Stability for compressible viscous and diffusive MHD equations with the Coulomb force

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Joint works with Xin Li and Shu Wang

Contents

$$\begin{cases} \partial_t \rho + \text{div}\,(\rho u) = 0, \\ \partial_t\,(\rho u) + \text{div}\,(\rho u \otimes u) + \nabla \left(p + \frac{1}{2}\,|\mathscr{B}|^2 \right) \\ \qquad\qquad = \text{div}\,(\mathscr{B} \otimes \mathscr{B}) + \rho \nabla \phi + \mu \Delta u + (\mu + \lambda)\,\nabla \text{div} u, \\ \partial_t \mathscr{B} + \text{div}\,(\mathscr{B} \otimes u) - \text{div}\,(u \otimes \mathscr{B}) - \nu \Delta \mathscr{B} = 0, \quad \text{div} \mathscr{B} = 0, \\ -\Delta \phi = b - \rho, \quad \lim_{|x| \to \infty} \phi = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^3, \end{cases} \tag{1}$$

** ρ, $p = p(\rho)$ and u : density, pressure and velocity

** $\mathscr{B}$ and ϕ : magnetic field and electric potential

**$\nu > 0$: magnetic diffusivity

** μ and λ : constant viscosity coefficients satisfy

$$\mu > 0, \quad 3\lambda + 2\mu \geq 0.$$

** $b = b(x)$: the doping profile, $b \geq$ const. > 0

initial conditions in $\mathbb{R}^3$

$$t = 0: \ (\rho, u, \mathscr{B}) = (\rho_0, u_0, \mathscr{B}_0), \quad x \in \mathbb{R}^3. \tag{2}$$

the second equation in (1) can be written as

$$\partial_t u + u \cdot \nabla u + \nabla h(\rho) = \frac{1}{\rho}(\nabla \times \mathscr{B}) \times \mathscr{B} + \nabla\phi + \frac{1}{\rho}(\mu\Delta u + (\mu + \lambda)\nabla \mathrm{div} u),$$

where h is the enthalpy function satisfying $\nabla p(\rho) = \rho\nabla h(\rho)$. Since p is smooth and strictly increasing on $(0, +\infty)$, so is h.

Let $\left(\bar{\rho}, \bar{u}, \overline{\mathscr{B}}, \bar{\phi}\right)$ be such a solution of variable x with $\bar{u} = \mathbf{0}$ and $\overline{\mathscr{B}} = \mathbf{0}$. We get

$$\begin{cases} \nabla h(\bar{\rho}) = \nabla \bar{\phi}, \\ -\Delta \bar{\phi} = b - \bar{\rho}, \end{cases} \tag{3}$$

which implies $\bar{\rho}$ satisfies an elliptic equation :

$$-\Delta h(\bar{\rho}) = b - \bar{\rho}, \quad \text{in } \mathbb{R}^3. \tag{4}$$

By using a variational method or the classical fixed-point theorem.

Proposition 1. Existence of equilibrium solutions.

Let $s_1 \geq 1$. Assume $b \in L^\infty(\mathbb{R}^3), \nabla b \in H^{s_1-1}(\mathbb{R}^3)$ and $b \geq$ const. > 0 a.e. $x \in \mathbb{R}^3$. Then problem (4) admits a unique solution $\bar{\rho} = \bar{\rho}(x)$ satisfying $\bar{\rho} - b \in H^{s_1}(\mathbb{R}^3)$, $\bar{\rho} \geq$ const. > 0.

Neglecting the Coulomb force, system (1) becomes the general compressible barotropic MHD equations. Then the equilibrium solution with zero velocity will be constant.

Chen-Tan, Nonlinear Anal. 2010

– the optimal convergence rates of the small smooth solutions in L^q, $2 \le q \le 6$, provided that the initial data in L^p, $1 \le p < 6/5$.

Hu-Wang, Arch. Ration. Mech. Anal. 2010

– the existence and large-time behavior of global weak solutions for the initial-boundary value problem with large data.

Jiang-Jiang, SIAM J. Math. Anal. 2018

– Rayleigh-Taylor stability, presented a sufficient condition for the linear ideal instability of plane parallel equilibria with antisymmetric shear flow and symmetric or antisymmetric magnetic field.

Kang-Kim, J. Funct. Anal. 2014

– a regularity criteria for suitable weak solutions of MHD equations near boundary in dimension three,

– suitable weak solutions are Hölder continuous near boundary.

Kwon-Trivisa, J. Differential Equations, 2011

– the incompressible limits of weak solutions to the governing equations for MHD flows on both bounded and unbounded domains.

Li-Xu-Zhang, SIAM J. Math. Anal. 2013

– the Cauchy problem to barotropic MHD equations in $\mathbb{R}^3$,

– the global well-posedness of classical solution provided that regular initial data satisfying small energy.

Wang, SIAM J. Appl. Math. 2003

– the initial-boundary value problem for MHD equations in one space dimension,

– The existence, uniqueness, and regularity of global solutions with large initial data in H^{-1}.

Wang-Wang-Liu-Wang, J. Differential Equations, 2017

– the boundary layer problem and zero viscosity-diffusion limit of the initial boundary value problem for the incompressible viscous and diffusive MHD system with (no-slip characteristic) Dirichlet boundary conditions,

– the corresponding Prandtl's type boundary layer are stable with respect to small viscosity-diffusion coefficients.

Liu-Xie-Yang, CPAM 2019

—- 2D MHD Boundary layer system, Well-posedness and high Reynolds number limit

Li-Yang, 2020

—- Well-posedness of 3D MHD Boundary layer system without any structure assumption in Gevrey function space

Liu-Wang-Xie-Yang, JFA 2020

—- MHD system that show critical Gevrey index could be 2

Liu-Zhang-Yang, 2020

—-2D MHD system admits a unique solution, high Reynolds number limit of steady MHD in Sobolev space

J.H. Wu, Y.F. Wu, X.J. Xu,...

Motivation

All these results hold when the solution is near a constant equilibrium state of the MHD system.

Wang-Tan, 2019

– the stability on the non-constant equilibrium solutions of MHD equations (1) with an additional friction force $-\alpha\rho u$,

– the existence and uniqueness of the global solution.

A nature question is that how about the solution behave if there is no friction force?

Theorem 1. F. Li-Wang, JDDE. 2020

Let $s \geq 3$ be an integer. Then there exist constants $\delta_0 > 0, C > 0$ s. t. if

$$\|(\rho_0 - \bar{\rho}, u_0, \mathscr{B}_0, \nabla\phi_0 - \nabla\bar{\phi})\|_s \leq \delta_0,$$

Problem (1)-(2) has a unique global solution $(\rho, u, \mathscr{B}, \phi)$ satisfying

$$\begin{aligned} & \left\|(\rho(t) - \bar{\rho}, u(t), \mathscr{B}(t), \nabla\phi(t) - \nabla\bar{\phi})\right\|_s^2 \\ & + \int_0^t \left(\|\rho(\tau) - \bar{\rho}\|_s^2 + \|\nabla u(\tau)\|_s^2 + \|\nabla\mathscr{B}(\tau)\|_s^2\right) d\tau \\ \leq & C\left\|(\rho_0 - \bar{\rho}, u_0, \mathscr{B}_0, \nabla\phi_0 - \nabla\bar{\phi})\right\|_s^2, \qquad \forall\ t \geq 0, \end{aligned} \tag{5}$$

which implies

$$\lim_{t\to\infty} \|\rho(t) - \bar{\rho}\|_{s-1} = 0, \tag{6}$$

and

$$\lim_{t\to\infty} \|\nabla u(t)\|_{s-3} = 0, \quad \lim_{t\to\infty} \|\nabla \mathscr{B}(t)\|_{s-3} = 0. \tag{7}$$

We pointed out that gradients of both the velocity and the magnetic field converge to the equilibrium states with the same norm $\|\cdot\|_{H^{s-3}}$, while the density converges with stronger norm $\|\cdot\|_{H^{s-1}}$.

Remark :

The equilibrium solution is large

Since b is large, the techniques used for constant equilibrium solution no longer work due to the appearance of lower order terms which will make essential difficulties in energy estimates.

The friction force is lost

By using the Theorem of the decomposition of divergence and curl, Wang-Tan solve the problem in Wang-Tan (2019) Comm. Math. Sci. Different from their work, we remove the friction force (the velocity dissipation term).

We solve this problem by using an anti-symmetric matrix technique and employing an induction argument on the order of the derivatives of solutions in energy estimates.

2. Techniques and an induction argument.

Barotropic Euler-Maxwell equations for electrons

$$\begin{cases} \partial_t n + \mathrm{div}(nu) = 0 \\ \partial_t(nu) + \mathrm{div}(nu \otimes u) + \nabla p(n) = -n(E + u \times B) - nu \\ \partial_t E - \nabla \times B = nu, \quad \mathrm{div}\, E = b - n \\ \partial_t B + \nabla \times E = 0, \quad \mathrm{div}\, B = 0 \end{cases}$$

** n and u : density and velocity

** E and B : electric and magnetic fields

** p : pressure function, $p'(n) > 0, \ \forall\, n > 0$

** $b = b(x)$ is a given smooth periodic function, $b \geq$ const. > 0

initial conditions in a torus $\mathbb{T}^3$

$$t = 0: \quad (n, u, E, B) = (n^0, u^0, E^0, B^0)$$

which satisfies the compatibility condition

$$\operatorname{div} E^0 = b - n^0, \quad \operatorname{div} B^0 = 0$$

(a) Equivalent momentum equation for $n > 0$:

$$\partial_t u + (u \cdot \nabla) u + \nabla h(n) = -(E + u \times B) - u$$

the enthalpy h :

$$h'(n) = p'(n)/n > 0, \quad \forall\, n > 0$$

(b) All physical parameters are set equal to 1.

Otherwise, perform asymptotic analysis with small parameters

B. Texier (2005-2007)

– convergence of Euler-Maxwell to Zakharov equation

– to Davey-Stewartson equation

Y.J. Peng - S. Wang (2008-2009)

– convergence of Euler-Maxwell to incompressible Euler equations

– to e-MHD equations

Y. Guo - X.K. Pu (2012)

– convergence of Euler-Poisson to KdV equations

– to KP equations

J.W. Yang

– series works on non-isentropic Euler-Maxwell systems

Symmetrizable hyperbolic system :

$$\partial_t w + \sum_{j=1}^{d} A_j(w)\partial_{x_j} w = g(w), \quad w(0,x) = w^0(x), \quad x \in \mathbb{R}^d$$

(a) $\exists$ symmetrizer $A_0(w)$, symmetric positive definite matrix

(b) $\tilde{A}_j(w) \overset{def}{=} A_0(w)A_j(w)$ is symmetric for all $1 \le j \le d$

Consequence : energy estimate

$$\frac{d}{dt}\int A_0(w)w\cdot w\,dx = \int (\mathrm{div}_{t,x}\vec{A}w\cdot w + 2A_0(w)g(w)\cdot w)dx$$

where

$$\int A_0(w)w\cdot w\,dx \approx \|w\|_{L^2}^2, \quad \mathrm{div}_{t,x}\vec{A} = \partial_t A_0(w) + \sum_{j=1}^{d}\partial_{x_j}\tilde{A}_j(w)$$

Theorem T. Kato, ARMA, 1975

Let $s > d/2 + 1$ be an integer, $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$, $w^0 \in H^s(\Omega)$.

