Vanishing viscosity limit to the planar rarefaction wave for the multi-dimensional compressible Navier-Stokes equations

Lin-an Li

joint work with Professor Dehua Wang and Yi Wang

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Contents

Background

Compressible Navier-Stokes equations(CNS):

$$\begin{cases} \rho_t + \mathrm{div}(\rho \boldsymbol{u}) = 0, \\ (\rho \boldsymbol{u})_t + \mathrm{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = \mathrm{div}\mathcal{T}, \\ (\rho E)_t + \mathrm{div}(\rho E \boldsymbol{u} + p\boldsymbol{u} + \boldsymbol{q}) = \mathrm{div}(\boldsymbol{u}\mathcal{T}), \end{cases} \tag{1}$$

- $t > 0$; $x = (x_1, x_2, \cdots, x_n) \in \Omega \subset \mathbb{R}^n$; $n = 2, 3$;
- $\rho = \rho(t, x) > 0$: density;
- $\boldsymbol{u} = \boldsymbol{u}(t, x) = (u_1, u_2, \cdots, u_n)(t, x)$: velocity;
- $p = p(t, x)$: pressure;
- $E = e + \frac{1}{2}|\boldsymbol{u}|^2$: total energy; $e = e(t, x)$: internal energy;
- $\mathcal{T} = 2\mu_1 \mathbb{D}(\boldsymbol{u}) + \lambda_1 \mathrm{div}\boldsymbol{u}\mathbb{I}$: viscous stress tensor;
- $\mathbb{D}(\boldsymbol{u}) = \frac{\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^\top}{2}$: deformation tensor;
- μ_1 : shear viscosity and λ_1 : bulk viscosity satisfying the physical restrictions
$$\mu_1 > 0, \quad 2\mu_1 + n\lambda_1 \geq 0.$$
- $\boldsymbol{q} = -\kappa_1 \nabla \theta$: heat flux; $\kappa_1 > 0$: heat-conductivity; θ: absolute temperature.

- Polytropic fluids:

$$p = R\rho\theta = A\rho^{\gamma}\exp\left(\frac{\gamma-1}{R}S\right), \quad e = \frac{R}{\gamma-1}\theta + const.,$$

where S: entropy, $\gamma > 1$: adiabatic exponent, $A > 0$ and $R > 0$: fluid constants.

For isentropic fluids, i.e. $S = const.$, CNS (1) reduce to **isentropic CNS:**

$$\begin{cases} \rho_t + \mathrm{div}(\rho \boldsymbol{u}) = 0, \\ (\rho \boldsymbol{u})_t + \mathrm{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p(\rho) = \mu_1 \Delta \boldsymbol{u} + (\mu_1 + \lambda_1)\nabla \mathrm{div} \boldsymbol{u}, \end{cases} \tag{2}$$

- $p = p(\rho) = A\rho^{\gamma}$: pressure; $\gamma > 1$: adiabatic exponent, $A > 0$: fluid constants.

Non-isentropic CNS (1) and isentropic CNS (2) are both **viscous conservation laws**:

$$\mathbf{U}_t + \mathrm{div}\mathbf{F}(\mathbf{U}) = \varepsilon \mathrm{div}(\mathbf{B}(\mathbf{U})\nabla \mathbf{U}).$$

Formally, as $\varepsilon \to 0$, converge to **hyperbolic conservation laws**:

$$\mathbf{U}_t + \mathrm{div}\mathbf{F}(\mathbf{U}) = 0.$$

Main feature of **hyperbolic conservation laws**:

- Formation of shock no matter how smooth and small the initial values are!

Riemann problem:(three basic wave patterns)

- Shock wave (genuinely nonlinear characteristic field);
- Rarefaction wave (genuinely nonlinear characteristic field);
- Contact discontinuity (linearly degenerate characteristic field).

Riemann (1860):

$$\begin{cases} \rho_t + (\rho u_1)_{x_1} = 0, \\ (\rho u_1)_t + (\rho u_1^2 + p(\rho))_{x_1} = 0, \\ p(\rho) = A\rho^\gamma. \end{cases}$$

$$(\rho_0, u_{10})(x_1) = \begin{cases} (\rho_-, u_{1-}), & x_1 < 0, \\ (\rho_+, u_{1+}), & x_1 > 0. \end{cases}$$

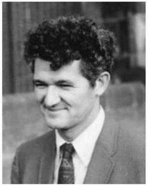
Georg Friedrich Bernhard Riemann
(1826 – 1866)

Lax (1957):

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_{x_1} = 0,$$

$$\mathbf{U}_0(x_1) = \begin{cases} \mathbf{U}_-, & x_1 < 0, \\ \mathbf{U}_+, & x_1 > 0. \end{cases}$$

Mathematical Theory of Conservation Laws. Commun. Pure Appl. Math. **10**, (1957).



Peter David Lax (1926--)

A simple example:

Riemann problem for inviscid Burgers equation:

$$\begin{cases} w_t + (\frac{1}{2}w^2)_{x_1} = 0, \\ w(0,x_1) = w_0(x_1) = \begin{cases} w_-, & x_1 < 0, \\ w_+, & x_1 > 0. \end{cases} \end{cases}$$

- $w_- < w_+$, Rarefaction wave:

$$w^r(\frac{x_1}{t}) = \begin{cases} w_-, & x_1 < w_- t, \\ \frac{x_1}{t}, & w_- t \le x_1 \le w_+ t, \\ w_+, & x_1 > w_+ t. \end{cases}$$

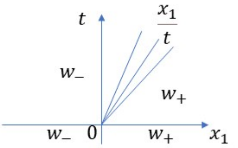


Figure 1. Rarefaction wave w^r

- $w_- > w_+$, Shock wave:

$$w^s(t,x_1) = \begin{cases} w_-, & x_1 < st, \\ w_+, & x_1 > st. \end{cases}$$

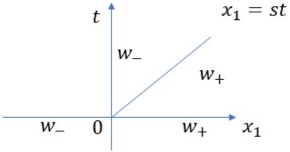


Figure 2. Shock wave w^s

R-H condition: $s[\mathbf{U}] = [\mathbf{F}(\mathbf{U})], s = \frac{w_+ + w_-}{2}$.

Burgers equation:

$$\begin{cases} w_t + (\frac{1}{2}w^2)_{x_1} = \varepsilon w_{x_1 x_1}, \\ w(0,x_1) = w_0^\varepsilon(x_1) \to \begin{cases} w_-, & x_1 \to -\infty, \\ w_+, & x_1 \to +\infty. \end{cases} \end{cases}$$

- $w_- < w_+$,

$$w^\varepsilon(t,x_1) \to w^r(\frac{x_1}{t}), \quad \text{as} \quad \varepsilon \to 0+, \quad \text{or} \quad \varepsilon = 1, t \to +\infty.$$

- $w_- > w_+$,

$$w^\varepsilon(t,x_1) \to w^s(t,x_1), \ \text{as} \ \varepsilon \to 0+, \ \text{away from the discontinuity}.$$

$$w^\varepsilon(t,x_1) \to w^s(x_1 - st), \quad \text{as} \quad \varepsilon = 1, t \to +\infty.$$

Travelling wave $w^s(x_1 - st)$, Anti-derivative, Zero mass condition $\cdots$

Hopf(CPAM, 1950), Il'in-Oleinik(1960) $\cdots$

Relation between **viscous conservation laws** and **hyperbolic conservation laws**:

- Vanishing viscosity limit ($\varepsilon \to 0+$):
  (1D: Hopf, Goodman-Xin, Bressan-Bianchini, Huang-Wang-Wang-Yang, $\cdots$)
- Large-time behavior($\varepsilon = 1, t \to +\infty$):
  (1D: Il'in-Oleinik, Goodman, Huang, Matsumura-Nishihara, Liu, Xin, Zhu, Zhao, $\cdots$)

Determined by Riemann solution to the corresponding inviscid hyperbolic conservation laws.

