot \nabla \theta - \Delta \theta = \mu, \quad \theta|_{t=0} = \theta_0.$$

Proposition 2

Let $\mu_0 \in \mathcal{M}_+(\mathbb{R}^2)$ *satisfy* $\mathrm{supp}\, \mu_0 \subset B_{R_0}(0)$ *for some* $R_0 > 0$, *and* $\theta_0 \in L^1(\mathbb{R}^2)$ *be with* $\theta_0 \geq 0$.
For $T > 0$ *any given, assume* (μ, θ, v) *are smooth functions on* $\mathbb{R}^2 \times [0, T]$ *solving system* (2), *and also* θ *has the point-wise spatial decay.*

Then we have that $\theta(t) \geq 0$ *for every* $t \in [0, T]$ *and*

$$\sup_{t \in [0,T]} \|\theta(t)\|_{L^1} \leq \|\theta_0\|_{L^1} + T\|\mu_0\|_{\mathcal{M}}.$$

A priori estimates

Proposition 3

Let $\mu_0 \in \mathcal{M}_+(\mathbb{R}^2)$ *satisfy* $\operatorname{supp}\mu_0 \subset B_{R_0}(0)$ *for some* $R_0 > 0$. *For each* $\sigma \in (0,2)$, *let* $\theta_0 \in L^1 \cap B^{2-\sigma}_{\frac{2}{2-\sigma},\infty}(\mathbb{R}^2)$ *be with* $\theta_0 \geq 0$, *and* $v_0 \in H^1(\mathbb{R}^2)$ *be a divergence-free vector field with initial vorticity* $\omega_0 := \partial_1 v_{2,0} - \partial_2 v_{1,0} \in B^{3-\sigma}_{\frac{2}{2-\sigma},\infty}(\mathbb{R}^2)$. *Let* $T > 0$ *be any given, and* (μ, θ, v) *be smooth functions on* $\mathbb{R}^2 \times [0, T]$ *solving system* (2). *Then we have*

$$\|\theta\|_{L^\infty_T(B^{2-\sigma}_{\frac{2}{2-\sigma},\infty})} + \|v\|_{L^\infty_T(H^1)} + \|v\|_{L^2_T(H^2)} \leq Ce^{\exp(C(1+T)^8)}, \tag{5}$$

and

$$\|\nabla v\|_{L^\infty_T(B^{3-\sigma}_{\frac{2}{2-\sigma},\infty})} + \|v\|_{L^\infty_T(W^{1,\infty})} + \|(\nabla p, \partial_t v, \nabla^2 v)\|_{L^\infty_T(B^{2-\sigma}_{\frac{2}{2-\sigma},\infty})} \leq Ce^{\exp(C(1+T)^8)}, \tag{6}$$

where $C > 0$ *depends only on* σ *and norms of* (μ_0, θ_0, v_0).

Proof of Proposition 3

- Energy estimate yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_{L^2}^2 + \|\nabla v(t)\|_{L^2}^2 \le \left|\int_{\mathbb{R}^2} \theta\, v_2(t,x)\mathrm{d}x\right|$$

$$\le \|\theta(t)\|_{L^1}\|v(t)\|_{L^\infty} \le C(1+t)\|v(t)\|_{L^2}^{1/2}\|\nabla^2 v\|_{L^2}^{1/2}.$$

- Consider the equation of vorticity $\omega := \mathrm{curl}\, v = \partial_1 v_2 - \partial_2 v_1$, which is

$$\partial_t \omega + v\cdot\nabla\omega - \Delta\omega = \partial_1\theta.$$

Energy estimate also yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t)\|_{L^2}^2 + \|\nabla\omega(t)\|_{L^2}^2 \le \left|\int_{\mathbb{R}^2} \theta\,\partial_1\omega(t,x)\mathrm{d}x\right| \le \|\theta(t)\|_{L^2}\|\nabla\omega(t)\|_{L^2}.$$

$$\Longrightarrow \frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t)\|_{L^2}^2 + \|\nabla\omega(t)\|_{L^2}^2 \le \|\theta(t)\|_{L^2}^2.$$

We get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2\right) + \|\nabla v\|_{L^2}^2 + \frac{1}{2}\|\nabla\omega\|_{L^2}^2 \le \|\theta(t)\|_{L^2}^2 + C(1+t)^{\frac{4}{3}}\|v(t)\|_{L^2}^{\frac{2}{3}}. \quad (7)$$

- In order to control the norm $\|\theta(t)\|_{L^2(\mathbb{R}^2)}$, we use the equation of θ

$$\partial_t \theta + v \cdot \nabla \theta - \Delta \theta = \mu.$$

Since $B^{\sigma}_{2/\sigma,1}(\mathbb{R}^2) \subset C_0(\mathbb{R}^2)$ for $\sigma \in (0,2)$, observe that

$$\mu(t) \in \mathcal{M}(\mathbb{R}^2) = (C_0(\mathbb{R}^2))^* \subset B^{-\sigma}_{\frac{2}{2-\sigma},\infty}(\mathbb{R}^2).$$

Lemma 2 (cf. Theorem 2.2.5 of Danchin05')