There exist $T > 0$ and a unique smooth solution

$$w \in C^1([0, T]; H^{s-1}(\Omega)) \cap C([0, T]; H^s(\Omega))$$

Regularity :

$$w \in \bigcap_{k=0}^{s} C^k([0, T]; H^{s-k}(\Omega))$$

The Euler-Maxwell system is symmetrizable hyperbolic for $n > 0$

$$w = (n, u, E, B)^T, \quad d = 3 \implies s \geq 3$$

Then we have local existence of smooth solutions

Steady states solutions with zero velocity

$$\bar{w} = (\bar{n}(x), 0, \bar{E}(x), \bar{B}(x))^T$$

Substitute $\bar{w}$ into

$$\begin{cases} \partial_t n + \operatorname{div}(nu) = 0 \\ \partial_t u + (u \cdot \nabla) u + \nabla h(n) = -(E + u \times B) - u \\ \partial_t E - \nabla \times B = nu, \quad \operatorname{div} E = b - n \\ \partial_t B + \nabla \times E = 0, \quad \operatorname{div} B = 0, \end{cases}$$

$$\Longrightarrow \begin{cases} \nabla h(\bar{n}) = -\bar{E} \\ \nabla \times \bar{B} = 0, \quad \operatorname{div} \bar{B} = 0 \implies \bar{B} \text{ is a constant} \\ \nabla \times \bar{E} = 0, \quad \operatorname{div} \bar{E} = b - \bar{n} \end{cases}$$

$$\operatorname{div} \bar{E} = b - \bar{n} \implies -\Delta h(\bar{n}) = b - \bar{n}$$

Let

$$\bar{\phi} = h(\bar{n}), \quad q = h^{-1}$$

Then $\bar{\phi}$ satisfies a semilinear monotone elliptic equation in $\mathbb{T}^3$:

$$-\Delta\bar{\phi} = b - q(\bar{\phi}), \quad \bar{n} = q(\bar{\phi})$$

Consequence : there is a unique steady state periodic smooth solution

$$b \geq \text{const.} > 0 \implies \quad \bar{n} \geq \text{const.} > 0$$

$$b = 1 \implies \quad \bar{n} = 1, \ \bar{E} = 0$$

Stability problem :

$\|\cdot\|_m$ is a norm of $H^m(\mathbb{T}^3)$

$\forall\, s \geq 3, \quad \|w^0 - \bar{w}\|_s$ is small

$\Longrightarrow$ global existence of solution w and stability estimate

$$\|w(t,\cdot) - \bar{w}\|_s \leq C\|w^0 - \bar{w}\|_s, \quad \forall\, t > 0$$

The unique steady state solution is constant $\bar{w} = (1, 0, 0, \bar{B})^T$

Denote

$$U = (n-1, u)^T, \quad W = w - \bar{w} = (n-1, u, E, B - \bar{B})^T$$

When $\|w^0 - \bar{w}\|_s$ is small, a classical energy estimate yields

$$\frac{d}{dt}\|W(t)\|_s^2 + C_0\|u(t)\|_s^2 \leq C\|U(t)\|_s^2\,\|W(t)\|_s$$

Next, using the system and $p'(n) > 0$ yields

$$\|n(t)-1\|_s^2 \leq C\|u(t)\|_s^2 + C\|U(t)\|_s^2\,\|W(t)\|_s + \frac{de}{dt}, \quad |e| \leq C_2\|U\|_s^2$$

Therefore, for $\varepsilon > 0$ small

$$\frac{d}{dt}(\|W(t)\|_s^2 - \varepsilon e) + C_1\|U(t)\|_s^2 \leq C\|U(t)\|_s^2\,\|W(t)\|_s, \quad t > 0$$

For $\|W\|_s$ small, we obtain

$$\|W(t)\|_s^2 + \int_0^t \|U(\tau)\|_s^2\,d\tau \leq C\|W(0)\|_s^2, \quad \forall\, t > 0$$

which yields global existence of solutions

Theorem Y.J. Peng - S. Wang - Q.L. Gu, SIAM JMA, 2011

Let $s \geq 3$ be an integer. If $\|W(0)\|_s$ is sufficiently small, the Euler-Maxwell system admits a unique global solution

$$W \in C^1(\mathbb{R}^+; H^{s-1}(\mathbb{T}^3)) \cap C(\mathbb{R}^+; H^s(\mathbb{T}^3))$$

Theorem R.J. Duan, J. Hyper. Differ. Equations, 2011

Let $s \geq 4$. If $\|W(0)\|_{H^{s+2}(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)}$ is small, then

$$\left\|n(t) - 1\right\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-1}, \quad \|u(t)\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{5}{8}},$$

$$\|E(t)\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{4}} \ln(3+t), \quad \|B(t)\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{8}}.$$

Theorem Y. Ueda - S. Wang - S. Kawashima, SIAM JMA, 2012

Let $s \geq 6$. If $\|W(0)\|_{H^s(\mathbb{R}^3)}$ is small, then

$$\|W(t)\|_{H^{s-2k}(\mathbb{R}^3)} \leq C\|W(0)\|_{H^s(\mathbb{R}^3)}(t+1)^{-k/2}, \quad \forall\, 0 \leq k \leq [s/2]$$

For b is a small perturbation of 1

Theorem Q.Q. Liu - C.J. Zhu, Indiana. Univ. Math. J., 2013

Let $s \geq 3$ be an integer. Suppose $\|b(x) - 1\|_{W_0^{s+1,2}}$ is small enough. If $\|(n_0 - n_{st}, u_0, E_0 - E_{st}, B_0)\|_s$ is sufficiently small, the Euler-Maxwell system admits a unique global solution

$$(n - n_{st}, u, E - E_{st}, B) \in C^1(\mathbb{R}^+; H^{s-1}(\mathbb{R}^3)) \cap C(\mathbb{R}^+; H^s(\mathbb{R}^3))$$

Theorem W.K. Wang and X. Xu, Z. Angew. Math. Phys., 2016

Let $s \geq 3$ be an integer. Suppose $\|b(x) - 1\|_{s+1}$ is small enough. If $\|(n_0 - n_{st}, u_0, \theta_0 - 1, E_0 - E_{st}, B_0)\|_s$ is sufficiently small, the non-isentropic Euler-Maxwell system admits a unique global solution

$$(n - n_{st}, u, \theta - 1, E - E_{st}, B) \in C^1(\mathbb{R}^+; H^{s-1}(\mathbb{R}^3)) \cap C(\mathbb{R}^+; H^s(\mathbb{R}^3))$$

A more general framework

Quasilinear symmetrizable hyperbolic system

$$\partial_t w + \sum_{j=1}^{d} \partial_{x_j} f_j(w) = g(w), \quad w(0,x) = w^0(x), \quad x \in \mathbb{R}^d$$

$$w(t,x) \in \mathbb{R}^N, \quad \bar{w} = 0 \quad \text{and} \quad g(0) = 0$$

Entropy-flux (E,F) : $\quad F'(w) = E'(w)(f_1'(w), \cdots, f_d'(w))$

- partial dissipation
- Shizuta-Kawashima condition

$\Longrightarrow$ global existence

Two stability conditions

partial dissipation :

there exist a strictly convex entropy $E(w)$ and a change of variables $w \longmapsto (u, v)^T$, $u \in \mathbb{R}^{N-r}$, $v \in \mathbb{R}^r$, such that

$$(E'(w) - E'(0))g(w) \le -c_0|v|^2, \quad |\cdot| \text{ is a norm of } \mathbb{R}^r$$

which implies

$$\|w(t)\|_s^2 + \int_0^t \|v(\tau)\|_s^2 \, d\tau \le C\|w^0\|_s^2 + C\int_0^t \|w(\tau)\|_s (\|\nabla u(\tau)\|_{s-1}^2 + \|v(\tau)\|_s^2) \, d\tau$$

Shizuta-Kawashima condition :

$$\forall\, \nu \in S^{d-1}, \quad \forall\, \lambda \in \mathbb{C}, \qquad \mathcal{N}(\lambda I_N - A(\bar{w}, \nu)) \cap \mathcal{N}(g'(\bar{w})) = \{0\}$$

$$A(w, \nu) = \sum_{j=1}^{d} \nu_j A_j(w), \quad A_j(w) = f_j'(w), \quad \nu = (\nu_1, \cdots, \nu_d) \in S^{d-1}$$

This condition implies

$$\int_0^t \|\nabla u(\tau)\|_{s-1}^2 \, d\tau \le C\|w^0\|_s^2 + C \int_0^t \|w(\tau)\|_s (\|\nabla u(\tau)\|_{s-1}^2 + \|v(\tau)\|_s^2) \, d\tau$$

Thus, for small solutions, we have

$$\|w(t)\|_s^2 + \int_0^t (\|\nabla u(\tau)\|_{s-1}^2 + \|v(\tau)\|_s^2) \, d\tau \le C\|w^0\|_s^2, \quad t > 0$$

which yields global existence of solutions when $\|w^0\|$ is small

B. Hanouzet - R. Natalini, ARMA, 2003,

global existence 1-d

W.A. Yong, ARMA, 2004,

global existence $d \geq 1$

S. Bianchini - B. Hanouzet - R. Natalini, CPAM, 2007

algebraic decay of solutions $O(t^{-\mu}), \mu > 0$

K. Beauchard - E. Zuazua, ARMA, 2011

refined results $O(t^{-\mu})$

Remark

Shizuta-Kawashima condition is not fulfilled by Euler-Maxwell systems

Y. Lv, 2019

The zakharov approximation to Euler-Maxwell is unstable

J. Xu, 2011

stability of Euler-Maxwell in Besov space

EM system without velocity dissipation

$$
\begin{cases}
\partial_t n + \operatorname{div}(nu) = 0 \\
\partial_t u + (u \cdot \nabla) u + \nabla h(n) = -(E + u \times B) \\
\partial_t E - \nabla \times B = nu, \quad \operatorname{div} E = 1 - n \\
\partial_t B + \nabla \times E = 0, \quad \operatorname{div} B = 0
\end{cases}
$$