Multi-D?

cf. Time-asymptotic stability of planar rarefaction wave

- For scalar case: Xin (1990), Ito (1996), Nishikawa-Nishihara (2000).
- For artificial 2×2 system with uniformly positive viscosities: Hokari-Matsumura (1997).
- For relaxation approximation of conservation laws: Luo (JDE, 1997), Zhao (JDE, 2000).
- For radiating gas model: Gao-Ruan-Zhu (JDE, 2008).
- For 2D/3D CNS: Li-Wang (SIAM, 2018), Li-Wang-Wang (ARMA, 2018)

Goal: Vanishing viscosity limit to the planar rarefaction wave for the multi-dimensional(MD) CNS (2) and (1).
Take dissipation coefficients

$$\mu_1 = \mu\varepsilon, \quad \lambda_1 = \lambda\varepsilon, \quad \kappa_1 = \kappa\varepsilon,$$

where $\varepsilon > 0$: vanishing dissipation parameter, μ, λ and κ: the given uniform-in-ε constants.

Initial data:

- 2D isentropic CNS (2):

$$(\rho, \boldsymbol{u})(0, x) = (\rho_0, \boldsymbol{u}_0)(x) \to (\rho_\pm, \boldsymbol{u}_\pm), \quad \text{as } x_1 \to \pm\infty, \tag{3}$$

where $\rho_0 > 0$, $\boldsymbol{u}_0 := (u_{10}, u_{20})$, $\boldsymbol{u}_\pm := (u_{1\pm}, 0)$ and $\rho_\pm > 0$, $u_{1\pm}$ are prescribed constant states.

- 3D non-isentropic CNS (1):

$$(\rho, \boldsymbol{u}, \theta)(0, x) = (\rho_0, \boldsymbol{u}_0, \theta_0)(x) \to (\rho_\pm, \boldsymbol{u}_\pm, \theta_\pm), \quad \text{as } x_1 \to \pm\infty, \tag{4}$$

where $\rho_0, \theta_0 > 0$, $\boldsymbol{u}_0 := (u_{10}, u_{20}, u_{30})$, $\boldsymbol{u}_\pm := (u_{1\pm}, 0, 0)$ and $\rho_\pm$, $\theta_\pm > 0$, $u_{1\pm}$ are prescribed constant states.

The vanishing viscosity limit of CNS is expected to be determined by the Riemann problem to the corresponding **compressible Euler equations**:

- The Riemann problem to the isentropic compressible Euler equations

$$\begin{cases} \rho_t + \mathrm{div}(\rho \boldsymbol{u}) = 0, \\ (\rho \boldsymbol{u})_t + \mathrm{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p(\rho) = 0, \end{cases} \tag{5}$$

$$(\rho_0^r, u_{10}^r, u_{20}^r)(x_1) = \begin{cases} (\rho_-, u_{1-}, 0), & x_1 < 0, \\ (\rho_+, u_{1+}, 0), & x_1 > 0. \end{cases} \tag{6}$$

- The Riemann problem to the full compressible Euler equations

$$\begin{cases} \rho_t + \mathrm{div}(\rho \boldsymbol{u}) = 0, \\ (\rho \boldsymbol{u})_t + \mathrm{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = 0, \\ (\rho E)_t + \mathrm{div}(\rho E \boldsymbol{u} + p \boldsymbol{u}) = 0. \end{cases} \tag{7}$$

$$(\rho, \boldsymbol{u}, \theta)(0, x) = (\rho_0^r, \boldsymbol{u}_0^r, \theta_0^r)(x_1) = \begin{cases} (\rho_-, \boldsymbol{u}_-, \theta_-), & x_1 < 0, \\ (\rho_+, \boldsymbol{u}_+, \theta_+), & x_1 > 0. \end{cases} \tag{8}$$

The above MD Riemann problem (5)-(6) or (7)-(8) has the essential differences from the corresponding 1D Riemann problem:

- 1D Riemann problem to the isentropic compressible Euler equations

$$\begin{cases} \rho_t + (\rho u_1)_{x_1} = 0, & x_1 \in \mathbb{R},\ t > 0, \\ (\rho u_1)_t + (\rho u_1^2 + p(\rho))_{x_1} = 0, \end{cases} \tag{9}$$

$$(\rho_0^r, u_{10}^r)(x_1) = \begin{cases} (\rho_-, u_{1-}), & x_1 < 0, \\ (\rho_+, u_{1+}), & x_1 > 0. \end{cases} \tag{10}$$

- 1D Riemann problem to the full compressible Euler equations

$$\begin{cases} \rho_t + (\rho u_1)_{x_1} = 0, & x_1 \in \mathbb{R},\ t > 0, \\ (\rho u_1)_t + (\rho u_1^2 + p)_{x_1} = 0, \\ (\rho E)_t + (\rho E u_1 + p u_1)_{x_1} = 0, \end{cases} \tag{11}$$

$$(\rho, u_1, \theta)(0, x_1) = (\rho_0^r, u_{10}^r, \theta_0^r)(x_1) = \begin{cases} (\rho_-, u_{1-}, \theta_-), & x_1 < 0, \\ (\rho_+, u_{1+}, \theta_+), & x_1 > 0. \end{cases} \tag{12}$$

1D Riemann problem:

The weak solution to 1D Riemann problem is not unique (infinitely many).
With the additional entropy condition:

$$(\rho\frac{|u_1|^2}{2}+\rho\mathcal{E}(\rho))_t+((\rho\frac{|u_1|^2}{2}+\rho\mathcal{E}(\rho)+p(\rho))u_1)_{x_1}\le 0,$$

where the internal energy $\mathcal{E}$ is given by

$$p(r)=r^2\mathcal{E}'(r),$$

then the weak solution to the 1D Riemann problem is unique in $BV_{\rm loc}$ or L^∞.

MD Riemann problem:

With the additional entropy condition:

$$\left[\rho\frac{|\mathbf{u}|^2}{2}+\rho\mathcal{E}(\rho)\right]_t+\mathrm{div}\left[\left(\rho\frac{|\mathbf{u}|^2}{2}+\rho\mathcal{E}(\rho)+p(\rho)\right)\mathbf{u}\right]\leq 0,$$

- Shock wave case
  Exist infinitely many bounded admissible weak solutions:
  Chiodaroli - De Lellis - Kreml (CPAM, 2015);
  Chiodaroli - Kreml (Nonlinearity., 2018).
  Global instability:
  Lai - Xiang - Zhou (2018).
- Shock + contact discontinuity or rarefaction case
  Exist infinitely many bounded admissible weak solutions:
  Klingenberg - Markfelder (ARMA, 2018);
  Brezina - Chiodaroli - Kreml (Electron. J. Differential Equations., 2018).

MD Riemann problem:

- Rarefaction wave case
  Unique even the rarefaction wave connected with vacuum states:
  Chen - Chen (J. Hyperbolic Differ. Equ., 2007);
  Feireisl - Kreml (J. Hyperbolic Differ. Equ., 2015);
  Feireisl - Kreml - Vasseur (SIAM J. Math. Anal., 2015).

Some known results on vanishing viscosity limit of CNS(1D):

- Viscous Burgers equation:
  - general initial data: Hopf (CPAM, 1950);
- Conservation laws system with uniformly artificial viscosity:
  - piecewise smooth shock: Goodman - Xin (ARMA, 1992);
  - piecewise smooth shock with initial layer: Yu (ARMA, 1999);
  - general small BV data: Bianchini - Bressan (Ann. of Math., 2005);
- Isentropic CNS:
  - piecewise constant shock with initial layer: Hoff - Liu(Indiana Univ. Math. J., 1989);
  - rarefaction: Xin(CPAM, 1993), Huang - Li - Wang(SIAM, 2012);
  - piecewise smooth shock: Wang(J. Math. Anal. Appl., 2004).