Let $s \in \mathbb{R}$ and $p \in [1,\infty]$. Let $T > 0$, $u_0 \in B^s_{p,\infty}(\mathbb{R}^d)$, and $f \in L^\infty_T(B^{s-2}_{p,\infty}(\mathbb{R}^d))$. Then

$$\partial_t u - \Delta u = f, \qquad u|_{t=0} = u_0, \quad x \in \mathbb{R}^d,$$

has a unique solution $u \in L^\infty_T(B^s_{p,\infty})$ and there exists a constant $C = C(d)$ such that

$$\|u\|_{L^\infty_T(B^s_{p,\infty})} \le C\left(\|u_0\|_{B^s_{p,\infty}} + (1+T)\|f\|_{L^\infty_T(B^{s-2}_{p,\infty})}\right).$$

- Hence we infer that for every $t \in [0, T]$,

$$\|\theta\|_{L^\infty(0,t;B^{2-\sigma}_{\frac{2}{2-\sigma},\infty})} \leq C_0\Big(\|\theta_0\|_{B^{2-\sigma}_{\frac{2}{2-\sigma},\infty}} + (1+t)\|\mu\|_{L^\infty_t(B^{-\sigma}_{\frac{2}{2-\sigma},\infty})} + (1+t)\|v\cdot\nabla\theta\|_{L^\infty_t(B^{-\sigma}_{\frac{2}{2-\sigma},\infty})}\Big).$$

Lemma 3

Let $v : \mathbb{R}^2 \to \mathbb{R}^2$ be divergence-free and $\theta : \mathbb{R}^2 \to \mathbb{R}$. Let $s \in (0, 1)$, $p \in [1, \infty]$. Then there exists $C = C(s) > 0$ such that

$$\|v \cdot \nabla\theta\|_{B^{-s}_{p,\infty}(\mathbb{R}^2)} \leq C\|v\|_{H^1(\mathbb{R}^2)}\Big(\sup_{k\geq -1} 2^{k(1-s)}\sqrt{k+2}\|\Delta_k\theta\|_{L^p(\mathbb{R}^2)}\Big).$$

- Owing to Lemma 3, it follows that

$$\|\theta\|_{L^\infty_t(B^{2-\sigma}_{\frac{2}{2-\sigma},\infty})} \leq C_0\Big(\|\theta_0\|_{B^{2-\sigma}_{\frac{2}{2-\sigma},\infty}} + (1+t)\|\mu_0\|_{\mathcal{M}}\Big)$$
$$+ C_0(1+t)\|(v,\omega)\|_{L^\infty_t(L^2)}\Big(\sup_{k\geq -1} 2^{k(1-\sigma)}\sqrt{2+k}\|\Delta_k\theta\|_{L^\infty_t(L^{\frac{2}{2-\sigma}})}\Big).$$

- We first derive a rough estimate of $\|\theta\|_{L_t^\infty(B^{2-\sigma}_{\frac{2}{2-\sigma},\infty})}$ by $\|(v,\omega)\|_{L_t^\infty(L^2)}$.

By interpolation and Young inequality, it gives

$$\begin{aligned}
\|\theta\|_{L_t^\infty(B^{2-\sigma}_{\frac{2}{2-\sigma},\infty})} &\leq C(1+t) + C(1+t)\|(v,\omega)\|_{L_t^\infty(L^2)}\|\theta\|_{L_t^\infty(B^{1-\frac{\sigma}{2}}_{\frac{2}{2-\sigma},\infty})} \\
&\leq C(1+t) + C(1+t)\|(v,\omega)\|_{L_t^\infty(L^2)}\|\theta\|^{\frac{2-\sigma}{4}}_{L_t^\infty(L^1)}\|\theta\|^{\frac{2+\sigma}{4}}_{L_t^\infty(B^{2-\sigma}_{\frac{2}{2-\sigma},\infty})} \\
&\leq C(1+t) + C\Big((1+t)^{\frac{4}{2-\sigma}}\|(v,\omega)\|^{\frac{4}{2-\sigma}}_{L_t^\infty(L^2)}\|\theta\|_{L_t^\infty(L^1)}\Big) + \frac{1}{2}\|\theta\|_{L_t^\infty(B^{2-\sigma}_{\frac{2}{2-\sigma},\infty})},
\end{aligned}$$

and

$$\|\theta\|_{L_t^\infty(B^{2-\sigma}_{\frac{2}{2-\sigma},\infty})} \leq C(1+t)^{\frac{6-\sigma}{2-\sigma}}\Big(1+\|(v,\omega)\|^{\frac{4}{2-\sigma}}_{L_t^\infty(L^2)}\Big), \tag{8}$$

with C depending on the norms $\|\theta_0\|_{L^1\cap B^{2-\sigma}_{\frac{2}{2-\sigma},\infty}}$ and $\|\mu_0\|_{\mathcal{M}}$.

- Then we show refined estimate of (8) by reducing power index of $\|(v,\omega)\|_{L^\infty_t L^2}$.