P. Germain-N. Masmoudi (Ann. Sci. Ec. Norm. Supér 2014)

one-fluid

$$
\partial_t (B - \nabla \times u) = \nabla \times (u \times (B - \nabla \times u))
$$

$$
B^0 - \nabla \times u^0 = 0 \implies B - \nabla \times u = 0
$$

Guo-Ionescu-Pausader (Ann. math. 2016)

two-fluid

i) linearized system around constant states is of Klein-Gordon type

$$\text{time decay} \quad O(t^{-\frac{3}{2}})$$

ii) link with Euler-Poisson system for potential flows (Y. Guo, 1998)

$$B = 0 \implies \nabla \times u = 0$$

Deng-Ionescu-Pausader (ARMA 2017)

2D

2.2 Stability of non constant states for EM system

Notation :

$$\|\cdot\| = \|\cdot\|_0, \quad \text{a norm of } L^2(\mathbb{T}^3)$$

For $m \in \mathbb{N}$ and $v \in \cap_{k=0}^{m} C^k([0,T]; H^{m-k}(\mathbb{T}^3))$, define

$$|\!|\!| v(t,\cdot) |\!|\!|_m = \Big(\sum_{k+|\alpha|\le m} \|\partial_t^k \partial_x^\alpha v(t,\cdot)\|^2 \Big)^{\frac{1}{2}}, \qquad t \in [0,T]$$

$$\alpha \in \mathbb{N}^3, \quad |\alpha| = \alpha_1 + \alpha_2 + \alpha_3$$

$$\Longrightarrow |\!|\!|\cdot|\!|\!|_m \text{ is a norm and } \|\cdot\|_m \le |\!|\!|\cdot|\!|\!|_m$$

unknown: $w = (n, u, E, B)^T$, initial data w^0

steady state solution: $\bar{w}(x) = (\bar{n}(x), 0, \bar{E}(x), \bar{B}(x))^T$

Theorem Y.J. Peng, JMPA 2015

Let $s \geq 3$. If $\|w^0 - \bar{w}\|_s$ is small, the periodic problem for the barotropic Euler-Maxwell equations admits a unique global smooth solution :

$$w - \bar{w} \in \bigcap_{k=0}^{s} C^k(\mathbb{R}^+; H^{s-k}(\mathbb{T}^3))$$

$$\|\|w(t) - \bar{w}\|\|_s^2 + \int_0^t \|\|(n(\tau) - \bar{n}, u(\tau))\|\|_s^2 \leq C\|w^0 - \bar{w}\|_s^2, \quad t > 0$$

Moreover, if

$$\int_{\mathbb{T}^3} B^0(x)\, dx = \bar{B}$$

then

$$\lim_{t \to +\infty} \|\|w(t) - \bar{w}\|\|_{s-1} = 0$$

Let

$$N = n - \bar{n}, \quad F = E - \bar{E}, \quad G = B - \bar{B}$$

The Euler-Maxwell system is

$$\begin{cases} \partial_t N + u \cdot \nabla N + n \operatorname{div} u + \nabla \bar{n} \cdot u = 0 \\ \partial_t u + (u \cdot \nabla) u + \nabla\big(h(n) - h(\bar{n})\big) + u \times G + (u + F + u \times \bar{B}) = 0 \\ \partial_t F - \nabla \times G = (N + \bar{n}) u, \quad \operatorname{div} F = -N \\ \partial_t G + \nabla \times F = 0, \quad \operatorname{div} G = 0 \end{cases}$$

where

$$\nabla(h(n) - h(\bar{n})) = h'(n) \nabla N + \nabla h'(\bar{n}) N + r$$

$$r = \big(h'(N + \bar{n}) - h'(\bar{n}) - h''(\bar{n}) N\big) \nabla \bar{n} = O(N^2)$$

$$U=(n-\bar{n},u)^T,\quad W=w-\bar{w}$$

Then Euler equations are written as

$$\partial_t U+\sum_{j=1}^{3}A_j(n,u)\partial_{x_j}U+L(x)U+M(W)=f$$

$$f=-\begin{pmatrix}0\\ r+u\times G\end{pmatrix}=O(U)\,O(W),\quad M(W)=\begin{pmatrix}0\\ u+F+u\times\bar{B}\end{pmatrix}$$

In $M(W)$

- u stands for velocity dissipation
- F can be treated together with Maxwell equations
- $u\cdot(u\times\bar{B})=0$

$$A_j(n,u) = \begin{pmatrix} u_j & n e_j^T \\ h'(n) e_j & u_j \mathbf{I}_3 \end{pmatrix}, \quad L(x) = \begin{pmatrix} 0 & (\nabla \bar{n})^T \\ \nabla h'(\bar{n}) & 0 \end{pmatrix}$$

Symmetrizer

$$A_0(n) = \begin{pmatrix} h'(n) & 0 \\ 0 & n\mathbf{I}_3 \end{pmatrix} \Longrightarrow \langle A_0(n)U, U\rangle \approx \|U\|^2$$

Case $b = 1$

$$\bar{n} = 1 \Longrightarrow L(x) = 0 \Longrightarrow \text{ no linear term in the system}$$

$$\nabla_x n = \nabla_x(n-1) \Longrightarrow \partial_x A_j(n,u) = O(\partial_x U)$$

Case $b = b(x)$: main difficulties in higher order energy estimates

(1) $L(x) \neq 0 \Longrightarrow L(x)U = O(U)$

(2) $\partial_x A_j(n,u) = \partial_x A_j(N + \bar{n}, u) = O(1)$

L^2 estimates

L^2-inner product $\langle \cdot, \cdot \rangle$

$$\frac{d}{dt}\langle A_0(n)U, U\rangle = \langle \partial_t A_0(n)U, U\rangle + 2\langle A_0(n)f, U\rangle - 2\langle A_0(n)M(W), U\rangle + \langle Q(x,n,u)U, U\rangle$$

1. Estimate for $\langle \partial_t A_0(n)U, U\rangle$

$$\|\partial_t n\|_\infty = \|\partial_t N\|_\infty \le C\|U\|_s \implies \langle \partial_t A_0(n)U, U\rangle \le C\|U\|_s^3$$

2. Estimate for $\langle A_0(n)f, U\rangle$

$$f = O(U)\,O(W) \implies 2\langle A_0(n)f, U\rangle \le C\|U\|_s^2\,\|W\|_s$$

3. Estimate for $\langle A_0(n)M(W), U\rangle$

$$-2\langle A_0(n)M(W), U\rangle = -2\langle nu, u\rangle - 2\langle nu, F\rangle$$

4. Estimate for $\langle Q(x,n,u)U, U\rangle$ (Guo-Strauss, ARMA 2005)

$$
\begin{aligned}
Q(x,n,u) &= \sum_{j=1}^{3} \partial_{x_j}\tilde{A}_j(n,u) - 2A_0(n)L(x) \\
&= \begin{pmatrix} \operatorname{div}(h'(n)u) & (\nabla p'(n) - 2h'(n)\nabla\bar{n})^T \\ \nabla p'(n) - 2n\nabla h'(\bar{n}) & \operatorname{div}(nu)\,\mathbf{I}_3 \end{pmatrix}
\end{aligned}
$$

is an anti-symmetric matrix at $(n,u) = (\bar{n}, 0)$, because

$$\nabla p'(n) - 2h'(n)\nabla n = -(\nabla p'(n) - 2n\nabla h'(n))$$

$$\Longrightarrow \left|\langle Q(x,n,u)U, U\rangle\right| \leq C\|U\|_s^3$$

Similar treatment for Q in higher order estimates

Energy estimates for Maxwell equations

$$\frac{d}{dt}(\|F\|^2 + \|G\|^2) = 2\langle nu, F\rangle$$

L^2-estimate

$$\frac{d}{dt}\|W\|^2 + C_0\|u\|^2 \leq C\|U\|_s^2 \, \|W\|_s$$

Higher order estimates

For $\alpha \in \mathbb{N}^3$, $|\alpha| \le s$, we have

$$\frac{d}{dt}\langle A_0(n)\partial_x^\alpha U, \partial_x^\alpha U\rangle = 2\sum_{j=1}^{3}\langle A_j\partial_{x_j}(\partial_x^\alpha U) - \partial_x^\alpha(A_j\partial_{x_j}U), A_0\partial_x^\alpha U\rangle + \cdots$$

Noticing $\partial_x A_j(n,u) = O(1)$ and using Moser inequality, we have

$$\langle A_j\partial_{x_j}(\partial_x^\alpha U) - \partial_x^\alpha(A_j\partial_{x_j}U), A_0\partial_x^\alpha U\rangle = O(\|U\|_s^2)$$

$$\frac{d}{dt}\|W(t)\|_{|\alpha|}^2 + C_0\|u(t)\|_{|\alpha|}^2 \le C\|N(t)\|_{|\alpha|}^2 + C\|U(t)\|_s^2\,\|W(t)\|_s + \cdots$$

$$\|N(t)\|_{|\alpha|}^2 \le C\|u(t)\|_{|\alpha|}^2 + C\|U(t)\|_s^2\,\|W(t)\|_s + \cdots$$

$\Longrightarrow$ these energy estimates are not sufficient to conclude

Idea : estimates on $\partial_t^k \partial_x^\alpha W$ + induction argument

Time derivative estimates : $\alpha = 0$

$$\frac{d}{dt}\|\partial_t^k W\|^2 + C_0\|\partial_t^k u\|^2 \le C\|U\|_s^2\, \|W\|_s, \quad \forall\, 0 \le k \le s$$

The momentum equation implies

$$\|\partial_t^k N\|_1^2 \le C\|\partial_t^k u\|^2 + C\|\partial_t^{k+1} u\|^2 + C\|U\|_s^2\, \|W\|_s, \quad 0 \le k \le s-1$$

The density equation implies

$$\|\partial_t^s N\|^2 \le C\|\partial_t^{s-1} u\|_1^2 + C\|U\|_s^2\, \|W\|_s \quad (k = s)$$

Time-space derivative estimates : $|\alpha| \ge 1$, $\forall\, k + |\alpha| \le s$

$$\frac{d}{dt}\|\partial_t^k W\|_{|\alpha|}^2 + C_0\|\partial_t^k U\|_{|\alpha|}^2 \le C(\|\partial_t^k U\|_{|\alpha|-1}^2 + \|\partial_t^{k+1} u\|_{|\alpha|-1}^2) + C\|U\|_s^2\, \|W\|_s$$