Some known results on vanishing viscosity limit of CNS(1D):

- Non-isentropic CNS:
  - rarefaction:
    Jiang - Ni - Sun(SIAM, 2006), Xin - Zeng(JDE, 2010), Li - Wang(Commun. Math. Sci., 2014);
  - shock:
    Wang(Acta Math. Sci. Ser. B., 2008);
  - contact discontinuity:
    Ma(JDE, 2010);
  - rarefaction wave + contact discontinuity:
    Huang - Wang - Yang(KRM, 2010), Huang - Jiang - Wang(CIS, 2013);
  - rarefaction wave + shock wave:
    Huang - Wang - Yang(ARMA, 2012);
  - rarefaction wave + shock wave + contact discontinuity:
    Huang - Wang - Wang - Yang(SIAM, 2013).

2D isentropic CNS case

Goal:

Vanishing viscosity limit to the planar rarefaction wave for the 2D isentropic CNS on $\mathbb{R} \times \mathbb{T}$.

The planar 2-rarefaction wave $(\rho^r, u_1^r, 0)(\frac{x_1}{t})$ for Riemann problem of 2D Euler (5)-(6).

Where $(\rho^r, u_1^r)(\frac{x_1}{t})$ is the 2-rarefaction wave solution for the Riemann problem of 1D Euler (9)-(10), satisfying

$$u_{1+} - \int_{\rho_-}^{\rho_+} \frac{\sqrt{p'(s)}}{s} ds = u_{1-}, \quad u_{1+} > u_{1-}.$$

Theorem 1: (Lin-an Li, Dehua Wang, Yi Wang, Comm. Math. Phys. (2020))

Let $(\rho^r, u_1^r, 0)(\frac{x_1}{t})$ be the planar 2-rarefaction wave to the 2D Euler system (5), $T > 0$ be any arbitrarily large but fixed time. Then there exists a positive constant ε_0 such that for any $\varepsilon \in (0, \varepsilon_0)$, we can construct a family of smooth solutions $(\rho^\varepsilon, \boldsymbol{u}^\varepsilon) = (\rho^\varepsilon, u_1^\varepsilon, u_2^\varepsilon)(t, x_1, x_2)$ up to time T with the initial value (23) to the CNS (2) satisfying

$$\begin{cases} (\rho^\varepsilon - \rho^r, u_1^\varepsilon - u_1^r, u_2^\varepsilon) \in C^0(0, T; L^2(\mathbb{R} \times \mathbb{T})), \\ (\nabla \rho^\varepsilon, \nabla \mathbf{u}^\varepsilon) \in C^0(0, T; H^1(\mathbb{R} \times \mathbb{T})), \\ \nabla^3 \mathbf{u}^\varepsilon \in L^2(0, T; L^2(\mathbb{R} \times \mathbb{T})). \end{cases}$$

Moreover, for any small positive constant h, there exists a constant $C_{h,T}$ independent of ε, such that:

$$\sup_{h \le t \le T} \|(\rho^\varepsilon, u_1^\varepsilon, u_2^\varepsilon)(t, x_1, x_2) - (\rho^r, u_1^r, 0)(\frac{x_1}{t})\|_{L^\infty(\mathbb{R} \times \mathbb{T})} \le C_{h,T} \varepsilon^{\frac{2}{9}} |\ln \varepsilon|.$$

As $\varepsilon \to 0+$,

$$(\rho^\varepsilon, \mathbf{u}^\varepsilon)(t, x_1, x_2) \to (\rho^r, u_1^r, 0)(\frac{x_1}{t}), \quad \text{a.e. in } \mathbb{R}^+ \times \mathbb{R} \times \mathbb{T}.$$

Proof of the 2D isentropic case:

Since the rarefaction wave is only Lipschitz continuous, we shall construct a smooth approximate rarefaction wave solution to the Euler system (9) through the Burgers equation.

Consider the Riemann problem for the inviscid Burgers equation:

$$\begin{cases} w_t + ww_{x_1} = 0, \\ w(0,x_1) = w_0^r(x_1) = \begin{cases} w_-, & x_1 < 0, \\ w_+, & x_1 > 0. \end{cases} \end{cases} \tag{13}$$

If $w_- < w_+$, then (13) has the self-similar rarefaction wave fan:

$$w^r(t,x_1) = w^r(\frac{x_1}{t}) = \begin{cases} w_-, & x_1 < w_- t, \\ \frac{x_1}{t}, & w_- t \le x_1 \le w_+ t, \\ w_+, & x_1 > w_+ t. \end{cases} \tag{14}$$

The smooth rarefaction wave solution of the Burgers equation:

$$\begin{cases} w_t + ww_{x_1} = 0, \\ w(0,x_1) = w_0(x_1) = \dfrac{w_+ + w_-}{2} + \dfrac{w_+ - w_-}{2}\tanh\dfrac{x_1}{\delta}, \end{cases} \tag{15}$$

where $\delta = \delta(\varepsilon) > 0$. In fact, we take $\delta = \varepsilon^{\frac{2}{9}}$ finally.

Lemma 1: Suppose $w_+ > w_-$.Then the problem (15) has a unique smooth global solution $w(t, x_1)$ such that
(1) $w_- < w(t, x_1) < w_+,\ w_{x_1} > 0$ for $x_1 \in \mathbb{R}$ and $t \geq 0, \delta > 0$.
(2) The following estimates hold for all $t > 0, \delta > 0$ and $p \in [1, +\infty]$:

$$\begin{aligned}
&\|w_{x_1}(t,\cdot)\|_{L^p(\mathbb{R})} \leq C(\delta + t)^{-1+1/p},\\
&\|w_{x_1x_1}(t,\cdot)\|_{L^p(\mathbb{R})} \leq C(\delta + t)^{-1}\delta^{-1+1/p},\\
&\|w_{x_1x_1x_1}(t,\cdot)\|_{L^p(\mathbb{R})} \leq C(\delta + t)^{-1}\delta^{-2+1/p},\\
&|w_{x_1x_1}(t,x_1)| \leq \frac{4}{\delta} w_{x_1}(t,x_1).
\end{aligned}$$

(3) There exists a constant $\delta_0 \in (0,1)$ such that for $\delta \in (0, \delta_0]$ and $t > 0$,

$$\|w(t,\cdot) - w^r(\frac{\cdot}{t})\|_{L^\infty(\mathbb{R})} \leq C\delta t^{-1}[\ln(1+t) + |\ln\delta|].$$

Approximate rarefaction wave for the Euler system (9)-(10):

Set $w_\pm = \lambda_2(\rho_\pm, u_{1\pm})$.
The 2-rarefaction wave $(\rho^r, u_1^r)(t, x_1) = (\rho^r, u_1^r)(x_1/t)$:

$$\lambda_2(\rho^r, u_1^r)(t, x_1) = w^r(t, x_1),$$
$$z_2(\rho^r, u_1^r)(t, x_1) = z_2(\rho_\pm, u_{1\pm}),$$

where z_2 is 2-Riemann invariant.
Smooth approximate rarefaction wave $(\bar{\rho}, \bar{u}_1)(t, x_1)$:

$$\lambda_2(\bar{\rho}, \bar{u}_1)(t, x_1) = w(t, x_1),$$
$$z_2(\bar{\rho}, \bar{u}_1)(t, x_1) = z_2(\rho_\pm, u_{1\pm}).$$

$(\bar{\rho}, \bar{u}_1)$ satisfies the system:

$$\begin{cases} \bar{\rho}_t + (\bar{\rho}\bar{u}_1)_{x_1} = 0, \\ (\bar{\rho}\bar{u}_1)_t + (\bar{\rho}\bar{u}_1^2 + p(\bar{\rho}))_{x_1} = 0, \\ (\bar{\rho}, \bar{u}_1)(0, x_1) := (\bar{\rho}_0, \bar{u}_{10})(x_1). \end{cases}$$

