Zero-viscosity limit of the Navier-Stokes equations in thin domain

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The incompressible Navier-Stokes equations in domain Ω :

$$\begin{cases} \partial_t u^\varepsilon + u^\varepsilon \cdot \nabla u^\varepsilon + \nabla p^\varepsilon - \varepsilon^2 \triangle u^\varepsilon = 0, \\ \text{div } u^\varepsilon = 0. \end{cases} \tag{1}$$

If there is boundary, we need add boundary conditions:

- Dirichlet boundary condition:

$$u^\varepsilon|_{\partial\Omega} = 0,$$

- Navier (slip) boundary condition:

$$u^\varepsilon \cdot n = 0, \quad ((\nabla u^\varepsilon + (\nabla u^\varepsilon)^t) \cdot n)_\tau = -\alpha u_\tau, \qquad x \in \partial\Omega.$$

Zero viscosity limit for Non-slip B.C

Formally, letting $\varepsilon \to 0$ system (1) is convergent to Euler equations:

$$\begin{cases} \partial_t u^e + u^e \cdot \nabla u^e + \nabla p^e = 0, \\ \text{div } u^e = 0. \end{cases} \tag{2}$$

If there is boundary, we need add boundary conditions

$$u^e \cdot n|_{\partial\Omega} = 0.$$

However, there exists mismatch on the boundary condition between $u^\varepsilon|_{\partial\Omega} = 0$ and $u^e \cdot n|_{\partial\Omega} = 0$, which leads to strong boundary layer.

Prandtl boundary layer theory(1904):
According to the Prandtl' assertion, one formally has

$$\begin{pmatrix} u^\varepsilon \\ v^\varepsilon \end{pmatrix}(t,x,y) = \begin{pmatrix} u^e \\ v^e \end{pmatrix}(t,x,y) + \begin{pmatrix} u^p \\ \varepsilon v^p \end{pmatrix}(t,x,\frac{y}{\varepsilon}) + O(\varepsilon), \qquad (3)$$

and derive the Prandtl equation:

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u - \partial_y^2 u + \partial_x P = 0, \quad y>0, \\ \partial_x u + \partial_y v = 0, \quad y>0, \\ u|_{y=0} = v|_{y=0} = 0, \\ \lim\limits_{y\to\infty} u = U(t,x), \quad u|_{t=0} = u_0(x,y), \end{cases} \qquad (4)$$

where $(U(t,x), P(t,x))$ satisfies Bernoulli's Law :

$$\partial_t U + U\partial_x U + \partial_x P = 0.$$

Known results on the classical inviscid limit

- Results on the Local well-posedness of Prandtl equations:
  – Monotonic data:
  Oleinik(1966): Xin-Zhang(2004),
  Alexandre-Wang-Xu-Yang(2015), Masmoudi-Wong(2015),
  – Gevrey (analytic) class:
  Sammartino-Caflish(1998), Gerard-Varet-Masmoudi(2016),
  Chen-Wang-Zhang(2018), Li-Yang(2020),
  Dietert-Gerard-Varet(2018), etc
- Results on the life span of Prandtl equations:
  Zhang-Zhang(2016), Igatova-Vicol(2016), etc

Known results on the classical inviscid limit

- Results on the ill-posedness of Prandtl equations:
  E-Enquist(1998), Gerard-Varet-Dormy(2010), etc.
- Results on the inviscid limit:
  Sammartino-Caflish(1998), Meakawa(2014),
  Wang-Wang-Zhang(2018), Fei-Tao-Zhang(2018),
  Grenier-Guo-Nguyen(2016), Chen-Wu-Zhang(2020),
  Gerard-Varet-Meakawa-Masmoudi(2018, 2020), etc

Navier-Stokes equation in a thin domain

The incompressible Navier-Stokes equations in thin domain $\mathcal{S}^\varepsilon \stackrel{\text{def}}{=} \{(x,y) \in \mathbb{T} \times \mathbb{R} : 0 < y < \varepsilon\}$

$$\begin{cases} \partial_t U + U \cdot \nabla U - \varepsilon^\alpha \triangle U + \nabla P = 0, \\ \text{div } U = 0, \\ U|_{y=0,\varepsilon} = 0, \end{cases} \tag{5}$$

with initial data $U|_{t=0} = (u_0^\varepsilon(x, \frac{y}{\varepsilon}), \varepsilon v_0^\varepsilon(x, \frac{y}{\varepsilon})) = U_0^\varepsilon$. Here $\alpha > 0$.