Lemma 4

Let $\theta : \mathbb{R}^2 \times [0,T] \to \mathbb{R}$ be a scalar function. Let $s \in (0,1)$, $p \in [1,\infty]$, then there is a positive constant $C = C(s,p)$ such that

$$\sup_{k\ge -1} 2^{k(1-s)}\sqrt{k+2}\|\Delta_k\theta\|_{L^\infty_T(L^p)}$$
$$\le C\|\theta\|^{\frac{1}{4-s-2/p}}_{L^\infty_T(L^1)}\|\theta\|^{\frac{3-s-2/p}{4-s-2/p}}_{L^\infty_T(B^{2-s}_{p,\infty})}\sqrt{\log\left(e+\frac{\|\theta\|_{L^\infty_T(B^{2-s}_{p,\infty})}}{\|\theta\|_{L^\infty_T(L^1)}}\right)}+C\|\theta\|_{L^\infty_T(L^1)}. \tag{9}$$

Applying (9) and the fact $z \mapsto z^{\frac12}\sqrt{\log(e+\frac{1}{z})}$ is increasing on $(0,\infty)$,

$$\|\theta\|_{L^\infty_t(B^{2-\sigma}_{\frac{2}{2-\sigma},\infty})} \le C(1+t)+C(1+t)\|(v,\omega)\|_{L^\infty_t(L^2)}\|\theta\|_{L^\infty_t(L^1)}$$
$$+C(1+t)\|(v,\omega)\|_{L^\infty_t(L^2)}\|\theta\|^{\frac12}_{L^\infty_t(L^1)}\|\theta\|^{\frac12}_{L^\infty_t(B^{2-\sigma}_{\frac{2}{2-\sigma},\infty})}\sqrt{\log\left(e+\frac{\|\theta\|_{L^\infty_t(B^{2-\sigma}_{2/(2-\sigma),\infty})}}{\|\theta\|_{L^\infty_t(L^1)}}\right)}$$
$$\le C(1+t)+C(1+t)^2\|(v,\omega)\|_{L^\infty_t(L^2)}\left(\|\theta\|^{\frac12}_{L^\infty_t(B^{2-\sigma}_{\frac{2}{2-\sigma},\infty})}\sqrt{\log\left(e+\|\theta\|_{L^\infty_t(B^{2-\sigma}_{\frac{2}{2-\sigma},\infty})}\right)}+1\right).$$

Using rough estimate (8), we see

$$\sqrt{\log\Big(e+\|\theta\|_{L_t^\infty(B^{2-\sigma}_{\frac{2}{2-\sigma},\infty})}\Big)} \le \sqrt{\log\Big(\Big(e+C(1+t)^{\frac{6-\sigma}{2-\sigma}}\Big)\Big(e+\|(v,\omega)\|^{\frac{4}{2-\sigma}}_{L_t^\infty(L^2)}\Big)\Big)}$$
$$\le C(1+t)\sqrt{\log\Big(e+\|(v,\omega)\|^2_{L_t^\infty(L^2)}\Big)},$$

thus

$$\|\theta\|_{L_t^\infty(B^{2-\sigma}_{\frac{2}{2-\sigma},\infty})} \le C(1+t)+C(1+t)^3\|(v,\omega)\|_{L_t^\infty(L^2)}\Big(\sqrt{\log\Big(e+\|(v,\omega)\|^2_{L_t^\infty(L^2)}\Big)}\|\theta\|^{\frac{1}{2}}_{L_t^\infty(B^{2-\sigma}_{\frac{2}{2-\sigma},\infty})}+1\Big).$$

We then obtain

$$\|\theta\|_{L_t^\infty(B^{2-\sigma}_{\frac{2}{2-\sigma},\infty})} \le C(1+t)^6\Big(1+\|(v,\omega)\|^2_{L_t^\infty(L^2)}\Big)\log\Big(e+\|(v,\omega)\|^2_{L_t^\infty(L^2)}\Big). \tag{10}$$

- Now we go back to inequality (7).

By (10) and interpolation $\|\theta\|^2_{L^\infty_T(L^2)} \leq C\|\theta\|_{L^\infty_T(L^1)}\|\theta\|_{L^\infty_T(B^{2/p}_{p,\infty})}$, $\forall p \in [1,\infty)$, we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(v,\omega)(t)\|^2_{L^2} + \frac{1}{2}\|(\nabla v,\nabla\omega)(t)\|^2_{L^2} \leq C\|\theta\|_{L^\infty_t(L^1)}\|\theta\|_{L^\infty_t(B^{2-\sigma}_{\frac{2}{2-\sigma},\infty})} + C(1+t)^{\frac{4}{3}}\|v(t)\|^{\frac{2}{3}}_{L^2}$$
$$\leq C(1+t)^7\Big(1 + \|(v,\omega)\|^2_{L^\infty_t(L^2)}\log\Big(e + \|(v,\omega)\|^2_{L^\infty_t(L^2)}\Big)\Big).$$

Integrating on time yields

$$\|(v,\omega)\|^2_{L^\infty_t(L^2)} + \|(\nabla v,\nabla\omega)\|^2_{L^2_t(L^2)}$$
$$\leq C(1+t)^8 + C\int_0^t (1+\tau)^7\|(v,\omega)\|^2_{L^\infty_\tau(L^2)}\log\Big(e + \|(v,\omega)\|^2_{L^\infty_\tau(L^2)}\Big)\mathrm{d}\tau.$$