Lemma 2: The smooth approximate 2-rarefaction wave $(\bar{\rho}, \bar{u}_1)$ satisfies the following properties:
(1) $\bar{u}_{1x_1} = \frac{2}{\gamma+1} w_{x_1} > 0$ for all $x_1 \in \mathbb{R}$ and $t \geq 0, \bar{\rho}_{x_1} = \frac{1}{\sqrt{A\gamma}} \bar{\rho}^{\frac{3-\gamma}{2}} \bar{u}_{1x_1} > 0$, and $\bar{\rho}_{x_1x_1} = \frac{1}{\sqrt{A\gamma}} \bar{\rho}^{\frac{3-\gamma}{2}} \bar{u}_{1x_1x_1} + \frac{3-\gamma}{2A\gamma} \bar{\rho}^{2-\gamma} (\bar{u}_{1x_1})^2$.
(2) The following estimates hold for all $t \geq 0, \delta > 0$ and $p \in [1, +\infty]$:

$$\|(\bar{\rho}_{x_1}, \bar{u}_{1x_1})\|_{L^p(\mathbb{R})} \leq C(\delta + t)^{-1+1/p},$$
$$\|(\bar{\rho}_{x_1x_1}, \bar{u}_{1x_1x_1})\|_{L^p(\mathbb{R})} \leq C(\delta + t)^{-1}\delta^{-1+1/p},$$
$$\|(\bar{\rho}_{x_1x_1x_1}, \bar{u}_{1x_1x_1x_1})\|_{L^p(\mathbb{R})} \leq C(\delta + t)^{-1}\delta^{-2+1/p}.$$

(3) There exists a constant $\delta_0 \in (0, 1)$ such that for $\delta \in (0, \delta_0]$ and $t > 0$,

$$\|(\bar{\rho}, \bar{u}_1)(t, \cdot) - (\rho^r, u_1^r)(\frac{\cdot}{t})\|_{L^\infty(\mathbb{R})} \leq C\delta t^{-1}[\ln(1 + t) + |\ln \delta|].$$

Main Difficulty: Error terms $(2\mu + \lambda)\bar{u}_{1x_1x_1}$

- Sufficient time-decay in 1D time-asymptotic stability: Matsumura-Nishihara(JJM 1986, CMP 1992), Liu-Xin(CMP 1988), Nishihara-Yang-Zhao(SIAM 2004)
- Sufficient time-decay in 2D/3D time-asymptotic stability: Li-Wang (SIAM 2018), Li-Wang-Wang (ARMA 2018), Wang-Wang (KRM, 2019)
- Enough decay rate with respect to the viscosity in 1D vanishing viscosity limit:
  Xin (CPAM 1993), Jiang-Ni-Sun (SIAM 2006)
- NOT enough decay rate with respect to the viscosity in 2D vanishing viscosity limit for the scaled variables.

Key Observations (Hyperbolic Wave):

Introduce the hyperbolic wave $(d_1, d_2)(t, x_1)$ to recover the physical viscosities for inviscid approximate rarefaction wave profile.(Motivated by Huang-Wang-Yang, ARMA, 2012)

$$\begin{cases} d_{1t} + d_{2x_1} = 0, \\ d_{2t} + \left(-\frac{\bar{m}_1^2}{\bar{\rho}^2} d_1 + p'(\bar{\rho}) d_1 + \frac{2\bar{m}_1}{\bar{\rho}} d_2 \right)_{x_1} = (2\mu + \lambda)\varepsilon \bar{u}_{1x_1x_1}, \\ (d_1, d_2)(0, x_1) = (0, 0), \end{cases} \tag{16}$$

where $\bar{m}_1 := \bar{\rho}\bar{u}_1$ is the momentum of the approximate rarefaction wave.

We shall solve this linear hyperbolic system (16) on $[0, T]$.
Diagonalize the above system. Rewrite the system (16) as

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}_t + \left(\bar{A} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \right)_{x_1} = \begin{bmatrix} 0 \\ (2\mu+\lambda)\varepsilon \bar{u}_{1x_1x_1} \end{bmatrix},$$

where

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ -\frac{\bar{m}_1^2}{\bar{\rho}^2} + p'(\bar{\rho}) & \frac{2\bar{m}_1}{\bar{\rho}} \end{bmatrix}$$

with two distinct eigenvalues

$$\bar{\lambda}_1(\bar{\rho}, \bar{m}_1) = \frac{\bar{m}_1}{\bar{\rho}} - \sqrt{p'(\bar{\rho})}, \quad \bar{\lambda}_2(\bar{\rho}, \bar{m}_1) = \frac{\bar{m}_1}{\bar{\rho}} + \sqrt{p'(\bar{\rho})},$$

the corresponding left and right eigenvectors $\bar{l}_i, \bar{r}_i (i = 1, 2)$ satisfying

$$\bar{L}\bar{A}\bar{R} = diag(\bar{\lambda}_1, \bar{\lambda}_2) := \bar{\Lambda}, \quad \bar{L}\bar{R} = I,$$

where $\bar{L} = (\bar{l}_1, \bar{l}_2)^\top, \bar{R} = (\bar{r}_1, \bar{r}_2)$ and I is 2×2 identity matrix.

Set

$$(D_1, D_2)^\top = \bar{L}(d_1, d_2)^\top,$$

then

$$(d_1, d_2)^\top = \bar{R}(D_1, D_2)^\top,$$

and (D_1, D_2) satisfies the diagonalized system

$$\begin{bmatrix} D_1 \\ D_2 \end{bmatrix}_t + \left(\bar{\Lambda}\begin{bmatrix} D_1 \\ D_2 \end{bmatrix}\right)_{x_1} = (\bar{L}_t\bar{R} + \bar{L}_{x_1}\bar{A}\bar{R})\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} + \bar{L}\begin{bmatrix} 0 \\ (2\mu+\lambda)\varepsilon\bar{u}_{1x_1x_1} \end{bmatrix}. \tag{17}$$

Since the 2-Riemann invariant z_2 is constant along the approximate 2-rarefaction wave curve, we have the structure relation

$$\bar{L}_t = -\bar{\lambda}_2\bar{L}_{x_1}, \tag{18}$$

Substituting (18) into (17), we obtain the diagonalized system

$$\begin{cases} D_{1t} + (\bar{\lambda}_1 D_1)_{x_1} = \dfrac{\sqrt{2}}{2}(2\mu+\lambda)\varepsilon \bar{u}_{1x_1x_1} + (a_{11}(\bar{\rho})\bar{\rho}_{x_1} + a_{12}(\bar{\rho})\bar{u}_{1x_1})D_1, \\ D_{2t} + (\bar{\lambda}_2 D_2)_{x_1} = \dfrac{\sqrt{2}}{2}(2\mu+\lambda)\varepsilon \bar{u}_{1x_1x_1} + (a_{21}(\bar{\rho})\bar{\rho}_{x_1} + a_{22}(\bar{\rho})\bar{u}_{1x_1})D_1, \\ (D_1, D_2)(0, x_1) = (0, 0). \end{cases} \tag{19}$$

In the diagonalized system (19), the equation of D_1 is decoupled with D_2 due to the rarefaction wave structure of the system as in (18).

Lemma 3: There exists a positive constant C_T independent of δ and ε, such that

$$\|\frac{\partial^k}{\partial x_1^k}(d_1, d_2)(t,\cdot)\|^2_{L^2(\mathbb{R})} \le C_T(\frac{\varepsilon}{\delta^{k+1}})^2, \quad k = 0, 1, 2, 3.$$

Sketch of the Proof:

- Fully using the expanding properties of 2-rarefaction wave $(\bar{\lambda}_{2x_1} > 0)$ and the decoupling structure of the diagonalized system.
- Choosing the suitable weight function.