Scaled anisotropic Navier-Stokes equations

Write

$$U(t,x,y) = (u^\varepsilon(t,x,\frac{y}{\varepsilon}), \varepsilon v^\varepsilon(t,x,\frac{y}{\varepsilon})) \text{and} \quad P(t,x,y) = p^\varepsilon(t,x,\frac{y}{\varepsilon})$$

and introduce domain $\mathcal{S} \overset{\text{def}}{=} \{(x,y) \in \mathbb{T} \times \mathbb{R} : 0 < y < 1\}$. System (5) becomes

$$\begin{cases} \partial_t u^\varepsilon + u^\varepsilon \partial_x u^\varepsilon + v^\varepsilon \partial_y u^\varepsilon - \varepsilon^\alpha \partial_x^2 u^\varepsilon - \varepsilon^{\alpha-2} \partial_y^2 u^\varepsilon + \partial_x p^\varepsilon = 0, \\ \varepsilon^2 (\partial_t v^\varepsilon + u^\varepsilon \partial_x v^\varepsilon + v^\varepsilon \partial_y v^\varepsilon - \varepsilon^\alpha \partial_x^2 v^\varepsilon - \varepsilon^{\alpha-2} \partial_y^2 v^\varepsilon) + \partial_y p^\varepsilon = 0, \\ \partial_x u^\varepsilon + \partial_y v^\varepsilon = 0, \\ (u^\varepsilon, v^\varepsilon)|_{y=0,1} = 0, \quad (u^\varepsilon, v^\varepsilon)|_{t=0} = (u_0^\varepsilon, v_0^\varepsilon). \end{cases} \tag{6}$$

Formal limit equation

Formally, letting $\varepsilon \to 0$, (6) is convergent to:

Case 1: $0 < \alpha < 2$.

$$\partial_y^2 u = 0, \quad \partial_y p = 0, \quad \partial_x u + \partial_y v = 0, \quad (u, v)|_{y=0,1} = 0,$$

which implies

$$u = v = 0, \quad p(t, x, y) = p(t, x).$$

Hydrostatic Euler equation

Case 2: $\alpha > 2$.

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u + \partial_x p = 0, \\ \partial_y p = 0, \\ \partial_x u + \partial_y v = 0, \\ v|_{y=0,1} = 0, \quad u|_{t=0} = u_0, \end{cases} \tag{7}$$

with compatibility condition $\int_0^1 \partial_x u(t,x,y)dy = 0$.

- Local well-posedness: $\partial_{yy} u \geq \sigma > 0$, Brenier(1999), Masmoudi-Wong(2012)
- ill-posedness: u has inflection points, the system is Lipschitz ill-posedness, Grenier(2000), Renardy(2009)

Case 3: $\alpha = 2$.

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u - \partial_y^2 u + \partial_x p = 0, \\ \partial_y p = 0, \\ \partial_x u + \partial_y v = 0, \\ u|_{y=0,1} = v|_{y=0,1} = 0, \quad u|_{t=0} = u_0, \end{cases} \tag{8}$$

with compatibility condition $\int_0^1 \partial_x u(t,x,y)dy = 0$.

- Paicu-Zhang-Zhang (2019): GWP in analytic space.
- Renardy (2009): High-frequency instability in presence of inflexion point.
- Gérard-Varet-Masmoudi-Vicol (2018): LWP in Gevrey class $\frac{9}{8}$ under convexity condition.