Gronwall inequality guarantees

$$\|(v,\omega)\|^2_{L^\infty_T(L^2)} + \|(\nabla v,\nabla\omega)\|^2_{L^2_T(L^2)} \leq Ce^{\exp(C(1+T)^8)}. \tag{11}$$

Plugging (11) into (8) leads to

$$\|\theta\|_{L^\infty_t(B^{2-\sigma}_{\frac{2}{2-\sigma},\infty})} \leq Ce^{\exp(C(1+T)^8)}.$$

- By viewing the equation of ω as $\partial_t\omega - \Delta\omega = \partial_1\theta - v\cdot\nabla\omega$, we use Lemma 2 and product estimate to get

$$\|\nabla v\|_{L^\infty_T(B^{3-\sigma}_{\frac{2}{2-\sigma},\infty})} \approx \|\omega\|_{L^\infty_T(B^{3-\sigma}_{\frac{2}{2-\sigma},\infty})} \leq Ce^{\exp(C(1+T)^8)},$$

and

$$\begin{aligned}\|v\|_{L^\infty_T(W^{1,\infty})} &\leq C_0\|\Delta_{-1}v\|_{L^\infty_T(L^\infty)} + C_0\sum_{q\in\mathbb{N}}\|\Delta_q\nabla v\|_{L^\infty_T(L^\infty)}\\ &\leq C_0\|v\|_{L^\infty_T(L^2)} + C_0\sum_{q\in\mathbb{N}}2^{-q}2^{q(3-\sigma)}\|\Delta_q\nabla v\|_{L^\infty_T(L^{\frac{2}{2-\sigma}})}\\ &\leq C_0\|v\|_{L^\infty_T(L^2)} + C_0\|\nabla v\|_{L^\infty_T(B^{3-\sigma}_{\frac{2}{2-\sigma},\infty})} \leq Ce^{\exp(C(1+T)^8)},\end{aligned}$$

- Furthermore,

$$\|(\partial_t v, \nabla^2 v, \nabla p)\|_{L^\infty_T(B^{2-\sigma}_{\frac{2}{2-\sigma},\infty})} \leq Ce^{\exp(C(1+T)^8)}.$$

Global Existence

- In order to construct a suitable approximation we consider the system with smooth initial data.

We assume

$$\mu^n|_{t=0}, \quad \theta^n|_{t=0}, \quad v^n|_{t=0} \text{ belong to the Schwartz class over } \mathbb{R}^2, \tag{12}$$

where $n \in \mathbb{N}^+$ ($n \to \infty$ in the end) and they converge to μ_0, θ_0, v_0 in spaces prescribed by Theorem 1 (at least in a weak sentence).

- By Galerkin's approximation and energy estimates, we can get the solution to system (2) with initial data given by (12), that is:

$$\begin{cases} \partial_t \mu^n + v^n \cdot \nabla \mu^n = 0, & \text{in } \mathbb{R}^2 \times (0,T], \\ \partial_t \theta^n + v^n \cdot \nabla \theta^n - \Delta \theta^n = \mu^n, & \text{in } \mathbb{R}^2 \times (0,T], \\ \partial_t v^n + v^n \cdot \nabla v^n - \Delta v^n + \nabla p^n = \theta^n, & \text{in } \mathbb{R}^2 \times (0,T], \\ \operatorname{div} v^n = 0, & \text{in } \mathbb{R}^2 \times (0,T], \\ \mu^n|_{t=0}, \quad \theta^n|_{t=0}, \quad v^n|_{t=0} \in \mathcal{S}(\mathbb{R}^2). \end{cases} \tag{13}$$

• Using the standard bootstrap method, we obtain that for every $n \in \mathbb{N}^+$ and for any $1 < q, p < \infty$,

$$
\begin{aligned}
&\mu^n \in L^\infty(0, T; L^1 \cap L^\infty), \\
&\theta^n \in L^q(0, T; W^{2,p}) \cap W^{1,q}(0, T; L^p), \\
&v^n \in L^q(0, T; W^{4,p}) \cap W^{2,q}(0, T; L^p).
\end{aligned}
\tag{14}
$$

Due to $\mu^n|_{t=0} \in \mathcal{S}$, we get $\mu^n \in L^\infty(0, T; W^{4,p}) \cap W^{1,\infty}(0, T; W^{3,p})$.

• Iteration ensures that (μ^n, θ^n, v^n) has sufficient smoothness so that the regularity assumptions in Propositions 1 - 3 are fulfilled.

Hence we have the uniform-in-n estimates

$$
\|\mu^n\|_{L^\infty(0,T;\mathcal{M})} \le \|\mu_0\|_{\mathcal{M}}, \quad \mu^n \ge 0, \quad \text{and} \quad \operatorname{supp} \mu^n \subset B_{R_0+C}(0).
$$

$$
\|\theta^n\|_{L^\infty_T(L^1 \cap B^{2-\sigma}_{\frac{2}{2-\sigma},\infty})} + \|v^n\|_{L^\infty_T(H^1 \cap W^{1,\infty})} + \|(\partial_t v^n, \nabla p^n, \nabla^2 v^n)\|_{L^\infty_T(B^{2-\sigma}_{\frac{2}{2-\sigma},\infty})} \le C,
$$

where C is depending on T and norms of (μ_0, θ_0, v_0) but independent of $n \in \mathbb{N}^+$.