Sketch of the Proof on Lemma 3:

Multiplying the second equation of (19) by D_2 and integrating the resulting equation over $[0,t]\times\mathbb{R}$ with $t\in(0,T)$ imply

$$\int_{\mathbb{R}} D_2^2(t,x_1)dx_1+\int_0^t\int_{\mathbb{R}}\bar{u}_{1x_1}D_2^2dx_1dt_1\le C_T(\frac{\varepsilon}{\delta})^2+C_T\int_0^t\int_{\mathbb{R}}\bar{u}_{1x_1}D_1^2dx_1dt_1. \tag{20}$$

Multiplying the first equation of (19) by $\bar{\rho}^N D_1$ with N a sufficiently large positive constant to be determined, and integrate the resulting equation over $[0,t]\times\mathbb{R}$ with $t\in(0,T)$ to get

$$\begin{aligned}&\int_{\mathbb{R}}\bar{\rho}^N\frac{D_1^2}{2}(t,x_1)dx_1+\int_0^t\int_{\mathbb{R}}N\bar{\rho}^N\bar{u}_{1x_1}D_1^2dx_1dt_1\\ &\le C\int_0^t\int_{\mathbb{R}}\bar{\rho}^N D_1^2dx_1dt_1+C\varepsilon^2\int_0^t\int_{\mathbb{R}}\bar{u}_{1x_1x_1}^2dx_1dt_1+C\int_0^t\int_{\mathbb{R}}\bar{\rho}^N\bar{u}_{1x_1}D_1^2dx_1dt_1\\ &\le C\int_0^t\int_{\mathbb{R}}\bar{\rho}^N D_1^2dx_1dt_1+C(\frac{\varepsilon}{\delta})^2+C\int_0^t\int_{\mathbb{R}}\bar{\rho}^N\bar{u}_{1x_1}D_1^2dx_1dt_1.\end{aligned}$$

Choosing N large enough and using Gronwall's inequality give

$$\int_{\mathbb{R}} D_1^2(t,x_1)dx_1 + \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1} D_1^2 dx_1 dt_1 \leq C_T(\frac{\varepsilon}{\delta})^2. \tag{21}$$

Combining (20) and (21), we can get

$$\int_{\mathbb{R}} (D_1^2 + D_2^2)(t,x_1)dx_1 + \int_0^t \int_{\mathbb{R}} \bar{u}_{1x_1}(D_1^2 + D_2^2)dx_1 dt_1 \leq C_T(\frac{\varepsilon}{\delta})^2.$$

This completes the proof of case $k = 0$ in Lemma 3. The other cases $k = 1, 2, 3$ can be considered similarly to the differentiated system with respect to x_1 by k times.

Approximate Solution Profile:

Define

$$\tilde{\rho}(t,x_1) = (\bar{\rho} + d_1)(t,x_1), \quad \tilde{m}_1(t,x_1) = (\bar{m}_1 + d_2)(t,x_1) := \tilde{\rho}\tilde{u}_1(t,x_1).$$

Then the approximate wave profile $(\tilde{\rho}, \tilde{u}_1)$ satisfies the system

$$\begin{cases} \tilde{\rho}_t + (\tilde{\rho}\tilde{u}_1)_{x_1} = 0, \\ (\tilde{\rho}\tilde{u}_1)_t + (\tilde{\rho}\tilde{u}_1^2 + p(\tilde{\rho}))_{x_1} = (2\mu + \lambda)\varepsilon \bar{u}_{1x_1x_1} + Q \end{cases} \tag{22}$$

with the initial data

$$(\tilde{\rho}, \tilde{u}_1)(0,x_1) = (\bar{\rho}_0, \bar{u}_{10})(x_1).$$

Where Q is the error terms

$$\begin{aligned} Q &= (\tilde{\rho}\tilde{u}_1^2 - \bar{\rho}\bar{u}_1^2 + \bar{u}_1^2 d_1 - 2\bar{u}_1 d_2)_{x_1} + (p(\tilde{\rho}) - p(\bar{\rho}) - p'(\bar{\rho})d_1)_{x_1} \\ &= O(1)\Big[|(d_1,d_2)||(d_{1x_1}, d_{2x_1})| + |(\bar{\rho}_{x_1}, \bar{u}_{x_1})||(d_1,d_2)|\Big] \end{aligned}$$

Reformulation of the Problem:

Set the perturbation around the approximate wave profile $(\tilde{\rho}, \tilde{u}_1, 0)(t, x_1)$:

$$\phi(t, x_1, x_2) := \rho^\varepsilon(t, x_1, x_2) - \tilde{\rho}(t, x_1),$$
$$\Psi(t, x_1, x_2) = (\psi_1, \psi_2)^\top(t, x_1, x_2) := (u_1^\varepsilon, u_2^\varepsilon)^\top(t, x_1, x_2) - (\tilde{u}_1, 0)^\top(t, x_1),$$

with $(\rho^\varepsilon, u_1^\varepsilon, u_2^\varepsilon)$ being the solution to the problem (2) with the following initial data:

$$(\rho^\varepsilon, u_1^\varepsilon, u_2^\varepsilon)(0, x_1, x_2) := (\bar{\rho}_0, \bar{u}_{10}, 0)(x_1) + (\phi_0, \psi_{10}, \psi_{20})(x_1, x_2). \quad (23)$$

Introducing a scaling for the independent variables

$$\tau = \frac{t}{\varepsilon}, \quad y_1 = \frac{x_1}{\varepsilon}, \quad y_2 = \frac{x_2}{\varepsilon}.$$

The superscription of $(\rho^\varepsilon, u_1^\varepsilon, u_2^\varepsilon)$ will be omitted as (ρ, u_1, u_2). We still use the notations $(\rho, u_1, u_2)(\tau, y_1, y_2), (\tilde{\rho}, \tilde{u}_1)(\tau, y_1), (\bar{\rho}, \bar{u}_1)(\tau, y_1)$ and $(\phi, \Psi)(\tau, y_1, y_2)$ in the scaled independent variables.

The system for the perturbation (ϕ, Ψ) :

$$\begin{cases} \phi_\tau + \boldsymbol{u} \cdot \nabla\phi + \rho \mathrm{div}\Psi + \tilde{\rho}_{y_1}\psi_1 + \tilde{u}_{1y_1}\phi = 0, \\ \rho\Psi_\tau + \rho\boldsymbol{u} \cdot \nabla\Psi + p'(\rho)\nabla\phi + (\rho\tilde{u}_{1y_1}\psi_1, 0)^\top + ((p'(\rho) - \frac{\rho}{\tilde{\rho}}p'(\tilde{\rho}))\tilde{\rho}_{y_1}, 0)^\top \\ = \mu\Delta\Psi + (\mu+\lambda)\nabla\mathrm{div}\Psi + ((2\mu+\lambda)(\frac{-d_1\bar{u}_1+d_2}{\tilde{\rho}})_{y_1y_1}, 0)^\top \\ \quad -((2\mu+\lambda)\frac{\bar{u}_{1y_1y_1}}{\tilde{\rho}}\phi, 0)^\top - (\frac{\rho}{\tilde{\rho}}Q, 0)^\top, \end{cases} \tag{24}$$

$$(\phi, \Psi)(0, y_1, y_2) = (\phi, \psi_1, \psi_2)(0, y_1, y_2) = (\phi_0, \psi_{10}, \psi_{20})(y_1, y_2), \tag{25}$$

where the initial perturbation is chosen to satisfy

$$\|(\phi_0, \psi_{10}, \psi_{20})(y_1, y_2)\|_{H^2(\mathbb{R}\times\mathbb{T}_\varepsilon)} = O(\varepsilon^{\frac{2}{9}}).$$

Functional space $X(0, \frac{T}{\varepsilon})$. Where for $0 \le \tau_1 \le \frac{T}{\varepsilon}$, we define

$$X(0,\tau_1) \\ = \left\{(\phi,\Psi) \middle| (\phi,\Psi) \in C^0(0,\tau_1;H^2), \nabla\phi \in L^2(0,\tau_1;H^1), \nabla\Psi \in L^2(0,\tau_1;H^2)\right\}.$$

Proposition 1: There exists a positive constant $\varepsilon_0 < 1$ such that if $0 < \varepsilon \le \varepsilon_0$, then the reformulated problem (24)-(25) admits a unique solution $(\phi, \Psi) \in X(0, \frac{T}{\varepsilon})$ satisfying

$$\sup_{0\le\tau\le\frac{T}{\varepsilon}} \|(\phi,\Psi)(\tau)\|_2^2 + \int_0^{\frac{T}{\varepsilon}} \left[\|\bar{u}_{1y_1}^{1/2}(\phi,\psi_1)\|^2 + \|(\nabla\phi,\nabla\Psi)\|_1^2 + \|\nabla^3\Psi\|^2\right] d\tau$$
$$\le C_T \frac{\varepsilon^2}{\delta^7} + C\|(\phi_0,\Psi_0)\|_2^2,$$

where the constant C_T is independent of ε, δ, but may depend on T.