Main result

Theorem (Wang-W-Zhang, 2019)

Let initial data $(u_0^\varepsilon, v_0^\varepsilon, u_0)$ *fall into Gevrey class with* $\sigma \in [\frac{8}{9}, 1]$ *and convexity condition* $\inf_\Omega \partial_y^2 u_0 > 0$. *Then there exist* $T > 0$ *and* $C > 0$ *independent of* ε *such that there exists a unique solution of the Navier-Stokes equations (6) in* $[0, T]$, *which satisfies*

$$\|(u^\varepsilon - u^p, \varepsilon v^\varepsilon - \varepsilon v^p)\|_{L^2_{x,y} \cap L^\infty_{x,y}} \leq C\varepsilon^2,$$

where (u^p, v^p) *is solution of (8).*

Error equation

Define errors between solutions and approximate solutions:

$$u^R \overset{\text{def}}{=} u^\varepsilon - u^p, \quad v^R \overset{\text{def}}{=} v^\varepsilon - v^p, \quad p^R \overset{\text{def}}{=} p^\varepsilon - p^p, \tag{9}$$

then (u^R, v^R, p^R) satisfies

$$\begin{cases} \partial_t u^R - \triangle_\varepsilon u^R + u^p \partial_x u^R + u^R \partial_x u^p + v^R \partial_y u^p + v^p \partial_y u^R \\ \qquad\qquad + \partial_x p^R - \varepsilon^2 g_1 = 0, \\ \varepsilon^2 (\partial_t v^R - \triangle_\varepsilon v^R + u^p \partial_x v^R + u^R \partial_x v^p + v^R \partial_y v^p + v^p \partial_y v^R) \\ \qquad\qquad + \partial_y p^R - \varepsilon^2 g_2 = 0, \\ \partial_x u^R + \partial_y v^R = 0, \\ (u^R, v^R)|_{y=0,1} = 0, \quad (u^R, v^R)|_{t=0} = (u_0^\varepsilon - u_0, \varepsilon v_0^\varepsilon - \varepsilon v_0). \end{cases} \tag{10}$$

Here $\triangle_\varepsilon = \varepsilon^2 \partial_x^2 + \partial_y^2$.

Difficult

The difficult in error equation is that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(u^R,\varepsilon v^R)\|_{L^2}^2+\|\nabla_\varepsilon(u^R,\varepsilon v^R)\|_{L^2}^2\\
&\quad\leq C\|v^R\|_{L^2}\|u^R\|_{L^2}+good\\
&\quad\leq\begin{cases}C\|\partial_x u^R\|_{L^2}\|u^R\|_{L^2}, & \text{loss one tangential derivative}\\ \dfrac{C}{\varepsilon}\|\varepsilon v^R\|_{L^2}\|u^R\|_{L^2}, & \text{loss } \frac{1}{\varepsilon}.\end{cases}
\end{aligned}
$$

where we used $v^R=-\int_0^y\partial_x u^R dx$. Here $\nabla_\varepsilon=(\varepsilon\partial_x,\partial_y)$.

By convexity condition on u^p, we want to use trick in hydrostatic equation. Introduce $\omega^R \stackrel{\text{def}}{=} \partial_y u^R - \varepsilon^2 \partial_x v^R$ and satisfies

$$\partial_t \omega^R - \triangle_\varepsilon \omega^R + u^p \partial_x \omega^R + u^R \partial_x \omega^p + v^p \partial_y \omega^R + v^R \partial_y \omega^p - \varepsilon^2 g_3 = 0, \tag{11}$$

with boundary condition

$$(\partial_y + \varepsilon|D|)\omega^R|_{y=0} = \frac{1}{2}\mathcal{F}^{-1}\int_0^1 G_0 \mathcal{F}(u^p \partial_x \omega^R + v^R \partial_y \omega^p + good)dy,$$

$$(\partial_y - \varepsilon|D|)\omega^R|_{y=1} = \frac{1}{2}\mathcal{F}^{-1}\int_0^1 G_1 \mathcal{F}(u^p \partial_x \omega^R + v^R \partial_y \omega^p + good)dy,$$

where $\|(G_0, G_1)\|_{L^s} \leq C \min\{1, \frac{1}{(\varepsilon|k|)^{\frac{1}{s}}}\}$.