Induction argument : for $k \downarrow$ and $|\alpha| \uparrow$

$$(k, |\alpha|) = (s, 0),\ \frac{d}{dt}\|\partial_t^s W\|^2 + C_0\|\partial_t^s U\|^2 \le C\|\partial_t^{s-1} u\|_1^2 + C\|U\|_s^2\ \|W\|_s$$

$$(k, |\alpha|) = (s-1, 1),\ \frac{d}{dt}\|\partial_t^{s-1} W\|_1^2 + C_0\|\partial_t^{s-1} U\|_1^2 \le C\|\partial_t^{s-1} U\|^2 + C\|\partial_t^s u\|^2 + C\|U\|_s^2\ \|W\|_s$$

By induction we obtain

$$\frac{d}{dt}\|W\|_s^2 + 2C_1\|U\|_s^2 \le C\|U\|_s^2\ \|W\|_s$$

For small solutions, we have

$$\|W(t)\|_s^2 + C_1 \int_0^t \|U(\tau)\|_s^2\, d\tau \le C\|W(0)\|_s^2, \quad \forall\, t > 0$$

which yields the global existence of solutions

For all $k + |\alpha| \le s - 1$,

$$\partial_t^k \partial_x^\alpha U \in L^2(\mathbb{R}^+; L^2(\mathbb{T}^3)) \cap W^{1,\infty}(\mathbb{R}^+; L^2(\mathbb{T}^3))$$

which implies

$$\lim_{t\to+\infty} \|(n(t) - \bar{n}, u(t))\|_{s-1} = 0$$

Similarly for E and B

let us introduce the perturbation variables as

$$\xi = \rho - \bar{\rho}, \quad \Phi = \phi - \bar{\phi}, \tag{8}$$

$$\mathscr{U}^I = \begin{pmatrix} \xi \\ u \end{pmatrix}, \quad \mathscr{V}^I = \begin{pmatrix} \mathscr{U}^I \\ \mathscr{B} \end{pmatrix}, \quad \mathscr{W}^I = \begin{pmatrix} \mathscr{V}^I \\ \nabla\Phi \end{pmatrix},$$
$$\mathscr{U}^I_0 = \begin{pmatrix} \xi_0 \\ u_0 \end{pmatrix}, \quad \mathscr{V}^I_0 = \begin{pmatrix} \mathscr{U}^I_0 \\ \mathscr{B}_0 \end{pmatrix}, \quad \mathscr{W}^I_0 = \begin{pmatrix} \mathscr{V}^I_0 \\ \nabla\Phi_0 \end{pmatrix}, \tag{9}$$

where

$$\xi_0 = \rho_0 - \bar{\rho}, \quad \nabla\Phi_0 = \nabla\phi_0 - \nabla\bar{\phi}.$$

System (1) can be written as

$$
\begin{cases}
\partial_t \xi + u \cdot \nabla \xi + \rho \mathrm{div} u + u \cdot \nabla \bar{\rho} = 0, \\
\partial_t u + (u \cdot \nabla) u + \nabla h(\rho) - \nabla h(\bar{\rho}) \\
\quad = \frac{1}{\rho} (\nabla \times \mathscr{B}) \times \mathscr{B} + \nabla \Phi + \frac{1}{\rho} (\mu \Delta u + (\mu + \lambda) \nabla \mathrm{div} u), \\
\partial_t \mathscr{B} + \nabla \times (\mathscr{B} \times u) = \nu \Delta \mathscr{B}, \quad \mathrm{div} \mathscr{B} = 0, \\
\Delta \Phi = \xi, \quad (t, x) \in (0, \infty) \times \mathbb{R}^3.
\end{cases}
\tag{10}
$$

A straightforward computation implies

$$
\nabla h(\rho) - \nabla h(\bar{\rho}) = h'(\rho) \nabla \xi + \nabla h'(\bar{\rho}) \xi + \mathscr{R}^I,
$$

where

$$
\mathscr{R}^I = (h'(\rho) - h'(\bar{\rho}) - h''(\bar{\rho}) \xi) \nabla \bar{\rho} \sim \xi^2.
$$

The first two equations in (10) can be rewritten as

$$\partial_t \mathscr{U}^I + \sum_{j=1}^{3} \mathscr{A}_j^I (\rho, u) \partial_j \mathscr{U}^I + \mathscr{L}^I (x) \mathscr{U}^I + \mathscr{M}^I (u, \Phi) = \mathscr{F}^I(\mathscr{B}, \mathscr{R}^I), \quad (11)$$

with

$$\mathscr{A}_j^I (\rho, u) = \begin{pmatrix} u_j & \rho e_j^T \\ h'(\rho) e_j & u_j \mathbf{I_3} \end{pmatrix}, \quad j = 1, 2, 3, \quad (12)$$

$$\mathscr{L}^I (x) = \begin{pmatrix} 0 & (\nabla \bar{\rho})^T \\ \nabla h'(\bar{\rho}) & 0 \end{pmatrix}, \quad (13)$$

$$\mathscr{M}^I(u,\Phi) = \begin{pmatrix} 0 \\ -\nabla\Phi - \frac{1}{\rho}\left(\mu\Delta u + (\mu+\lambda)\nabla \mathrm{div} u\right) \end{pmatrix}, \tag{14}$$

and

$$\mathscr{F}^I\left(\mathscr{B}, \mathscr{R}^I\right) = \begin{pmatrix} 0 \\ \frac{1}{\rho}(\nabla\times\mathscr{B})\times\mathscr{B} - \mathscr{R}^I \end{pmatrix}. \tag{15}$$

It is clear that the matrix $\mathscr{A}_0^I(\rho)$ defined by

$$\mathscr{A}_0^I(\rho) = \begin{pmatrix} h'(\rho) & 0 \\ 0 & \rho \mathbf{I_3} \end{pmatrix}, \tag{16}$$

is symmetric and uniformly positive definite for $\rho > 0$, and

$$\widetilde{\mathscr{A}}_j^I(\rho, u) = \mathscr{A}_0^I(\rho)\mathscr{A}_j^I(\rho, u) = \begin{pmatrix} h'(\rho)u_j & h'(\rho)\rho e_j^T \\ h'(\rho)\rho e_j & \rho u_j \mathbf{I_3} \end{pmatrix} \tag{17}$$

is symmetric.

Local existence of smooth solution

Then it follows from Kato's theory that there exists an unique local smooth solution of the Cauchy problem to system (10).

Let $\mathscr{W}^I$ be smooth solution of (10) defined on time interval $[0, T]$ with initial data $\mathscr{W}_0^I$. From now on, we denote

$$\mathscr{V}_T^I = \sup_{0 \le t \le T} \|\mathscr{V}^I(t)\|_s. \tag{18}$$

For $s \ge 3$, we assume that $\mathscr{V}_T^I$ is uniformly sufficiently small with respect to T. Then from the continuous embedding $H^{s-1} \hookrightarrow L^\infty$, we get

$$\frac{1}{2}\bar{\rho} \le \rho \le \frac{3}{2}\bar{\rho}, \ h'(\rho) \ge m_0,$$

where m_0 is a positive constant independent of any time.

Lemma 1.

For all $t \in [0,T]$, it holds

$$\|\partial_t \xi\| + \|\partial_t \xi\|_\infty + \left\|\partial_t \mathscr{A}_0^I(n)\right\|_\infty \le C\|\nabla u\|_2, \tag{19}$$

and

$$\left|\left\langle \sum_{j=1}^{3} \partial_j \widetilde{\mathscr{A}}_j^I(\rho,u) - 2\mathscr{A}_0^I(\rho)\mathscr{L}^I(x)\mathscr{U}^I, \mathscr{U}^I \right\rangle\right| \le C\|\mathscr{U}^I\|_s \mathscr{D}_s^*(\mathscr{W}^I(t)), \tag{20}$$

where

$$\mathscr{D}_s^*(\mathscr{W}^I(t)) = \|\xi(t)\|_s^2 + \|\nabla u(t)\|_s^2. \tag{21}$$

Lemma 2.

For all $t \in [0, T]$, it holds

$$\begin{aligned} &\frac{d}{dt}\left(\left\langle \mathscr{A}_0^I(\rho)\mathscr{U}^I, \mathscr{U}^I\right\rangle + \|\mathscr{B}\|^2 + \|\nabla\Phi\|^2\right) + 2\mu\|\nabla u\|^2 \\ &+ 2(\mu+\lambda)\|\mathrm{div} u\|^2 + 2\nu\|\nabla\mathscr{B}\|^2 \\ \leq & C\left\|\mathscr{U}^I\right\|_s \mathscr{D}_s^*(\mathscr{W}^I(t)). \end{aligned} \tag{22}$$

Let $\alpha \in \mathbb{N}^3$ with $1 \leq |\alpha| \leq s$. Applying ∂^α on (11) gives

$$\begin{aligned}
&\partial_t \partial^\alpha \mathscr{U}^I + \sum_{j=1}^{3} \mathscr{A}_j^I \partial_j \partial^\alpha \mathscr{U}^I + \mathscr{L}^I \partial^\alpha \mathscr{U}^I + \partial^\alpha \mathscr{M}^I(u, \Phi) \\
=&\partial^\alpha \mathscr{F}^I(\mathscr{B}, \mathscr{R}^I) + \mathscr{G}^{I\alpha},
\end{aligned} \tag{23}$$

where

$$\mathscr{G}^{I\alpha} = \sum_{j=1}^{3} \left(\mathscr{A}_j^I \partial_j \partial^\alpha \mathscr{U}^I - \partial^\alpha \left(\mathscr{A}_j^I \partial_j \mathscr{U}^I \right) \right) + \mathscr{L}^I \partial^\alpha \mathscr{U}^I - \partial^\alpha \left(\mathscr{L}^I \mathscr{U}^I \right). \tag{24}$$

Lemma 3.

For all $t \in [0, T]$ and $\alpha \in \mathbb{N}^3$ with $1 \le |\alpha| \le s$, it holds

$$\begin{aligned} &\frac{d}{dt}\left(\left\langle \mathscr{A}_0^I \partial^\alpha \mathscr{U}^I, \partial^\alpha \mathscr{U}^I \right\rangle + \left\|\partial^\alpha \mathscr{B}\right\|^2 + \left\|\partial^\alpha \nabla \Phi\right\|^2\right) + \mu \left\|\partial^\alpha \nabla u\right\|^2 \\ &+ (\mu + \lambda) \left\|\partial^\alpha \mathrm{div} u\right\|^2 + 2\nu \left\|\partial^\alpha \nabla \mathscr{B}\right\|^2 \\ \le & C \mathscr{D}^*_{|\alpha|-1}(\mathscr{W}^I(t)) + C \left\|\mathscr{V}^I\right\|_s \mathscr{D}_s(\mathscr{W}^I(t)), \end{aligned} \tag{25}$$

where

$$\mathscr{D}_s(\mathscr{W}^I(t)) = \mathscr{D}_s^*(\mathscr{W}^I(t)) + \|\nabla \mathscr{B}\,(t)\,\|_s^2, \tag{26}$$

from Lemma 2 and Lemma 3, we obtain

Proposition 2.