• By passing $n \to \infty$, up to a subsequence, one has that for every $\varphi \in \mathcal{D}(\mathbb{R}^2)$,

$$\varphi v^n \to \varphi v, \quad \text{in } L^\infty(0,T; L^2 \cap W^{1,\infty}),$$
$$\varphi \theta^n \to \varphi \theta, \quad \text{in } L^\infty(0,T; L^2),$$

and[3]

$$\mu^n(t) \to \mu(t) \text{ in } d\text{-topology uniformly in time.} \tag{15}$$

• The bound (15) and strong convergence of velocity in $L^\infty(0,T; W^{1,\infty}_{\mathrm{loc}})$ guarantee that as $n \to \infty$,

$$v^n \mu^n \to v\mu \quad \text{in } \mathcal{D}'(\mathbb{R}^2 \times [0,T]).$$

[3]Denote $\mathcal{M} = \mathcal{M}(\mathbb{R}^d, d)$ as the space of finite Radon measures on $\mathbb{R}^d$ equipped with *bounded Lipschitz distance* topology, i.e.,

$$d(\mu,\nu) := \sup\left\{\left|\int_{\mathbb{R}^d} f\mathrm{d}\mu - \int_{\mathbb{R}^d} f\mathrm{d}\nu\right| : \|f\|_{L^\infty} \le 1 \text{ and } Lip(f) := \sup_{x \neq y \in \mathbb{R}^d} \frac{|f(x)-f(y)|}{|x-y|} \le 1\right\}.$$

Uniqueness: Lagrangian coordinates

- Let $X_v(t,y)$ be the particle trajectory

$$\frac{\mathrm{d}X_v(t,y)}{\mathrm{d}t} = v(t, X_v(t,y)), \quad X_v(t,y)|_{t=0} = y. \tag{16}$$

If $v \in L^1(0,T;\dot{W}^{1,\infty}(\mathbb{R}^d))$, then (16) has a unique solution $X_v(t,y)$ on $[0,T]$.

- Set $\bar{\mu}(t,y) = \mu(t, X_v(t,y))$, $\bar{\theta}(t,y) = \theta(t, X_v(t,y))$, $\bar{p}(t,y) = p(t, X_v(t,y))$,

then the system (2) recasts in

$$\begin{cases} \partial_t \bar{\mu} = 0, \\ \partial_t \bar{\theta} - \operatorname{div}\left(A_v A_v^{\mathrm{T}} \nabla_y \bar{\theta}\right) = \bar{\mu}, \\ \partial_t \bar{v} - \operatorname{div}\left(A_v A_v^{\mathrm{T}} \nabla_y \bar{v}\right) + A_v^{\mathrm{T}} \nabla_y \bar{p} = \bar{\theta}\, e_2, \\ \operatorname{div}(A_v \bar{v}) = 0, \\ \bar{\mu}|_{t=0} = \mu_0, \quad \bar{\theta}|_{t=0} = \theta_0, \quad \bar{v}|_{t=0} = v_0, \end{cases} \tag{17}$$

where $A_v(t,y) = (\nabla_y X_v(t,y))^{-1}$ which under condition $\int_0^t \|\nabla_y \bar{v}\|_{L^\infty} \mathrm{d}\tau \le \frac{1}{2}$ has

$$A_v(t,y) = \left(\mathrm{Id} + (\nabla_y X_v - \mathrm{Id})\right)^{-1} = \sum_{k=0}^{\infty} (-1)^k \left(\int_0^t \nabla_y \bar{v}(\tau, y)\mathrm{d}\tau\right)^k.$$

The equation $(17)_1$ implies $\bar{\mu}$ becomes time independent, i.e.

$$\bar{\mu}(t,y) \equiv \mu_0(y), \quad \forall t \in [0,T].$$

Then system (17) reduces to

$$\begin{cases} \partial_t \bar{\theta} - \Delta \bar{\theta} = \mu_0 + N_1(A_v, \nabla \bar{\theta}), \\ \partial_t \bar{v} - \Delta \bar{v} + \nabla \bar{p} = \bar{\theta}\, e_2 + N_2(A_v, \nabla \bar{v}) + N_3(A_v, \nabla \bar{p}), \\ \mathrm{div}\, \bar{v} = \mathrm{div}\, ((\mathrm{Id} - A_v)\bar{v}) = (\mathrm{Id} - A_v^{\mathrm{T}}) : \nabla \bar{v}, \\ \bar{\theta}|_{t=0} = \theta_0, \quad \bar{v}|_{t=0} = v_0. \end{cases} \tag{18}$$

Nonlinear terms N_1, N_2, N_3 are defined by

$$N_1(A_v, \nabla \bar{\theta}) := \mathrm{div}\Big((A_v A_v^{\mathrm{T}} - \mathrm{Id})\nabla \bar{\theta}\Big), \quad \text{and}$$

$$N_2(A_v, \nabla \bar{v}) := \mathrm{div}\Big((A_v A_v^{\mathrm{T}} - \mathrm{Id})\nabla \bar{v}\Big), \quad N_3(A_v, \nabla \bar{p}) := (\mathrm{Id} - A_v^{\mathrm{T}})\nabla \bar{p}.$$

Observe that the left-hand side of (18) fits perfectly to the needs of Lemma 5.