Vorticity formula

Hydrostatic Trick:

$$\int_{\mathcal{S}} v^R \partial_y \omega^p \frac{\omega^R}{\partial_y \omega^p} = \int_{\mathcal{S}} v^R(\partial_y u^R - \varepsilon^2 \partial_x v^R)$$
$$= \int_{\mathcal{S}} \partial_y v^R u^R - \frac{\varepsilon^2}{2} \partial_x (v^R)^2 = 0.$$

However, integrate by parts on dissipation term lefts boundary term:

$$\int_{\mathbb{T}} (\partial_y \omega^R \ \omega^R)|_{y=0,1} \sim \int_{\mathbb{T}} (\varepsilon |D| \omega^R \ \omega^R)|_{y=0,1} + \int_{\mathbb{T}} ((\partial_y + \varepsilon |D|) \omega^R \ \omega^R)|_{y=0,1}$$
$$\leq \underbrace{C \|\varepsilon \partial_x \omega^R\|_{L^2} \|\partial_y \omega^R\|_{L^2}}_{\text{same order to dissipation}} + \underbrace{(\|\partial_x \omega^R\| + \|v^R\|_{L^2}}_{\text{lose one tangential derivative}} \|\partial_y \omega^R\|_{L^2}$$

Motivated by Gérard-Varet-Masmoudi-Vicol(2018), we introduce boundary layer lift function $\omega^{b,0}$

$$\begin{cases} (\partial_t - \triangle_\varepsilon)\omega^{b,0} = 0, \\ \partial_y \omega^{b,i}|_{y=0} = \partial_x h^0, \\ \omega^{b,0}|_{t=0} = 0, \end{cases} \tag{12}$$

posed for $t \in [0,T]$, $x \in \mathbb{T}$ and $y > 0$. Here

$$h^0 = \frac{1}{2}\mathcal{F}^{-1}\int_0^1 G_0\mathcal{F}(u^p\omega^R - \int_0^y u^R dz\ \partial_y\omega^p)dy.$$

We also define

$$u^{b,0}(x,y) = \int_{+\infty}^y \omega^{b,0}(x,z)dz, \quad v^{b,0} = \int_y^{+\infty} \partial_x u^{b,0}(x,z)dz \quad \text{for} \quad y > 0 \tag{13}$$

Lemma

Let $T > 0$ and $r \in \mathbf{R}$. The boundary layer vorticity $\omega^{b,i}$ obeys that

$$\int_0^t \|\omega^{b,i}\|_{X^r}^2 + \|(y-i)\partial_y \omega^{b,i}\|_{X^r}^2 ds \le \frac{C}{\beta^{\frac{3}{2}}} \int_0^t |h^i|^2_{X^{r+1-\frac{3\sigma}{4}}} ds,$$

and the boundary layer velocity $u^{b,i}, v^{b,i}$ obeys that

$$\int_0^t \|u^{b,i}\|_{X^r}^2 ds \le \frac{C}{\beta^{\frac{5}{2}}} \int_0^t |h^i|^2_{X^{r+1-\frac{5\sigma}{4}}} ds,$$
$$\int_0^t \|v^{b,i}\|_{X^r}^2 ds \le \frac{C}{\beta^{\frac{7}{2}}} \int_0^t |h^i|^2_{X^{r+2-\frac{7\sigma}{4}}} ds,$$

for all $t \in [0, T]$, $i = 0, 1$ and any $M \ge 0$.