For all $t \in [0, T]$, $\alpha \in \mathbb{N}^3$ with $1 \le |\alpha| \le s$, it holds

$$\begin{aligned}
&\frac{d}{dt}\sum_{\beta\le\alpha}\left(\left\langle \mathscr{A}_0^I \partial^\beta \mathscr{U}^I, \partial^\beta \mathscr{U}^I\right\rangle + \left\|\partial^\beta \mathscr{B}\right\|^2 + \left\|\partial^\beta \nabla\Phi\right\|^2\right) + \mu\,\|\nabla u\|_{|\alpha|}^2 \\
&+ (\mu+\lambda)\,\|\mathrm{div} u\|_{|\alpha|}^2 + 2\nu\|\nabla\mathscr{B}\|_{|\alpha|}^2 \qquad (27) \\
\le\, & C\mathscr{D}_{|\alpha|-1}^*(\mathscr{W}^I(t)) + C\left\|\mathscr{V}^I\right\|_s \mathscr{D}_s(\mathscr{W}^I(t)).
\end{aligned}$$

Dissipation estimate for ξ

Estimate (27) contains a recurrence relation on the time dissipation of ∇u. It is clear that this estimate is not sufficient to control the higher order term in (27) and the dissipation estimates of ξ is necessary.

Proposition 3.

For all $t \in [0, T]$, $\alpha \in \mathbb{N}^3$ with $1 \leq |\alpha| \leq s$, it holds

$$\begin{aligned} &\frac{d}{dt} \sum_{|\beta|<|\alpha|} \langle \partial^\beta u, \partial^\beta \nabla \xi \rangle + C_0 \|\xi\|_{|\alpha|}^2 \\ \leq &C \|\nabla u\|_{|\alpha|}^2 + C \|\xi\|_{|\alpha|-1}^2 + C \left\|\mathscr{V}^I\right\|_s \mathscr{D}_s(\mathscr{W}^I(t)), \end{aligned} \tag{28}$$

$$\frac{d}{dt} \langle u, \nabla (h'(\bar{\rho}) \xi) \rangle + C_0 \|\xi\|_1^2 \leq C \|\nabla u\|_1^2 + C \left\|\mathscr{V}^I\right\|_s \mathscr{D}_s(\mathscr{W}^I(t)), \tag{29}$$

$$
\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha|\le 1}\left(\langle \mathscr{A}_0^I \partial^\alpha \mathscr{U}^I, \partial^\alpha \mathscr{U}^I\rangle + \left\|\partial^\alpha \mathscr{B}\right\|^2 + \|\partial^\alpha \nabla\Phi\|^2\right) + \mu\,\|\nabla u\|_1^2 \\
&+ (\mu+\lambda)\,\|\mathrm{div} u\|_1^2 + 2\nu\,\|\nabla \mathscr{B}\|_1^2 \\
\le & \varepsilon_0\,\|\xi\|_1^2 + C\,\|\nabla u\|^2 + C\left\|\mathscr{V}^I\right\|_s \mathscr{D}_s(\mathscr{W}^I(t)),
\end{aligned}
\tag{30}
$$

where C_0 is a positive constant independent of any time and the positive constant $\varepsilon_0 > 0$ is determined later.

Proposition 4.

There exists small positive constants η and C_1 such that, for all $t \in [0, T]$, $\alpha \in \mathbb{N}^3$ with $1 \le |\alpha| \le s$, it holds

$$\frac{d}{dt}\left(\sum_{\beta\le\alpha}\left(\left\langle \mathscr{A}_0^I \partial^\beta \mathscr{U}^I, \partial^\beta \mathscr{U}^I \right\rangle + \left\|\partial^\beta \mathscr{B}\right\|^2 + \left\|\partial^\beta \nabla\Phi\right\|^2\right) + \eta \sum_{|\gamma|\le|\alpha|-1} \langle \partial^\gamma u, \partial^\gamma \nabla\xi \rangle\right)$$

$$+ C_1 \mathscr{D}_{|\alpha|}(\mathscr{W}^I(t))$$

$$\le C\mathscr{D}^*_{|\alpha|-1}(\mathscr{W}^I(t)) + C\left\|\mathscr{V}^I\right\|_s \mathscr{D}_s(\mathscr{W}^I(t)). \tag{31}$$

We carry on the induction on $|\alpha|(1 \le |\alpha| \le s)$ of space derivatives for (31) and (29)-(30).

The step of the induction is increasing from $|\alpha| = 1$ to $|\alpha| = s$.

More precisely, for $|\alpha| = 1$, we first multiplies η on both sides of (29).

We point out that the term $C\eta\|\nabla u\|_1^2$ can be controlled by $\|\nabla u\|_1^2$ on the left-hand side of (30),

and the term $\varepsilon_0\|\xi\|_1^2$ on the right-hand side of (30) can be controlled by $\|\xi\|_1^2$ provided that $\varepsilon_0 > 0$ is small enough.

Thus, there exists a positive constant κ_1 such that

$$\begin{aligned}&\kappa_1 \frac{d}{dt}\Big(\sum_{|\alpha|\leq 1} (\langle \mathscr{A}_0^I \partial^\alpha \mathscr{U}^I, \partial^\alpha \mathscr{U}^I\rangle + \left\|\partial^\alpha \mathscr{B}\right\|^2 + \|\partial^\alpha \nabla\Phi\|^2) + \eta \langle u, \nabla (h'(\bar{\rho})\xi)\rangle \Big)\\ &+\mathscr{D}_1(\mathscr{W}^I(t))\\ \leq & C\|\nabla u\|^2 + C\|\mathscr{V}^I\|_s \mathscr{D}_s(\mathscr{W}^I(t)).\end{aligned} \tag{32}$$

In the same way, for $|\alpha| \geq 2$, $C\mathscr{D}^*_{|\alpha|-1}(\mathscr{W}^I(t))$ on the right-hand side of (31) can be controlled by $\mathscr{D}_{|\alpha|}(\mathscr{W}^I(t))$ in the preceding step on the left-hand side of (31) multiplying an appropriate large positive constant. Then we get

$$
\begin{aligned}
&\frac{d}{dt}\sum_{m=1}^{s}\kappa_m\Big(\sum_{|\alpha|\le m}\big(\langle\mathscr{A}_0^I\partial^\alpha\mathscr{U}^I,\partial^\alpha\mathscr{U}^I\rangle+\left\|\partial^\alpha\mathscr{B}\right\|^2+\|\partial^\alpha\nabla\Phi\|^2\\
&+\eta\sum_{|\gamma|\le|\alpha|-1}\langle\partial^\gamma u,\partial^\gamma\nabla\xi\rangle\big)\Big)+\mathscr{D}_s(\mathscr{W}^I(t)) \qquad (33)\\
\le& C\|\nabla u\|^2+C\|\mathscr{V}^I\|_s\mathscr{D}_s(\mathscr{W}^I(t)),
\end{aligned}
$$

where $\kappa_m > 0(m = 1,\cdots,s)$ are some constants.

By use of formulas (22) and (33) and noting $\|\mathscr{V}^I\|_s$ is small, we have

$$
\begin{aligned}
&\frac{d}{dt}\sum_{m=1}^{s}\kappa_m\Big(\sum_{|\alpha|\le m}\big(\langle \mathscr{A}_0^I\partial^\alpha\mathscr{U}^I,\partial^\alpha\mathscr{U}^I\rangle+\big\|\partial^\alpha\mathscr{B}\big\|^2+\|\partial^\alpha\nabla\Phi\|^2 \\
&+\eta\sum_{|\gamma|\le|\alpha|-1}\langle\partial^\gamma u,\partial^\gamma\nabla\xi\rangle\big)\Big) \\
&+\mathscr{D}_s(\mathscr{W}^I(t))\le 0,
\end{aligned}
\tag{34}
$$

where the constant $\kappa_m>0(m=1,\cdots,s)$ may be modified again.

When $\eta > 0$ is small enough,

$$\sum_{k=1}^{s} \kappa_m \Big(\sum_{|\alpha| \le m} (\langle \mathscr{A}_0^I \partial^\alpha \mathscr{U}^I, \partial^\alpha \mathscr{U}^I \rangle + \left\| \partial^\alpha \mathscr{B} \right\|^2 + \|\partial^\alpha \nabla \Phi\|^2 + \eta \sum_{|\gamma| \le |\alpha| - 1} \langle \partial^\gamma u, \partial^\gamma \nabla \xi \rangle) \Big)$$

is equivalent to

$$\mathscr{E}_s(\mathscr{W}^I(t)) = \|\xi\,(t)\,\|_s^2 + \|u\,(t)\,\|_s^2 + \|\mathscr{B}\,(t)\,\|_s^2 + \|\nabla \Phi\,(t)\,\|_s^2. \tag{35}$$

Integrating (34) from 0 to t, we obtain (5).