Lemma 5 (cf. Lemma 3 of Danchin-Mucha13')

Let R be a vector field satisfying $\partial_t R \in L^2(\mathbb{R}^d \times (0,T])$ and $\nabla \operatorname{div} R \in L^2(\mathbb{R}^d \times (0,T])$. Then the following system

$$\begin{cases} \partial_t u - \Delta u + \nabla P = f, & \text{in } \mathbb{R}^d \times (0,T], \\ \operatorname{div} u = \operatorname{div} R, & \text{in } \mathbb{R}^d \times (0,T], \\ u|_{t=0} = u_0, & \text{on } \mathbb{R}^d, \end{cases}$$

admits a unique solution $(u, \nabla P)$ which satisfies that

$$\|\nabla u\|_{L^\infty_T(L^2)} + \|(\partial_t u, \nabla^2 u, \nabla P)\|_{L^2_T(L^2)} \le C\Big(\|\nabla u_0\|_{L^2} + \|(f, \partial_t R)\|_{L^2_T(L^2)} + \|\nabla \operatorname{div} R\|_{L^2_T(L^2)}\Big),$$

where C is a positive constant independent of T.

- Hence the easiest framework in order to prove the uniqueness is via the $L^2(0, T; L^2)$ estimate for the difference of temperatures.
- Note that the change of the Lagrangian coordinates makes our system quasi-linear, and the input from matrix A_v is negligible as the time interval is short.

Remark 1

The uniqueness to system (2) could be proved directly in the Eulerian coordinates.

Based on the considerations in Besov spaces with negative regularity index (e.g. Hmidi et al [HKR10]), one may be able to control the part coming from the transport equation $(2)_1$.

However this approach for our system seems to be very technical with nontrivial considerations for convection terms.

Uniqueness

- Consider two solutions $(\mu_1, \theta_1, v_1, p_1)$ and $(\mu_2, \theta_2, v_2, p_2)$ to system (2) starting from same data (μ_0, θ_0, v_0).

We know that for $i = 1, 2$ and for any $T > 0$,

$$\|\theta_i\|_{L^\infty_T(L^1 \cap B^{2-\sigma}_{\frac{2}{2-\sigma},\infty})} < \infty, \quad \|v_i\|_{L^\infty_T(H^1 \cap W^{1,\infty})} + \|(\partial_t v_i, \nabla^2 v_i, \nabla p_i)\|_{L^\infty_T(B^{2-\sigma}_{\frac{2}{2-\sigma},\infty})} < \infty.$$

- Using the Lagrangian coordinates, we have

$$\|\nabla_y \bar{v}_i\|_{L^\infty_T(L^\infty)} < \infty,$$

and moreover by letting $T' > 0$ be small enough,

$$\int_0^{T'} \|\nabla_x v_i(t)\|_{L^\infty} \mathrm{d}t \le \frac{1}{2}, \quad \text{and} \quad \int_0^{T'} \|\nabla_y \bar{v}_i(t)\|_{L^\infty} \mathrm{d}t \le \frac{1}{2}. \tag{19}$$

We also infer

$$\|\bar{\theta}_i\|_{L^\infty_T(L^{\frac{4}{2-\sigma}})} + \|\nabla_y^2 \bar{v}_i\|_{L^\infty_T(L^{\frac{4}{2-\sigma}})} + \|\partial_t \bar{v}_i\|_{L^\infty_T(L^{\frac{4}{2-\sigma}})} + \|\nabla_y \bar{p}_i\|_{L^\infty_T(L^{\frac{4}{2-\sigma}})} < \infty. \tag{20}$$

The difference equations of $\delta\bar{\theta} := \bar{\theta}_1 - \bar{\theta}_2$, $\delta\bar{v} := \bar{v}_1 - \bar{v}_2$ and $\delta\bar{p} := \bar{p}_1 - \bar{p}_2$ read as follows

$$\begin{cases} \partial_t\delta\bar{\theta} - \Delta\delta\bar{\theta} = \delta N_1, \\ \partial_t\delta\bar{v} - \Delta\delta\bar{v} + \nabla\delta\bar{p} = (\delta\bar{\theta})e_2 + \delta N_2 + \delta N_3, \\ \operatorname{div}\delta\bar{v} = \operatorname{div}(\delta N_4), \\ \delta\bar{\theta}|_{t=0} = 0, \quad \delta\bar{v}|_{t=0} = 0, \end{cases} \tag{21}$$