Boundary layer lift

We define

$$\omega^{in} = \omega^R - \omega^{bl},$$

and recall

$$h^i = \frac{1}{2}\mathcal{F}^{-1}\int_0^1 G_i \mathcal{F}(u^p\omega^R - \int_0^y u^R dz\ \partial_y\omega^p)dy.$$

Lemma

There exists $\beta_* > 1$ *such that for* $\beta \geq \beta_*$ *and* $\sigma \in [\frac{4}{5}, 1]$*, there holds that*

$$\int_0^t |(h^0, h^1)|^2_{X^{r+\frac{\sigma}{2}}} ds \leq C \int_0^t \|\omega^{in}\|^2_{X^{r+\frac{\sigma}{2}}} ds, \tag{14}$$

and for $\sigma \in [\frac{8}{9}, 1]$,

$$\int_0^t \left|\varepsilon|D|(h^0, h^1)\right|^2_{X^{r+1-\frac{3\sigma}{4}}} ds$$
$$\leq C \int_0^t \left(\|P_{\geq N(\varepsilon)}(\partial_y u^R, \varepsilon^2\partial_x v^R)\|^2_{X^{r+1-\sigma}} + \|\omega^{in}\|^2_{X^{r+\frac{\sigma}{2}}}\right) ds.$$

Lemma

Under the assumptions of Lemma 2.3, there holds that

$$\sup_{s\in[0,t]} \|\omega^{bl}(s)\|^2_{X^{r-1+\frac{3\sigma}{4}}} + \int_0^t \|(\partial_y, \varepsilon\partial_x)\omega^{bl}\|^2_{X^{r-1+\frac{3\sigma}{4}}}\, ds$$
$$+\beta \int_0^t (\|\omega^{bl}\|^2_{X^{r-1+\frac{5\sigma}{4}}} + \|\varphi\omega^{bl}\|^2_{X^{r-1+\frac{7\sigma}{4}}}) ds$$
$$\leq C \int_0^t \|\omega^{in}\|^2_{X^{r+\frac{\sigma}{2}}}\, ds.$$

Equation of ω^{in}

By the construction of ω^{in}, we find that

$$\begin{cases} \partial_t \omega^{in} - \Delta_\varepsilon \omega^{in} + u^p \partial_x \omega^{in} + v^p \partial_y \omega^{in} + v^{in} \partial_y \omega^p \\ \qquad\qquad = N(\omega^R, \omega^R) + \text{easy terms}, \\ \partial_y \omega^{in}|_{y=0} = -\varepsilon |D| \omega^R|_{y=0} - \partial_y (\triangle_{\varepsilon,D})^{-1} (N(\omega^R, \omega^R))|_{y=0} \\ \qquad\qquad\qquad + \text{easy terms}, \\ \partial_y \omega^{in}|_{y=1} = \varepsilon |D| \omega^R|_{y=1} - \partial_y (\triangle_{\varepsilon,D})^{-1} (N(\omega^R, \omega^R))|_{y=1} \\ \qquad\qquad\qquad + \text{easy terms}, \\ \omega^{in}|_{t=0} = 0, \end{cases}$$

The estimates of ω^{in}

We use the hydrostatic trick to deal with $v^{in}\partial_y\omega^p$, and we derive the following energy estimate:

$$\sup_{s\in[0,t]}\|\omega^{in}(s)\|_{X^r}^2+\int_0^t\|(\partial_y,\varepsilon\partial_x)\omega^{in})\|_{X^r}^2ds+\beta\int_0^t\|\omega^{in}\|_{X^{r+\frac{\sigma}{2}}}^2ds$$
$$\leq C\varepsilon^2\int_0^t\|P_{\geq N(\varepsilon)}(\partial_y,\varepsilon\partial_x)(u^R,\varepsilon v^R)\|_{X^{r+1}}^2ds+\cdots$$

where the first term on the right comes from the following boundary term in the energy estimate

$$\Big|\int_0^t\int_{\mathbb{T}}\varepsilon|D|\langle D_x\rangle^r\omega_\Phi^R\,\frac{\langle D_x\rangle^r\omega_\Phi^{in}}{\partial_y\omega^p}|_{y=0,1}dxds\Big|,$$