Moreover, (5) implies that

$$\partial^\beta \xi \in L^\infty\left(\mathbb{R}^+, L^2(\mathbb{R}^3)\right), \quad \forall |\beta| \le s-1,$$
$$\partial^\gamma \nabla u, \quad \partial^\gamma \nabla \mathscr{B} \in L^\infty\left(\mathbb{R}^+, L^2(\mathbb{R}^3)\right), \quad \forall |\gamma| \le s-3,$$

and

$$\partial_t \partial^\beta \xi \in L^\infty\left(\mathbb{R}^+, L^2(\mathbb{R}^3)\right), \quad \forall |\beta| \le s-1,$$
$$\partial_t \partial^\gamma \nabla u, \quad \partial_t \partial^\gamma \nabla \mathscr{B} \in L^\infty\left(\mathbb{R}^+, L^2(\mathbb{R}^3)\right), \quad \forall |\gamma| \le s-3.$$

Then,

$$\partial^\beta \xi \in W^{1,\infty}\left(\mathbb{R}^+, L^2(\mathbb{R}^3)\right), \quad \forall |\beta| \le s-1,$$
$$\partial^\gamma \nabla u, \quad \partial^\gamma \nabla \mathscr{B} \in W^{1,\infty}\left(\mathbb{R}^+, L^2(\mathbb{R}^3)\right), \quad \forall |\gamma| \le s-3.$$

Furthermore,

$$\partial^\beta \xi \in L^2\left(\mathbb{R}^+, L^2(\mathbb{R}^3)\right), \quad \forall |\beta| \le s-1,$$
$$\partial^\gamma \nabla u, \quad \partial^\gamma \nabla \mathscr{B} \in L^2\left(\mathbb{R}^+, L^2(\mathbb{R}^3)\right), \quad \forall |\gamma| \le s-3.$$

We deduce that

$$\partial^\beta \xi \in L^2\left(\mathbb{R}^+, L^2(\mathbb{R}^3)\right) \cap W^{1,\infty}\left(\mathbb{R}^+, L^2(\mathbb{R}^3)\right), \quad \forall |\beta| \le s-1,$$
$$\partial^\gamma \nabla u, \quad \partial^\gamma \nabla \mathscr{B} \in L^2\left(\mathbb{R}^+, L^2(\mathbb{R}^3)\right) W^{1,\infty}\left(\mathbb{R}^+, L^2(\mathbb{R}^3)\right), \quad \forall |\gamma| \le s-3,$$

which implies (6)-(7),

$$\lim_{t\to\infty} \|\rho(t) - \bar{\rho}\|_{s-1} = 0,$$

and

$$\lim_{t\to\infty} \|\nabla u(t)\|_{s-3} = 0, \quad \lim_{t\to\infty} \|\nabla \mathscr{B}(t)\|_{s-3} = 0.$$

This completes the proof of Theorem 1.

4. The full MHD systems

$$
\begin{cases}
\partial_t \rho + \operatorname{div}(\rho u) = 0, \\
\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla\left(p + \frac{1}{2}|\mathscr{B}|^2\right) \\
\qquad\qquad = \operatorname{div}(\mathscr{B} \otimes \mathscr{B}) + \rho\nabla\phi + \mu\Delta u + (\mu+\lambda)\nabla \operatorname{div} u, \\
\partial_t \mathcal{E} + \operatorname{div}(\mathcal{E}u + pu) = \rho u \cdot \nabla\phi - (\mathcal{E} - \rho e_l) + u\left(\mu\Delta u + (\mu+\lambda)\nabla \operatorname{div} u\right), \\
\partial_t \mathscr{B} + \operatorname{div}(\mathscr{B} \otimes u) - \operatorname{div}(u \otimes \mathscr{B}) - \nu\Delta\mathscr{B} = 0, \quad \operatorname{div}\mathscr{B} = 0, \\
-\Delta\phi = b(x) - \rho, \quad \lim_{|x|\to\infty} \phi = 0,
\end{cases}
\tag{36}
$$

** p, e and θ : pressure, internal energy and absolute temperature

** $\mathcal{E} = \frac{1}{2}\rho|u|^2 + \rho e$: total energy

** $e_l > 0$: background internal energy

** $b = b(x)$: the doping profile, $b \geq$ const. > 0

The initial condition to system (36) is given as

$$(\rho, u, \theta, \mathscr{B})|_{t=0} = (\rho_0, u_0, \theta_0, \mathscr{B}_0), \quad x \in \mathbb{R}^3. \tag{37}$$

For convenience, we consider the case of ideal polytropic gas

$$p = \rho\theta, \quad e = \theta, \tag{38}$$

Then for smooth solutions in any non-vacuum field, the momentum and energy equations in (36) can be written as

$$\partial_t u + u \cdot \nabla u + \frac{1}{\rho} \nabla (n\theta) = \frac{1}{\rho} (\nabla \times \mathscr{B}) \times \mathscr{B} + \nabla \phi + \frac{1}{\rho} (\mu \Delta u + (\mu + \lambda) \nabla \mathrm{div} u), \tag{39}$$

and

$$\partial_t \theta + u \cdot \nabla \theta + \theta \mathrm{div} u + u \cdot ((\nabla \times \mathscr{B}) \times \mathscr{B}) + \frac{1}{2} |u|^2 + (\theta - \theta_l) = 0. \tag{40}$$

Let $(\bar{\rho}, \bar{u}, \bar{\theta}, \overline{\mathscr{B}}, \bar{\phi})$ be such a solution of variable x with $\bar{u} = \mathbf{0}$ and $\overline{\mathscr{B}} = \mathbf{0}$. It follows from (36) and (38)-(40) that

$$\begin{cases} \nabla \bar{p} = \bar{\rho} \nabla \bar{\phi}, \quad \bar{p} = \bar{\rho}\bar{\theta}, \\ \bar{\theta} = \theta_l, \\ -\Delta \bar{\phi} = b - \bar{\rho}. \end{cases} \tag{41}$$

This implies $\bar{\rho}$ satisfies an elliptic equation:

$$-\theta_l \Delta \ln \bar{\rho} + \bar{\rho} = b(x), \quad x \in \mathbb{R}^3. \tag{42}$$

Since the function $\bar{\rho} \mapsto \ln \bar{\rho}$ is strictly increasing, by use of the classical fixed-point theorem or a variational method, (42) admits a unique solution. The existence of solutions to the elliptic equation is also stated as Proposition 1.

Background

Hu-Wang, Comm. Math. Phys. 2008

–the initial-boundary value problem of the 3-d full MHD system,

– an approximation scheme and a weak convergence method,

– the existence of a global variational weak solution with large data.

Jiang-Ju-Li, Adv. Math. 2014

– the low Mach number limit for the full compressible MHD equations with general initial data in $\mathbb{R}^3$,

– by using a theorem due to Metivier-Schochet, Arch. Ration. Mech. Anal. 2001 for the Euler equations that gives the local energy decay of the acoustic wave equations.

Ju-Li-Li (2013) SIAM J. Math. Anal.

– as the Mach number, the viscosity coefficients, the heat conductivity, and the magnetic diffusion coefficient go to zero simultaneously

– for the general initial data, the weak solutions of the full compressible MHD equations in $\mathbb{R}^3$ converge to the strong solution of the ideal incompressible MHD equations

Pu-Guo, Z. Angew. Math. Phys. 2013

– by energy method, the full compressible MHD equations in $\mathbb{R}^3$,

– the global existence of smooth solutions near the constant state

– the convergence rates of the L^p norm of these solutions to this state when the L^q norm of the perturbation is bounded.

What do we want to do?

We want to prove the stability of the steady-state solution $(\bar{\rho}, \mathbf{0}, \theta_l, \mathbf{0}, \bar{\phi})$ provided that the initial data $(\rho_0, u_0, \theta_0, \phi_0)$ are close to this steady-state.

step 1. choose a new perturbation variable Q

$$Q = \ln p - \ln \bar{p}$$

with

$$\bar{p} = \bar{\rho}\theta_l,$$

step 2. use a non-diagonal positive matrix as symmetrizer

This allows us to use again the technique of anti-symmetric matrix in the energy estimates.

step 3. establish the recurrence relation

By the equations satisfied by the new variables, we establish the relation that the derivatives of $(\nabla u, \theta)$ can be controlled by lower order derivatives with respect to x of $(Q, \nabla u, \theta)$, and Q depends only on the same order derivative with respect to x of (u, θ) in refined estimates.

Theorem 2. F. Li-Wang (2020) JDDE

Let $s \geq 3$. Then there exist constants $\delta_0 > 0, C > 0$ such that if

$$\| (\rho_0 - \bar{\rho}, u_0, \theta_0 - \theta_l, \mathscr{B}_0, \nabla\phi_0 - \nabla\bar{\phi}) \|_s \leq \delta_0,$$

Problem (36)-(37) has a global smooth solution $(n, u, \theta, \mathscr{B}, \phi)$ satisfying

$$\begin{aligned}
&\left\|(\rho(t) - \bar{\rho}, u(t), \theta(t) - \theta_l, \mathscr{B}(t), \nabla\phi(t) - \nabla\bar{\phi})\right\|_s^2 \\
&+ \int_0^t \left(\|\rho(\tau) - \bar{\rho}\|_s^2 + \|\nabla u(\tau)\|_s^2 + \|\theta(\tau) - \theta_l\|_s^2 + \|\nabla\mathscr{B}(\tau)\|_s^2\right) d\tau \qquad (43) \\
\leq& C\left\|(\rho_0 - \bar{\rho}, u_0, \theta_0 - \theta_l, \mathscr{B}_0, \nabla\phi_0 - \nabla\bar{\phi})\right\|_s^2, \qquad \forall\, t \geq 0.
\end{aligned}$$

for all $t > 0$,

$$\lim_{t\to\infty} \|\rho(t) - \bar{\rho}\|_{s-1} = 0, \quad \lim_{t\to\infty} \|\theta(t) - \theta_l\|_{s-1} = 0, \tag{44}$$

and

$$\lim_{t\to\infty} \|\nabla u(t)\|_{s-3} = 0, \quad \lim_{t\to\infty} \|\nabla \mathscr{B}(t)\|_{s-3} = 0. \tag{45}$$

It should be pointed out that both the density and the temperature converge to the equilibrium states with the same norm $\|\cdot\|_{H^{s-1}}$.

let us set

$$\Theta = \theta - \theta_l, \quad \mathscr{U}^F = \begin{pmatrix} \xi \\ u \\ \Theta \end{pmatrix}, \quad \mathscr{U}_0^F = \begin{pmatrix} \xi_0 \\ u_0 \\ \Theta_0 \end{pmatrix}, \tag{46}$$

where $\Theta_0 = \theta_0 - \theta_l$. From (36), (38) and (40), it is easy to check that the pressure p satisfies the equation

$$\partial_t p + u \cdot \nabla p + 2p\mathrm{div}u + \frac{p}{\theta} u \cdot ((\nabla \times \mathscr{B}) \times \mathscr{B}) + \frac{p}{2\theta} |u|^2 + \frac{p}{\theta} \Theta = 0. \tag{47}$$