with

$$\begin{aligned} \delta N_1 &:= \operatorname{div}\Big((A_{v_1}A_{v_1}^{\mathrm{T}} - A_{v_2}A_{v_2}^{\mathrm{T}})\nabla\bar{\theta}_2\Big) - \operatorname{div}\Big((\mathrm{Id} - A_{v_1}A_{v_1}^{\mathrm{T}})\nabla\delta\bar{\theta}\Big), \\ \delta N_2 &:= \operatorname{div}\Big((A_{v_1}A_{v_1}^{\mathrm{T}} - A_{v_2}A_{v_2}^{\mathrm{T}})\nabla\bar{v}_2\Big) - \operatorname{div}\Big((\mathrm{Id} - A_{v_1}A_{v_1}^{\mathrm{T}})\nabla\delta\bar{v}\Big), \\ \delta N_3 &:= -(A_{v_1}^{\mathrm{T}} - A_{v_2}^{\mathrm{T}})\nabla\bar{p}_2 + \Big(\mathrm{Id} - A_{v_1}^{\mathrm{T}}\Big)\nabla\delta\bar{p}, \\ \delta N_4 &:= (A_{v_1} - A_{v_2})\bar{v}_2 + (\mathrm{Id} - A_{v_1})\delta\bar{v}. \end{aligned}$$

- The target is to show uniqueness in:

$$\delta\bar{\theta} \in L^2(0,T;L^2), \quad \text{and} \quad \nabla\delta\bar{v} \in L^\infty(0,T;L^2) \cap L^2(0,T;\dot{H}^1).$$

- In order to perform the $L^2_T(L^2)$-estimate of $\delta\bar{\theta}$, we introduce a function w which solves the backward heat equation

$$\partial_t w + \Delta w = \delta\bar{\theta}, \quad \text{and} \quad w|_{t=T'} = 0, \tag{22}$$

with $T' \in (0,T]$ being any given.

Lemma 6

Let $\delta\bar{\theta} \in L^2(0,T';L^2)$, then there exists a unique weak solution $w \in L^\infty(0,T';H^1) \cap L^2(0,T';H^2)$ which satisfies

$$\sup_{t\in[0,T']} \|\nabla w\|_{L^2}^2 + \int_0^{T'} \|\nabla^2 w, \partial_t w\|_{L^2}^2 \mathrm{d}t + \|\nabla w\|^2_{L^{\frac{2+\sigma}{\sigma}}_{T'}(L^{\frac{2(2+\sigma)}{2-\sigma}})} \leq C\|\delta\bar{\theta}\|^2_{L^2(0,T';L^2)},$$

with $\sigma \in (0,2)$ and $C = C(\sigma) > 0$ a constant independent of T'.

Now we take the space-time scalar product of $(21)_1$ with w, and observe

$$\int_0^{T'}\int_{\mathbb{R}^2}(\partial_t\delta\bar\theta-\Delta\delta\bar\theta)\,w\mathrm{d}x\mathrm{d}t=\int_0^{T'}\int_{\mathbb{R}^2}\delta\bar\theta\,(-\partial_t w-\Delta w)\mathrm{d}x\mathrm{d}t,$$

we find

$$\begin{aligned}
&\|\delta\bar\theta\|^2_{L^2(0,T';L^2(\mathbb{R}^2))}\le\Big|\int_0^{T'}\int_{\mathbb{R}^2}(\delta N_1)\;w\,\mathrm{d}x\mathrm{d}t\Big|\\
&\le\Big|\int_0^{T'}\int_{\mathbb{R}^2}\Big((A_{v_1}A_{v_1}^{\mathrm{T}}-A_{v_2}A_{v_2}^{\mathrm{T}})\nabla\bar\theta_2\Big)\cdot\nabla w\,\mathrm{d}x\mathrm{d}t\Big|+\Big|\int_0^{T'}\int_{\mathbb{R}^2}\Big((\mathrm{Id}-A_{v_1}A_{v_1}^{\mathrm{T}})\nabla\delta\bar\theta\Big)\cdot\nabla w\,\mathrm{d}x\mathrm{d}t\Big|\\
&\le\Big|\int_0^{T'}\int_{\mathbb{R}^2}\Big((A_{v_1}A_{v_1}^{\mathrm{T}}-A_{v_2}A_{v_2}^{\mathrm{T}})\,\bar\theta_2\Big)\cdot\nabla^2 w\mathrm{d}x\mathrm{d}t\Big|+\Big|\int_0^{T'}\int_{\mathbb{R}^2}\Big(\nabla(A_{v_1}A_{v_1}^{\mathrm{T}}-A_{v_2}A_{v_2}^{\mathrm{T}})\,\bar\theta_2\Big)\cdot\nabla w\mathrm{d}x\mathrm{d}t\Big|\\
&\quad+\Big|\int_0^{T'}\int_{\mathbb{R}^2}\Big((\mathrm{Id}-A_{v_1}A_{v_1}^{\mathrm{T}})\,\delta\bar\theta\Big)\cdot\nabla^2 w\mathrm{d}x\mathrm{d}t\Big|+\Big|\int_0^{T'}\int_{\mathbb{R}^2}\Big(\nabla(\mathrm{Id}-A_{v_1}A_{v_1}^{\mathrm{T}})\,\delta\bar\theta\Big)\cdot\nabla w\,\mathrm{d}x\mathrm{d}t\Big|.
\end{aligned}$$