which is bounded by

$$\int_0^t\big(\|\varepsilon|D|\omega^{in}\|_{X^r}+\|\varepsilon|D|\omega^{bl}\|_{X^r}\big)\big(\|\partial_y\omega^{in}\|_{X^r}+\|\omega^{in}\|_{X^r}\big)$$
$$+\big(\|\varepsilon|D|\omega^{bl}\|_{X^{r-\frac{\sigma}{2}}}+\|\varepsilon|D|\partial_y\omega^{bl}\|_{X^{r-\frac{\sigma}{2}}}\big)\|\omega^{in}\|_{X^{r+\frac{\sigma}{2}}}ds.$$

The estimates of ω^{in}

New trouble is to control the term $\int_0^t \|\varepsilon|D|\omega^{in}\|_{X^r}^2 dt$. For this, we need to make a high-low frequency decomposition for ω^{in} so that

$$\int_0^t \|P_{\leq 2N(\varepsilon)}\varepsilon|D|\omega^{in}\|_{X^r}^2 ds \leq C\int_0^t \|\omega^{in}\|_{X^{r+\frac{\sigma}{2}}}^2 ds$$

and

$$\int_0^t \|P_{\geq N(\varepsilon)}\varepsilon|D|\omega^{in}\|_{X^r}^2 ds \leq C\varepsilon^2\int_0^t \|P_{\geq N(\varepsilon)}(\partial_y, \varepsilon\partial_x)(u^R, \varepsilon v^R)\|_{X^{r+1}}^2 ds$$
$$+ \int_0^t \big(\|P_{\geq N(\varepsilon)}(\partial_y, \varepsilon\partial_x)(u^R, \varepsilon v^R)\|_{X^{r+1-\sigma}}^2 + \|\omega^{in}\|_{X^{r+\frac{\sigma}{2}}}^2\big) ds,$$

where $N(\varepsilon) = [\varepsilon^{-\frac{2}{2-\sigma}}]$.

The estimates of ω^{in}

This decomposition is the key observation of this paper, which is motivated by the fact that

$$\|P_{\geq N(\varepsilon)} f\|_{X^r} \leq C\|P_{\geq N(\varepsilon)} \varepsilon f\|_{X^{r+1-\frac{\sigma}{2}}},$$

which is very useful for the control of v^R instead of the usual control $\|v^R\|_{X^r} \leq \|u^R\|_{X^{r+1}}$(losing one derivative).

All we left it to give the estimates of high frequency part of $(u^R, \varepsilon v^R)$. Here, we notice that the factor ε^2 is useful in this case when $(u^R, \varepsilon v^R)$ in middle frequency($\varepsilon \sim \langle k\rangle^{\frac{\sigma}{2}-1}$ can gain derivative). Then we get

$$
\begin{aligned}
&\varepsilon^2\|P_{\geq N(\varepsilon)}(u^R, \varepsilon v^R)(t)\|^2_{X^{r+1}} + \beta\varepsilon^2\int_0^t \|P_{\geq N(\varepsilon)}(u^R, \varepsilon v^R)\|^2_{X^{r+1+\frac{\sigma}{2}}}\\
&\quad + \int_0^t \varepsilon^2\|P_{\geq N(\varepsilon)}(\partial_y, \varepsilon\partial_x)(u^R, \varepsilon v^R)\|^2_{X^{r+1}}\\
&\leq C\int_0^t \|\omega^{in}\|^2_{X^{r+\frac{\sigma}{2}}}\,ds + \delta\int_0^t \|P_{\geq N(\varepsilon)}(\mathcal{N}_u, \varepsilon\mathcal{N}_v)\|^2_{X^{r+1-\frac{\sigma}{2}}}\,ds.
\end{aligned}
$$

Combining all the above estimates, we get the inviscid limit.

Thank you !