Let

$$q = \ln p, \quad \bar{q} = \ln \bar{p}, \quad Q = q - \bar{q}, \quad \mathscr{V}^F = \begin{pmatrix} Q \\ u \\ \Theta \end{pmatrix}, \quad \mathscr{V}_0^F = \begin{pmatrix} Q_0 \\ u_0 \\ \Theta_0 \end{pmatrix}, \tag{48}$$

$$\mathscr{W}^F = \begin{pmatrix} \mathscr{V}^F \\ \mathscr{B} \end{pmatrix}, \quad \mathscr{W}_0^F = \begin{pmatrix} \mathscr{V}_0^F \\ \mathscr{B}_0 \end{pmatrix}, \quad \widetilde{\mathscr{W}}^F = \begin{pmatrix} \mathscr{W}^F \\ \nabla\Phi \end{pmatrix}, \quad \widetilde{\mathscr{W}}_0^F = \begin{pmatrix} \mathscr{W}_0^F \\ \nabla\Phi_0 \end{pmatrix},$$

where $Q_0 = \ln(\rho_0\theta_0) - \ln(\bar{\rho}\theta_l)$. By (36) and (48), and noticing (41), the perturbation variables $(Q, u, \Theta, \mathscr{B}, \Phi)$ satisfy

$$
\begin{cases}
\partial_t Q + u\cdot\nabla Q + 2\nabla\cdot u + u\cdot\nabla\bar{q} + \frac{1}{\theta}u\cdot((\nabla\times\mathscr{B})\times\mathscr{B}) = -\frac{1}{2\theta}|u|^2 - \frac{\Theta}{\theta},\\
\partial_t u + (u\cdot\nabla)u + \theta\nabla Q + \Theta\nabla\bar{q}\\
\qquad = \frac{1}{\rho}(\nabla\times\mathscr{B})\times\mathscr{B} + \nabla\Phi + \frac{1}{\rho}(\mu\Delta u + (\mu+\lambda)\nabla\mathrm{div}u),\\
\partial_t\Theta + u\cdot\nabla\Theta + \theta\mathrm{div}u + \frac{1}{\rho}u\cdot((\nabla\times\mathscr{B})\times\mathscr{B}) = -\frac{1}{2}|u|^2 - \Theta,\\
\partial_t\mathscr{B} + \nabla\times(\mathscr{B}\times u) = \nu\Delta\mathscr{B}, \quad \mathrm{div}\mathscr{B} = 0,\\
\Delta\Phi = \xi, \quad (t,x)\in(0,\infty)\times\mathbb{R}^3,
\end{cases}
\tag{49}
$$

with the initial condition

$$
\widetilde{\mathscr{W}}^F|_{t=0} = \widetilde{\mathscr{W}}_0^F. \tag{50}
$$

Here ξ is regarded as a function of Q and Θ

$$
\xi = \rho - \bar{\rho} = \frac{p}{\theta} - \frac{\bar{p}}{\bar{\theta}} = \frac{e^q}{\theta} - \frac{e^{\bar{q}}}{\theta_l} = \mathcal{O}(Q) + \mathcal{O}(\Theta). \tag{51}
$$

Next, we denote

$$\mathscr{A}_j^F(u,\theta)=\begin{pmatrix} u_j & 2e_j^T & 0 \\ \theta e_j & u_j\mathbf{I_3} & 0 \\ 0 & \theta e_j^T & u_j \end{pmatrix}, \quad j=1,2,3, \tag{52}$$

$$\mathscr{L}^F(x)=\begin{pmatrix} 0 & (\nabla\bar{q})^T & 0 \\ 0 & 0 & \nabla\bar{q} \\ 0 & 0 & 0 \end{pmatrix}, \tag{53}$$

$$\mathscr{K}(\rho,u,\theta,\mathscr{B},\nabla\Phi)=\begin{pmatrix} -\frac{1}{\theta}u\cdot((\nabla\times\mathscr{B})\times\mathscr{B})-\frac{1}{2\theta}|u|^2-\frac{\Theta}{\theta} \\ \frac{1}{\rho}(\nabla\times\mathscr{B})\times\mathscr{B}+\nabla\Phi+\frac{1}{\rho}(\mu\Delta u+(\mu+\lambda)\nabla\mathrm{div}u) \\ -\frac{1}{\rho}u\cdot((\nabla\times\mathscr{B})\times\mathscr{B})-\frac{1}{2}|u|^2-\Theta \end{pmatrix}. \tag{54}$$

Then the first three equations in (49) can be rewritten as:

$$\partial_t \mathscr{V}^F + \sum_{j=1}^{3} \mathscr{A}_j^F(u,\theta)\,\partial_j V^F + \mathscr{L}^F(x)\,\mathscr{V}^F = \mathscr{K}(\rho, u, \theta, \mathscr{B}, \nabla\Phi)\,. \tag{55}$$

From (37) and (48), the initial condition for (55) is

$$\mathscr{V}^F|_{t=0} = \mathscr{V}_0^F. \tag{56}$$

We denote by $\mathscr{A}_0^F(p,\theta)$ the symmetrizer defined by

$$\mathscr{A}_0^F(p,\theta) = \begin{pmatrix} p & 0 & -\rho \\ 0 & \rho \mathbf{I}_3 & 0 \\ -\rho & 0 & \dfrac{2\rho}{\theta} \end{pmatrix}, \tag{57}$$

which is symmetric and positive definite when $p > 0$ and $\theta > 0$.

It is easy to check that

$$\tilde{\mathscr{A}}_j^F(p,u,\theta) = \mathscr{A}_0^F(p,\theta)\,\mathscr{A}_j^F(u,\theta) = \begin{pmatrix} pu_j & pe_j^T & -\rho u_j \\ pe_j & \rho u_j \mathbf{I}_3 & 0 \\ -\rho u_j & 0 & \dfrac{2\rho}{\theta} u_j \end{pmatrix}$$

is symmetric.

Let us introduce matrix

$$\mathcal{B}(p,u,\theta,x) = \sum_{j=1}^{3} \partial_j \tilde{\mathscr{A}}_j^F(p,u,\theta) - 2\mathscr{A}_0^F(p,\theta)\,\mathscr{L}^F(x)\,.$$

Since

$$\nabla \bar{q} = \frac{1}{\bar{p}} \nabla \bar{p},$$

we have

$$\mathcal{B}(p,u,\theta,x) = \begin{pmatrix} \nabla\cdot(pu) & (\nabla p)^T - \frac{2p}{\bar{p}}(\nabla\bar{p})^T & -\nabla\cdot\left(\frac{pu}{\theta}\right) \\ \nabla p & \nabla\cdot\left(\frac{pu}{\theta}\right)\mathbf{I}_3 & -\frac{2p}{\theta\bar{p}}\nabla\bar{p} \\ -\nabla\cdot\left(\frac{pu}{\theta}\right) & \frac{2p}{\theta\bar{p}}(\nabla\bar{p})^T & 2\nabla\cdot\left(\frac{pu}{\theta^2}\right) \end{pmatrix}, \tag{58}$$

which is antisymmetric at the point $(p,u,\theta) = (\bar{p},0,\theta_l)$.

Let $T > 0$ and $\widetilde{\mathscr{W}}$ be a smooth solution to the Cauchy problem (49)-(50) defined on time interval $[0, T]$. Let

$$\mathscr{W}_T^F = \sup_{0 \leq t \leq T} \|\mathscr{W}^F(t)\|_s.$$

We suppose that $s \geq 3$ and $\mathscr{W}_T^F$ is sufficiently small with respect to T. Then it follows that

$$\frac{1}{2}\bar{\rho} \leq \rho \leq \frac{3}{2}\bar{\rho}, \quad \frac{1}{2}\theta_l \leq \theta \leq \frac{3}{2}\theta_l, \quad \frac{1}{2}\bar{p} \leq p \leq \frac{3}{2}\bar{p}. \tag{59}$$

lemma 5.

For all $t \in [0,T]$, it holds

$$\|\partial_t p\| + \|\partial_t \theta\| \le C\left(\|\nabla u\|_1 + \|\Theta\|_1\right), \tag{60}$$

$$\|\partial_t p\|_{L^\infty} + \|\partial_t \theta\|_{L^\infty} \le C\left(\|\nabla u\|_2 + \|\nabla\Theta\|_1\right), \tag{61}$$

and

$$\left|\langle \mathcal{B}(p,u,\theta,x)\mathscr{V}^F, \mathscr{V}^F\rangle\right| \le C\|\mathscr{V}^F\|_s \mathscr{D}_s^*(\widetilde{\mathscr{W}}^F(t)), \tag{62}$$

where

$$\mathscr{D}_s^*(\widetilde{\mathscr{W}}^F(t)) = \|Q(t)\|_s^2 + \|\nabla u(t)\|_s^2 + \|\Theta(t)\|_s^2. \tag{63}$$

Next, we want to establish an energy estimate of the form

$$\mathscr{E}_s(\widetilde{\mathscr{W}}^F(t)) + \int_0^t \mathscr{D}_s(\widetilde{\mathscr{W}}^F(\tau))d\tau \le C\mathscr{E}_s(\widetilde{\mathscr{W}}^F(0)), \quad t \in [0,T]. \tag{64}$$

where

$$\mathscr{E}_s(\widetilde{\mathscr{W}}^F(t)) = \|Q(t)\|_s^2 + \|u(t)\|_s^2 + \|\Theta(t)\|_s^2 + \|\mathscr{B}(t)\|_s^2 + \|\nabla\Phi(t)\|_s^2, \tag{65}$$

and

$$\mathscr{D}_s(\widetilde{\mathscr{W}}^F(t)) = \mathscr{D}_s^*(\widetilde{\mathscr{W}}^F(t)) + \|\nabla\mathscr{B}(t)\|_s^2. \tag{66}$$

lemma 6. (L^2 estimates for the full MHD system.)

For all $t \in [0,T]$, it holds

$$
\begin{aligned}
&\frac{d}{dt}\left(\left\langle \mathscr{A}_0^F(\rho)\mathscr{V}^F, \mathscr{V}^F\right\rangle + \|\mathscr{B}\|^2 + \|\nabla\Phi\|^2\right)\\
+&2\left(\mu\|\nabla u\|^2 + (\mu+\lambda)\|\mathrm{div} u\|^2\right) + 2\left\langle\frac{\rho}{\theta}, |\Theta|^2\right\rangle + 2\nu\|\nabla\mathscr{B}\|^2 \\
\leq& C\left\|\mathscr{W}^F\right\|_s \mathscr{D}_s(\widetilde{\mathscr{W}}^F(t)).
\end{aligned}
\tag{67}
$$

Lemma 7. (Higher order estimates for the full MHD system).