Once the Proposition 1 is proved, we have

$$\sup_{0\le t\le T}\|(\phi,\Psi)(t,x_1,x_2)\|_{L^\infty(\mathbb{R}\times\mathbb{T})}=\sup_{0\le\tau\le\frac{T}{\varepsilon}}\|(\phi,\Psi)(\tau,y_1,y_2)\|_{L^\infty(\mathbb{R}\times\mathbb{T}_\varepsilon)}$$
$$\le C\sup_{0\le\tau\le\frac{T}{\varepsilon}}\|(\phi,\Psi)(\tau)\|_2\le C_T\frac{\varepsilon}{\delta^{7/2}}+C\|(\phi_0,\Psi_0)\|_2.$$

Taking $\delta=\varepsilon^a$, we get

$$\|(\rho,u_1,u_2)(t,x_1,x_2)-(\rho^r,u_1^r,0)(\frac{x_1}{t})\|_{L^\infty(\mathbb{R}\times\mathbb{T})}$$
$$\le\|(\phi,\Psi)(t,x_1,x_2)\|_{L^\infty(\mathbb{R}\times\mathbb{T})}+C\|(d_1,d_2)(t,x_1)\|_{L^\infty(\mathbb{R})}$$
$$+\|(\bar\rho,\bar u_1)(t,x_1)-(\rho^r,u_1^r)(\frac{x_1}{t})\|_{L^\infty(\mathbb{R})}$$
$$\le C_T\frac{\varepsilon}{\delta^{7/2}}+C\|(\phi_0,\Psi_0)\|_2+C_T\frac{\varepsilon}{\delta^{3/2}}+C\delta t^{-1}[\ln(1+t)+|\ln\delta|]$$
$$=C_T\varepsilon^{1-\frac{7}{2}a}+C\|(\phi_0,\Psi_0)\|_2+C_T\varepsilon^{1-\frac{3}{2}a}+C\varepsilon^a t^{-1}[\ln(1+t)+|\ln\varepsilon|].$$

Taking $a=\frac{2}{9}$, i.e. $\delta=\varepsilon^{2/9}$, and then the proof of Theorem 1 is completed.

Proposition 2:(a priori estimates) Suppose that the reformulated problem (24)-(25) has a solution $(\phi, \Psi) \in X(0, \tau_1(\varepsilon))$ for some $\tau_1(\varepsilon)(>0)$. Then there exists a positive constant ε_1 which is independent of ε, δ and $\tau_1(\varepsilon)$, such that if $0 < \varepsilon \leq \varepsilon_1$ and

$$E = E(0, \tau_1(\varepsilon)) = \sup_{0 \leq \tau \leq \tau_1(\varepsilon)} \|(\phi, \Psi)(\tau)\|_2 \ll 1,$$

then it holds

$$\sup_{0 \leq \tau \leq \tau_1(\varepsilon)} \|(\phi, \Psi)(\tau)\|_2^2 + \int_0^{\tau_1(\varepsilon)} \left[\| \bar{u}_{1y_1}^{1/2} (\phi, \psi_1) \|^2 + \|(\nabla \phi, \nabla \Psi)\|_1^2 + \|\nabla^3 \Psi\|^2 \right] d\tau$$
$$\leq C_T \frac{\varepsilon^2}{\delta^7} + C \|(\phi_0, \Psi_0)\|_2^2. \tag{26}$$

Sketch of the Proof:

Step 1. elementary L^2 estimates:

$$\sup_{0\le\tau\le\tau_1(\varepsilon)}\|(\phi,\Psi)(\tau)\|^2+\int_0^{\tau_1(\varepsilon)}\left[\|\bar{u}_{1y_1}^{1/2}(\phi,\psi_1)\|^2+\|\nabla\Psi\|^2\right]d\tau$$
$$\le C_T\frac{\varepsilon^2}{\delta^7}+C\|(\phi_0,\Psi_0)\|^2.$$

- Crucially used the periodicity in x_2-direction.
- The hyperbolic wave (d_1,d_2) are crucially used in the estimate. The decay rate $\frac{\varepsilon^2}{\delta^7}$ comes from the error term $\frac{\rho}{\tilde{\rho}}Q$.

Step 2. First-order derivative estimates of density:

$$
\begin{aligned}
&\sup_{0\le\tau\le\tau_1(\varepsilon)}\left(\|(\phi,\Psi)(\tau)\|^2+\|\nabla\phi(\tau)\|^2\right)\\
&+\int_0^{\tau_1(\varepsilon)}\left[\|\bar{u}_{1y_1}^{1/2}(\phi,\psi_1)\|^2+\|(\nabla\phi,\nabla\Psi)\|^2\right]d\tau\\
&\le C_T\frac{\varepsilon^2}{\delta^7}+C(\|(\phi_0,\Psi_0)\|^2+\|\nabla\phi_0\|^2)+CE^2\int_0^{\tau_1(\varepsilon)}\|\nabla^2\Psi\|^2d\tau.
\end{aligned}
$$

Observations:

Applying the operator ∇ to the first equation of (24) and then multiplying the resulting equation by $\frac{\nabla\phi}{\rho^2}$ yield

$$\begin{aligned}&(\frac{|\nabla\phi|^2}{2\rho^2})_\tau+\mathrm{div}(\frac{\boldsymbol{u}|\nabla\phi|^2}{2\rho^2})+\frac{\nabla\mathrm{div}\Psi\cdot\nabla\phi}{\rho}\\&=-\frac{\phi_{y_i}\nabla\phi\cdot\nabla\psi_i}{\rho^2}+\frac{|\nabla\phi|^2\mathrm{div}\Psi}{2\rho^2}+\cdots\end{aligned}\tag{27}$$

Multiplying the second equation of (24) by $\frac{\nabla\phi}{\rho}$ gives

$$\begin{aligned}&(\Psi\cdot\nabla\phi)_\tau-\mathrm{div}(\Psi\phi_\tau)+\frac{p'(\rho)}{\rho}|\nabla\phi|^2-\frac{(\mu+\lambda)\nabla\mathrm{div}\Psi\cdot\nabla\phi+\mu\triangle\Psi\cdot\nabla\phi}{\rho}\\&=-\phi_\tau\mathrm{div}\Psi-u_i\Psi_{y_i}\cdot\nabla\phi+\cdots,\end{aligned}\tag{28}$$

Note that

$$\begin{aligned}\frac{\mu\Delta\Psi\cdot\nabla\phi}{\rho}&=\mathrm{div}(\frac{\mu}{\rho}\nabla\psi_i\phi_{y_i})-(\frac{\mu}{\rho}\nabla\psi_i\cdot\nabla\phi)_{y_i}+\frac{\mu\nabla\mathrm{div}\Psi\cdot\nabla\phi}{\rho}\\&+\frac{\mu}{\rho^2}\tilde\rho_{y_1}\Psi_{y_1}\cdot\nabla\phi-\frac{\mu}{\rho^2}\tilde\rho_{y_1}\nabla\psi_1\cdot\nabla\phi.\end{aligned}$$

Step 3. First-order derivative estimates of velocity:

$$\sup_{0\le\tau\le\tau_1(\varepsilon)}\|(\phi,\Psi)(\tau)\|_1^2+\int_0^{\tau_1(\varepsilon)}\left[\|\bar{u}_{1y_1}^{1/2}(\phi,\psi_1)\|^2+\|(\nabla\phi,\nabla\Psi)\|^2+\|\nabla^2\Psi\|^2\right]d\tau$$
$$\le C_T\frac{\varepsilon^2}{\delta^7}+C\|(\phi_0,\Psi_0)\|_1^2.$$