- Letting $T' > 0$ be sufficiently small so that

$$T'\|\nabla\bar{v}_1\|_{L^\infty_{T'}(L^\infty)} \le \frac{1}{32}, \quad \text{and} \quad CT'^{\frac{6+\sigma}{2(2+\sigma)}}\|\nabla^2\bar{v}_1\|_{L^\infty_T(L^{\frac{4}{2-\sigma}})} \le \frac{1}{8},$$

we get

$$\|\delta\bar{\theta}\|^2_{L^2_{T'}(L^2)} \le CT'^{\frac{6+\sigma}{2+\sigma}}\|\bar{\theta}_2\|^2_{L^\infty_{T'}(L^{\frac{2+\sigma}{\sigma}})}\Big(\|\nabla^2\delta\bar{v}\|^2_{L^2_{T'}(L^2)} + \|\nabla\delta\bar{v}\|^2_{L^{\frac{2+\sigma}{\sigma}}_{T'}(L^{\frac{2(2+\sigma)}{2-\sigma}})}\Big). \tag{23}$$

- Next we turn to the estimation of $\delta\bar{v}$. According to Lemma 5, we have

$$\begin{aligned}
&\|\nabla\delta\bar{v}\|_{L^\infty_{T'}(L^2)} + \|(\partial_t\delta\bar{v}, \nabla^2\delta\bar{v}, \nabla\delta\bar{p})\|_{L^2_{T'}(L^2)} + \|\nabla\delta\bar{v}\|_{L^{\frac{2+\sigma}{\sigma}}_{T'}(L^{\frac{2(2+\sigma)}{2-\sigma}})} \\
&\qquad \le C\|\delta\bar{\theta}\|_{L^2_{T'}(L^2)} + C\|\delta N_2\|_{L^2_{T'}(L^2)} + C\|\delta N_3\|_{L^2_{T'}(L^2)} \\
&\qquad\qquad + C\|\partial_t(\delta N_4)\|_{L^2_{T'}(L^2)} + C\|\nabla\,\mathrm{div}(\delta N_4)\|_{L^2_{T'}(L^2)}.
\end{aligned}$$

• We finally obtain

$$
\begin{aligned}
&\|\nabla\delta\bar{v}\|_{L^\infty_{T'}(L^2)} + \|(\partial_t\delta\bar{v}, \nabla^2\delta\bar{v}, \nabla\delta\bar{p})\|_{L^2_{T'}(L^2)} + \|\nabla\delta\bar{v}\|_{L^{\frac{2+\sigma}{\sigma}}_{T'}(L^{\frac{2(2+\sigma)}{2-\sigma}})} \\
&\quad\le CT'\|\nabla\bar{v}_1\|_{L^\infty_{T'}(L^\infty)}\|(\partial_t\delta\bar{v}, \nabla\delta\bar{p})\|_{L^2_{T'}(L^2)} + CT'^{\frac{1}{2}}\|v_2\|_{L^\infty_{T'}(L^\infty)}\|\nabla\delta\bar{v}\|_{L^\infty_{T'}(L^2)} \\
&\qquad + C\Big(T'\|(\nabla\bar{v}_1, \nabla\bar{v}_2)\|_{L^\infty_{T'}(L^\infty)} + T'^{\frac{6+\sigma}{2(2+\sigma)}}\|\bar{\theta}_2\|_{L^\infty_{T'}(L^{\frac{4}{2-\sigma}})}\Big)\|\nabla^2\delta\bar{v}\|_{L^2_{T'}(L^2)} \\
&\qquad\quad + CT'^{\frac{6+\sigma}{2(2+\sigma)}}\|(\bar{\theta}_2, \partial_t\bar{v}_2, \nabla\bar{p}_2, \nabla^2\bar{v}_1, \nabla^2\bar{v}_2)\|_{L^\infty_{T'}(L^{\frac{4}{2-\sigma}})}\|\nabla\delta\bar{v}\|_{L^{\frac{2+\sigma}{\sigma}}_{T'}(L^{\frac{2(2+\sigma)}{2-\sigma}})},
\end{aligned}
$$

where $C > 0$ is a universal constant.

• By letting $T' > 0$ small enough, we get $\|\nabla\delta\bar{v}\|_{L^\infty_{T'}(L^2)} = \|\nabla\delta\bar{v}\|_{L^{\frac{2+\sigma}{\sigma}}_{T'}(L^{\frac{2(2+\sigma)}{2-\sigma}})} \equiv 0$, and from (23), $\|\delta\bar{\theta}\|_{L^2_{T'}(L^2)} \equiv 0$. Since $\delta\bar{v}|_{t=0} = 0$, we conclude $\delta\bar{v} \equiv 0$, $\delta\bar{\theta} \equiv 0$ on $\mathbb{R}^2 \times [0, T']$.

• Repeating the above procedure, we finally get $\delta\bar{v} = \delta\bar{\theta} \equiv 0$ and also $X_{v_1} = X_{v_2}$ on $\mathbb{R}^2 \times [0, T]$. Going back to the Eulerian coordinates implies $(\mu_1, \theta_1, v_1) = (\mu_2, \theta_2, v_2)$ on $\mathbb{R}^2 \times [0, T]$.

Thanks for your attention!