For all $t \in [0, T]$ and $\alpha \in \mathbb{N}^3$ with $1 \le |\alpha| \le s$, there is a constant $C_0 > 0$ such that

$$\begin{aligned}
&\frac{d}{dt}\left(\left\langle \mathscr{A}_0^F \partial^\alpha \mathscr{V}^F, \partial^\alpha \mathscr{V}^F \right\rangle + \left\|\partial^\alpha \mathscr{B}\right\|^2 + \left\|\partial^\alpha \nabla \Phi\right\|^2\right) \\
&+ C_0\left(\left\|\partial^\alpha \nabla u\right\|^2 + \left\|\partial^\alpha \mathrm{div} u\right\|^2 + \left\|\partial^\alpha \Theta\right\|^2 + \left\|\partial^\alpha \nabla \mathscr{B}\right\|^2\right) \\
\le & C\mathscr{D}^*_{|\alpha|-1}(\widetilde{\mathscr{W}}^F(t)) + C\left\|\mathscr{W}^F\right\|_s \mathscr{D}_s(\widetilde{\mathscr{W}}^F(t)).
\end{aligned} \tag{68}$$

where we have used

$$\|\xi\|_m^2 \le C\left(\|Q\|_m^2 + \|\Theta\|_m^2\right). \tag{69}$$

which is deduced from $\dfrac{1}{\rho} = \dfrac{1}{\bar\rho} - \dfrac{\xi}{\rho\bar\rho}$ and (51).

From Lemma 6 and Lemma 7, we obtain

Proposition 5.

For all $t \in [0,T]$, $\alpha \in \mathbb{N}^3$ with $1 \le |\alpha| \le s$, it holds

$$\begin{aligned}
&\frac{d}{dt}\sum_{\beta\le\alpha}\left(\left\langle \mathscr{A}_0^F \partial^\beta \mathscr{V}^F, \partial^\beta \mathscr{V}^F\right\rangle + \left\|\partial^\beta \mathscr{B}\right\|^2 + \left\|\partial^\beta \nabla\Phi\right\|^2\right) \\
&+ C_0\left(\|\nabla u\|_{|\alpha|}^2 + \|\mathrm{div} u\|_{|\alpha|}^2 + \|\Theta\|_{|\alpha|}^2 + \|\nabla \mathscr{B}\|_{|\alpha|}^2\right) \\
\le & C\mathscr{D}_{|\alpha|-1}^*(\widetilde{\mathscr{W}}^F(t)) + C\left\|\mathscr{W}^F\right\|_s \mathscr{D}_s(\widetilde{\mathscr{W}}^F(t)).
\end{aligned} \tag{70}$$

Dissipation estimates for Q

Estimate (70) contains a recurrence relation on the time dissipation of ∇u and Θ. It is clear that this estimate is not sufficient to complete the proof of (64). The dissipation estimates of Q is necessary.

Proposition 6.

For all $t \in [0, T]$, $\alpha \in \mathbb{N}^3$ with $1 \le |\alpha| \le s$, it holds

$$\begin{aligned}&\frac{d}{dt}\sum_{|\beta|\le|\alpha|-1}\left\langle \partial^\beta u, \partial^\beta \nabla Q\right\rangle \\ &+ C_0 \|Q\|_{|\alpha|}^2 \\ \le & C\mathscr{D}^*_{|\alpha|-1}(\widetilde{\mathscr{W}}^F(t)) + C\left\|\mathscr{W}^F\right\|_s \mathscr{D}_s(\widetilde{\mathscr{W}}^F(t)),\end{aligned} \tag{71}$$

$$\frac{d}{dt}\langle u, \nabla Q\rangle + C_0\|Q\|_1^2 \le C\|\nabla u\|_1^2 + C\|\Theta\|^2 + C\left\|\mathscr{W}^F\right\|_s \mathscr{D}^*_s(\widetilde{\mathscr{W}}^F(t)), \tag{72}$$

and

$$
\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha|\le 1}\left(\left\langle \mathscr{A}_0^F\partial^\alpha\mathscr{V}^F,\partial^\alpha\mathscr{V}^F\right\rangle+\left\|\partial^\alpha\mathscr{B}\right\|^2+\left\|\partial^\alpha\nabla\Phi\right\|^2\right)\\
&+C_0\left(\|\nabla u\|_1^2+\|\Theta\|_1^2+\|\nabla\mathscr{B}\|_1^2\right)\\
\le&\varepsilon_0\left(\|Q\|^2+\|\Theta\|^2\right)+C\|\nabla u\|^2+C\left\|\mathscr{W}^F\right\|_s\mathscr{D}_s^*(\widetilde{\mathscr{W}}^F(t)),
\end{aligned}
\tag{73}
$$

where $\varepsilon_0>0$ is a small constant to be chosen later.

Proposition 7.

There exist positive constants π and C_1 such that, for all $t \in [0, T]$, $\alpha \in \mathbb{N}^3$ with $1 \leq |\alpha| \leq s$, it holds

$$
\begin{aligned}
&\frac{d}{dt}\Big(\sum_{\beta \leq \alpha} \Big(\Big\langle \mathscr{A}_0^F \partial^\beta \mathscr{V}^F, \partial^\beta \mathscr{V}^F \Big\rangle + \left\| \partial^\beta \mathscr{B} \right\|^2 + \left\| \partial^\beta \nabla \Phi \right\|^2 \Big) + \sum_{|\gamma| \leq |\alpha| - 1} \pi \left\langle \partial^\gamma u, \partial^\gamma \nabla Q \right\rangle \Big) \\
&\quad + C_1 \mathscr{D}_{|\alpha|}(\widetilde{\mathscr{W}}^F(t)) \\
\leq & C \mathscr{D}^*_{|\alpha|-1}(\widetilde{\mathscr{W}}^F(t)) + C \left\| \mathscr{W}^F \right\|_s \mathscr{D}_s(\widetilde{\mathscr{W}}^F(t)).
\end{aligned} \tag{74}
$$

We still carry on the induction on $|\alpha|$ for (74) and (72)-(73).

For $|\alpha| = 1$, we multiplies π on both sides of (72).

We observe that the term $C\pi(\|\nabla u\|_1^2 + \|\Theta\|^2)$ can be controlled by $\|\nabla u\|_1^2 + \|\Theta\|_1^2$ on the left-hand side of (73),

and the term $\varepsilon_0(\|Q\|^2 + \|\Theta\|^2)$ on the right-hand side of (73) can be controlled by $\|Q\|_1^2 + \|\Theta\|_1^2$ when $\varepsilon_0 > 0$ is sufficiently small.

Then, there is a constant $a_1 > 0$ such that

$$
\begin{aligned}
& a_1 \frac{d}{dt}\Big(\sum_{|\alpha|\leq 1} (\langle \mathscr{A}_0^F \partial^\alpha \mathscr{V}^F, \partial^\alpha \mathscr{V}^F\rangle + \left\|\partial^\alpha \mathscr{B}\right\|^2 + \|\partial^\alpha \nabla\Phi\|^2) \\
& + \pi \langle u, \nabla Q\rangle \Big) + \mathscr{D}_1(\widetilde{\mathscr{W}}^F(t)) \\
\leq & C\|\nabla u\|^2 + C \left\|\mathscr{W}^F\right\|_s \mathscr{D}_s(\widetilde{\mathscr{W}}^F(t)).
\end{aligned} \tag{75}
$$

In this way, for $|\alpha| \geq 2$, $\mathscr{D}^*_{|\alpha|-1}(\widetilde{\mathscr{W}^F}(t))$ on the right-hand side of (74) can be controlled by $\mathscr{D}_{|\alpha|}(\widetilde{\mathscr{W}^F}(t))$ in the preceding step on the left-hand side of (74) multiplying an appropriate large positive constant. So, we obtain that there are positive constants $a_k > 0 (1 \leq k \leq s)$ such that

$$\frac{d}{dt}\sum_{k=1}^{s} a_k\Big(\sum_{|\alpha|\leq k}(\langle \mathscr{A}_0^F \partial^\alpha \mathscr{V}^F, \partial^\alpha \mathscr{V}^F\rangle + \left\|\partial^\alpha \mathscr{B}\right\|^2 + \|\partial^\alpha \nabla\Phi\|^2 + \pi \sum_{|\gamma|\leq|\alpha|-1}\langle \partial^\gamma u, \partial^\gamma \nabla Q\rangle\Big)$$

$$+ \mathscr{D}_s(\widetilde{\mathscr{W}^F}(t)) \leq C\|\nabla u\|^2 + C\|\mathscr{W}^F\|_s \mathscr{D}_s(\widetilde{\mathscr{W}^F}(t)). \tag{76}$$

By use of formulas (22) and (33) and noting $\|\mathscr{W}^F\|_s$ is small, we have

$$\frac{d}{dt}\sum_{k=1}^{s} a_k\Big(\sum_{|\alpha|\le k}(\langle \mathscr{A}_0^F\partial^\alpha \mathscr{V}^F, \partial^\alpha \mathscr{V}^F\rangle + \left\|\partial^\alpha \mathscr{B}\right\|^2 + \|\partial^\alpha \nabla\Phi\|^2 + \pi \sum_{|\gamma|\le|\alpha|-1}\langle \partial^\gamma u, \partial^\gamma \nabla Q\rangle)\Big)$$
$$+\mathscr{D}_s(\widetilde{\mathscr{W}}^F(t)) \le 0, \tag{77}$$

where the constant $a_k > 0$ may be amended again.

When $\pi > 0$ is small enough,

$$\sum_{k=1}^{s} a_k\Big(\sum_{|\alpha|\le k} (\langle \mathscr{A}_0^F \partial^\alpha \mathscr{V}^F, \partial^\alpha \mathscr{V}^F\rangle + \left\|\partial^\alpha \mathscr{B}\right\|^2 + \|\partial^\alpha \nabla\Phi\|^2 + \pi \sum_{|\gamma|\le|\alpha|-1} \langle \partial^\gamma u, \partial^\gamma \nabla Q\rangle)\Big)$$

is equivalent to

$$\mathscr{E}_s(\widetilde{\mathscr{W}}^F(t)) = \|Q\,(t)\,\|_s^2 + \|u\,(t)\,\|_s^2 + \|\Theta\,(t)\,\|_s^2 + \|\mathscr{B}\,(t)\,\|_s^2 + \|\nabla\Phi\,(t)\,\|_s^2.$$

Integrating (77) from 0 to t, and with the help of (69), we get (43).

Moreover, (43) implies that, for all $|\beta| \leq s-1$ and $|\gamma| \leq s-3$,

$$\partial^\beta(\rho-\bar{\rho}), \partial^\beta(\theta-\theta_l) \in L^2\left(\mathbb{R}^+, L^2(\mathbb{R}^3)\right) \cap W^{1,\infty}\left(\mathbb{R}^+, L^2(\mathbb{R}^3)\right),$$

and

$$\partial^\gamma \nabla u, \partial^\gamma \nabla \mathscr{B} \in L^2\left(\mathbb{R}^+, L^2(\mathbb{R}^3)\right) \cap W^{1,\infty}\left(\mathbb{R}^+, L^2(\mathbb{R}^3)\right),$$

which implies (44)-(45). The proof of Theorem 2 is finished.

Thanks a lot for your attention

谢谢