Step 4. Second-order derivative estimates of density:

$$\sup_{0\le\tau\le\tau_1(\varepsilon)}(\|(\phi,\Psi)\|_1^2+\|\nabla^2\phi\|^2)+\int_0^{\tau_1(\varepsilon)}\left[\|\bar{u}_{1y_1}^{1/2}(\phi,\psi_1)\|^2+\|(\nabla\phi,\nabla\Psi)\|_1^2\right]d\tau$$
$$\le C_T\frac{\varepsilon^2}{\delta^7}+C(\|(\phi_0,\Psi_0)\|_1^2+\|\nabla^2\phi_0\|^2)+CE^2\int_0^{\tau_1(\varepsilon)}\|\nabla^3\Psi\|^2d\tau.$$

Step 5. Second-order derivative estimates of velocity:

$$\sup_{0\le\tau\le\tau_1(\varepsilon)}\|(\phi,\Psi)(\tau)\|_2^2+\int_0^{\tau_1(\varepsilon)}\left[\|\bar{u}_{1y_1}^{1/2}(\phi,\psi_1)\|^2+\|(\nabla\phi,\nabla\Psi)\|_1^2+\|\nabla^3\Psi\|^2\right]d\tau$$
$$\le C_T\frac{\varepsilon^2}{\delta^7}+C\|(\phi_0,\Psi_0)\|_2^2.$$

3D non-isentropic CNS case ($\mathbb{R}\times\mathbb{T}^2$)

Theorem 2: (Lin-an Li, Dehua Wang, and Yi Wang, Preprint.)

Let $(\rho^r,\mathbf{u}^r,\theta^r)(\frac{x_1}{t})$ be the planar 3-rarefaction wave to the 3D compressible Euler system (7) and $T>0$ be any fixed time. Then there exists a constant $\varepsilon_0>0$, such that for any $\varepsilon\in(0,\varepsilon_0)$, the system (1)-(4) has a family of smooth solutions $(\rho^\varepsilon,\mathbf{u}^\varepsilon,\theta^\varepsilon)(t,x)$ up to time T satisfying

$$\begin{cases}(\rho^\varepsilon-\rho^r,\mathbf{u}^\varepsilon-\mathbf{u}^r,\theta^\varepsilon-\theta^r)\in C^0(0,T;L^2(\mathbb{R}\times\mathbb{T}^2)),\\ (\nabla\rho^\varepsilon,\nabla\mathbf{u}^\varepsilon,\nabla\theta^\varepsilon)\in C^0(0,T;H^1(\mathbb{R}\times\mathbb{T}^2)),\\ (\nabla^3\mathbf{u}^\varepsilon,\nabla^3\theta^\varepsilon)\in L^2(0,T;L^2(\mathbb{R}\times\mathbb{T}^2)),\end{cases}$$

Moreover, for any small positive constant h, there exists a positive constant $C_{h,T}$ independent of ε, such that

$$\sup_{h\le t\le T}\left\|(\rho^\varepsilon,\mathbf{u}^\varepsilon,\theta^\varepsilon)(t,x)-(\rho^r,\mathbf{u}^r,\theta^r)(\frac{x_1}{t})\right\|_{L^\infty(\mathbb{R}\times\mathbb{T}^2)}\le C_{h,T}\,\varepsilon^{\frac{1}{6}}|\ln\varepsilon|^2.$$

As $\varepsilon\to0+$,

$$(\rho^\varepsilon,\mathbf{u}^\varepsilon,\theta^\varepsilon)(t,x)\to(\rho^r,\mathbf{u}^r,\theta^r)(\frac{x_1}{t}),\quad \text{a.e. in } \mathbb{R}^+\times\mathbb{R}\times\mathbb{T}^2.$$

Sketch of the Proof:

In the setting of classical solution, we can rewrite the 3D full CNS (1) as

$$\begin{cases} \rho_t + \mathrm{div}(\rho \boldsymbol{u}) = 0, \\ (\rho \boldsymbol{u})_t + \mathrm{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) + R\nabla(\rho\theta) = \mu\varepsilon\Delta \boldsymbol{u} + (\mu+\lambda)\varepsilon\nabla\mathrm{div}\boldsymbol{u}, \\ \dfrac{R}{\gamma-1}((\rho\theta)_t + \mathrm{div}(\rho\theta \boldsymbol{u})) + R\rho\theta\mathrm{div}\boldsymbol{u} = \kappa\varepsilon\Delta\theta \\ \qquad\qquad + \dfrac{\mu\varepsilon}{2}|\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^\top|^2 + \lambda\varepsilon(\mathrm{div}\boldsymbol{u})^2. \end{cases}$$

Define the approximate solution profile $(\tilde{\rho}, \tilde{u}_1, \tilde{\theta})$ by the rarefaction wave $(\bar{\rho}, \bar{u}_1, \bar{\theta})$ and the hyperbolic wave (d_1, d_2, d_3):

$$\tilde{\rho} = \bar{\rho} + d_1, \quad \tilde{m}_1 = \bar{m}_1 + d_2 := \tilde{\rho}\tilde{u}_1, \quad \tilde{\mathcal{E}} = \bar{\mathcal{E}} + d_3 := \tilde{\rho}\tilde{E} = \tilde{\rho}(\frac{R}{\gamma-1}\tilde{\theta} + \frac{1}{2}\tilde{u}_1^2).$$

Then the approximate wave profile $(\tilde{\rho}, \tilde{u}_1, \tilde{\theta})$ satisfies the system

$$\begin{cases} \tilde{\rho}_t + (\tilde{\rho}\tilde{u}_1)_{x_1} = 0, \\ (\tilde{\rho}\tilde{u}_1)_t + (\tilde{\rho}\tilde{u}_1^2 + R\tilde{\rho}\tilde{\theta})_{x_1} = (2\mu+\lambda)\varepsilon \bar{u}_{1x_1x_1} + Q_1, \\ \dfrac{R}{\gamma-1}[(\tilde{\rho}\tilde{\theta})_t + (\tilde{\rho}\tilde{u}_1\tilde{\theta})_{x_1}] + R\tilde{\rho}\tilde{\theta}\tilde{u}_{1x_1} = \kappa\varepsilon\bar{\theta}_{x_1x_1} + (2\mu+\lambda)\varepsilon\bar{u}_{1x_1}^2 + Q_2 \end{cases}$$

with the initial data

$$(\tilde{\rho}, \tilde{u}_1, \tilde{\theta})(0, x_1) = (\bar{\rho}_0, \bar{u}_{10}, \bar{\theta}_0)(x_1),$$

and the error terms

$$
\begin{aligned}
Q_1 &= \Big(\frac{\tilde{m}_1^2}{\tilde{\rho}} - \frac{\bar{m}_1^2}{\bar{\rho}} + \frac{\bar{m}_1^2}{\bar{\rho}^2}d_1 - \frac{2\bar{m}_1}{\bar{\rho}}d_2\Big)_{x_1} + (\tilde{p} - \bar{p} - \bar{p}_{\bar{\rho}}d_1 - \bar{p}_{\bar{m}_1}d_2 - \bar{p}_{\bar{\mathcal{E}}}d_3)_{x_1} \\
&= \Big[\frac{3-\gamma}{2\tilde{\rho}}(\bar{u}_1 d_1 - d_2)^2\Big]_{x_1} = O(1)\big[|\bar{u}_{1x_1}||(d_1,d_2)|^2 + |(d_1,d_2)||(d_{1x_1},d_{2x_1})|\big],
\end{aligned}
$$

$$
\begin{aligned}
Q_2 &= \Big(\frac{\tilde{m}_1\tilde{\mathcal{E}}}{\tilde{\rho}} - \frac{\bar{m}_1\bar{\mathcal{E}}}{\bar{\rho}} + \frac{\bar{m}_1\bar{\mathcal{E}}}{\bar{\rho}^2}d_1 - \frac{\bar{\mathcal{E}}}{\bar{\rho}}d_2 - \frac{\bar{m}_1}{\bar{\rho}}d_3\Big)_{x_1} - \tilde{u}_1 Q_1 \\
&\quad + \Big(\tilde{p}\frac{\tilde{m}_1}{\tilde{\rho}} - \bar{p}\frac{\bar{m}_1}{\bar{\rho}} - \bar{p}_{\bar{\rho}}\frac{\bar{m}_1}{\bar{\rho}}d_1 + \bar{p}\frac{\bar{m}_1}{\bar{\rho}^2}d_1 - \bar{p}_{\bar{m}_1}\frac{\bar{m}_1}{\bar{\rho}}d_2 - \frac{\bar{p}}{\bar{\rho}}d_2 - \bar{p}_{\bar{\mathcal{E}}}\frac{\bar{m}_1}{\bar{\rho}}d_3\Big)_{x_1} \\
&= \Big[\tfrac{-\bar{u}_1 d_1 + d_2}{\tilde{\rho}}\Big(\gamma d_3 - (\gamma-1)\bar{u}_1 d_2 - \tfrac{R\gamma}{\gamma-1}\bar{\theta}d_1 + \tfrac{\gamma-2}{2}\bar{u}_1^2 d_1\Big) \\
&\quad -(\gamma-1)\tilde{u}_1\frac{(-\bar{u}_1 d_1 + d_2)^2}{2\tilde{\rho}}\Big]_{x_1} - (2\mu+\lambda)\varepsilon\bar{u}_{1x_1x_1}\frac{-\bar{u}_1 d_1 + d_2}{\tilde{\rho}} \\
&\quad -\tilde{u}_1\Big(\frac{3-\gamma}{2\tilde{\rho}}(-\bar{u}_1 d_1 + d_2)^2\Big)_{x_1} \\
&= O(1)\Big[|\bar{u}_{1x_1}||(d_1,d_2,d_3)|^2 + |(d_1,d_2,d_3)||(d_{1x_1},d_{2x_1},d_{3x_1})| \\
&\quad + \varepsilon|\bar{u}_{1x_1x_1}||(d_1,d_2)|\Big].
\end{aligned}
$$

Perturbation:

$$\begin{aligned}
&\phi(t,x) := \rho^{\varepsilon}(t,x) - \tilde{\rho}(t,x_1),\\
&\Psi(t,x) = (\psi_1,\psi_2,\psi_3)^{\top}(t,x) := (u_1^{\varepsilon},u_2^{\varepsilon},u_3^{\varepsilon})^{\top}(t,x) - (\tilde{u}_1,0,0)^{\top}(t,x_1),\\
&\zeta(t,x) := \theta^{\varepsilon}(t,x) - \tilde{\theta}(t,x_1),
\end{aligned}$$

with $(\rho^{\varepsilon},\boldsymbol{u}^{\varepsilon},\theta^{\varepsilon})$ being the solution to the full CNS (1) with the following initial data:

$$(\rho^{\varepsilon},\boldsymbol{u}^{\varepsilon},\theta^{\varepsilon})(0,x) := (\bar{\rho}_0,\bar{\boldsymbol{u}}_0,\bar{\theta}_0)(x_1) + (\phi_0,\Psi_0,\zeta_0)(x),$$

Perturbation system (ϕ, Ψ, ζ):

$$
\begin{cases}
\phi_t + \boldsymbol{u}\cdot\nabla\phi + \rho\mathrm{div}\Psi + \tilde{\rho}_{x_1}\psi_1 + \tilde{u}_{1x_1}\phi = 0, \\
\rho\Psi_t + \rho\boldsymbol{u}\cdot\nabla\Psi + R\theta\nabla\phi + R\rho\nabla\zeta + (\rho\tilde{u}_{1x_1}\psi_1, 0, 0)^\top + (R\tilde{\rho}_{x_1}(\theta - \frac{\rho}{\tilde{\rho}}\tilde{\theta}), 0, 0)^\top \\
= \mu\varepsilon\Delta\Psi + (\mu+\lambda)\varepsilon\nabla\mathrm{div}\Psi + ((2\mu+\lambda)\varepsilon(\frac{-\bar{u}_1 d_1 + d_2}{\tilde{\rho}})_{x_1x_1}, 0, 0)^\top \\
\quad -((2\mu+\lambda)\varepsilon\frac{\bar{u}_{1x_1x_1}}{\tilde{\rho}}\phi, 0, 0)^\top - (Q_1\frac{\rho}{\tilde{\rho}}, 0, 0)^\top, \\
\dfrac{R}{\gamma-1}(\rho\zeta_t + \rho\boldsymbol{u}\cdot\nabla\zeta) + R\rho\theta\mathrm{div}\Psi + \dfrac{R}{\gamma-1}\rho\tilde{\theta}_{x_1}\psi_1 + R\rho\tilde{u}_{1x_1}\zeta \\
= \kappa\varepsilon\Delta\zeta + \frac{\mu\varepsilon}{2}|\nabla\Psi + (\nabla\Psi)^\top|^2 + \lambda\varepsilon(\mathrm{div}\Psi)^2 + 2\tilde{u}_{1x_1}(2\mu\varepsilon\psi_{1x_1} + \lambda\varepsilon\mathrm{div}\Psi) \\
\quad + F_1 + F_2 - \frac{\rho}{\tilde{\rho}}Q_2,
\end{cases}
$$

- The analysis of the 3D case is carried out in the original non-scaled variables, and consequently the dissipation terms are more singular compared with the 2D scaled case.

where

$$
\begin{aligned}
F_1 &= -\kappa\varepsilon(\tilde{\theta}_{x_1x_1} - \bar{\theta}_{x_1x_1}) - (2\mu+\lambda)\varepsilon(\tilde{u}_{1x_1}^2 - \bar{u}_{1x_1}^2) \\
&= \frac{\gamma-1}{R}\kappa\varepsilon\{\frac{1}{\tilde{\rho}}[(\frac{1}{2}\bar{u}_1^2 - \frac{R}{\gamma-1}\bar{\theta})d_1 - \bar{u}_1 d_2 + d_3]\}_{x_1x_1} \\
&\quad - \frac{\gamma-1}{2R}\kappa\varepsilon[(\frac{-\bar{u}_1 d_1 + d_2}{\tilde{\rho}})^2]_{x_1x_1} + 2(2\mu+\lambda)\varepsilon\bar{u}_{1x_1}(\frac{-\bar{u}_1 d_1 + d_2}{\tilde{\rho}})_{x_1} \\
&\quad + (2\mu+\lambda)\varepsilon[(\frac{-\bar{u}_1 d_1 + d_2}{\tilde{\rho}})_{x_1}]^2 \\
&= O(\varepsilon)[|(d_{1x_1x_1}, d_{2x_1x_1}, d_{3x_1x_1})| + |(d_{1x_1}, d_{2x_1})|^2 + |\bar{u}_{1x_1}(d_{1x_1}, d_{2x_1}, d_{3x_1})| \\
&\qquad\qquad + |\bar{u}_{1x_1}(d_1, d_2, d_3)|^2], \\
F_2 &= -\kappa\varepsilon\frac{\bar{\theta}_{x_1x_1}}{\tilde{\rho}}\phi - (2\mu+\lambda)\varepsilon\frac{\bar{u}_{1x_1}^2}{\tilde{\rho}}\phi,
\end{aligned}
$$

- More accurate a priori assumptions with respect to the dissipation coeffcients.

$$
\begin{aligned}
\sup_{t\in[0,t_1(\varepsilon)]} \|(\phi,\Psi,\zeta)(t)\|_{L^\infty} &\ll 1,\\
\sup_{t\in[0,t_1(\varepsilon)]} \|(\nabla\phi,\nabla\Psi,\nabla\zeta)(t)\| &\le \varepsilon^{\frac{3}{4}}|\ln\varepsilon|^{-1},\\
\sup_{t\in[0,t_1(\varepsilon)]} \|(\nabla^2\phi,\nabla^2\Psi,\nabla^2\zeta)(t)\| &\le \varepsilon^{\frac{1}{4}}|\ln\varepsilon|^{-1}.
\end{aligned}
$$

- Some new observations on the cancellations of the physical structures for the flux terms.
- Here the decay rate is determined by the nonlinear terms, while for the 2D case the decay rate is fully determined by the error terms in the scaled variables.

Thank you!