Regularity of optimal transport maps and the associated free boundary problem

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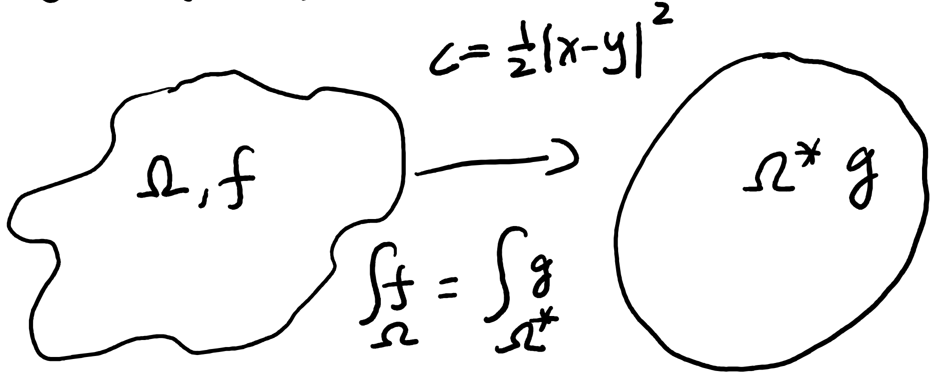
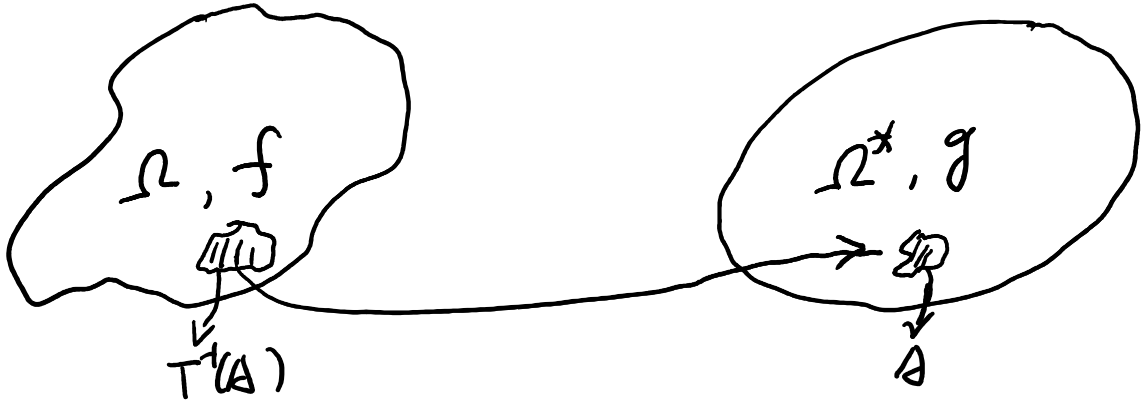
optimal transport:



Monge's problem: $\min\limits_{T_\# f = g} \int_\Omega \frac{1}{2}|x - Tx|^2 f(x)\, dx$



$$T_\# f = g \iff \forall \text{ Borel set } A \subset \Omega^*, \ \int_{T^{-1}(A)} f = \int_A g$$

$$\iff \forall\, h \in C_b(\Omega), \ \int_\Omega h(Tx) f(x) = \int_{\Omega^*} h(y)\, g(y)$$

$$\implies \det DT = \frac{f(x)}{g(Tx)}$$

(when T is C^1)

The cost $\frac{1}{2}|x-y|^2$ is equivalent to $-x \cdot y$

Reason: $\int_\Omega \frac{1}{2}|x-Tx|^2 f(x)\,dx$

$$= \underbrace{\int_\Omega \frac{1}{2}|x|^2 f(x)\,dx}_{\text{const}} - \int_\Omega x \cdot Tx\, f(x)\,dx + \underbrace{\frac{1}{2}\int_\Omega |Tx|^2 f(x)\,dx}_{\frac{1}{2}\int_{\Omega^*} |y|^2 g(y)\,dy = \text{const}}$$

Monge's problem: $\min\limits_{T_\# f = g} \int_\Omega -x \cdot Tx\, f(x)\,dx$

Existence:

Kantorovich duality:

$$\min_{T_\# f = g} \int_\Omega -x\cdot Tx\, f(x) = \sup_{\substack{u(x)+v(y)\geq x\cdot y \\ \forall x\in\Omega,\ \forall y\in\Omega^*}} \int_\Omega -uf + \int_{\Omega^*} -vg$$

$$u(x) = \sup_{y\in\Omega^*} x\cdot y - v(y), \quad v(y) = \sup_{x\in\Omega} y\cdot x - u(x)$$

Brenier: The optimal map $T = Du$ for some convex u.

If u is smooth, then $\det D^2 u = \dfrac{f(x)}{g(Du(x))}$

Regularity Theory.

Thm (Caffarelli, 90~96)

Suppose Ω^* is convex, $\frac{1}{C} < f, g < C$ in Ω, Ω^* respectively

Then: 1). u is strictly convex and $C^{1,\alpha}$ in Ω

2) If $f, g \in C^{\alpha}_{loc}(\Omega)$, Then $u \in C^{2,\alpha}_{loc}(\Omega)$

3) If Ω is also convex, then $u \in C^{1,\alpha}(\bar{\Omega})$

4) If Ω, Ω^* are uniformly convex, C^2, and $f \in C^{\alpha}(\bar{\Omega})$
$g \in C^{\alpha}(\overline{\Omega^*})$, then $u \in C^{2,\alpha'}(\bar{\Omega})$ for some $\alpha' \in (0, \alpha)$

Rmk: 4) was proved by Delanoe $(n=2)$, Urbas $(n \geq 2)$
under stronger conditions.

Thm [De Philippis, Figalli]

Suppose Ω^* is convex, $\frac{1}{C} < f, g < C$ in Ω, Ω^* respectively

Then $u \in W^{2,1+\varepsilon}_{loc}(\Omega)$.

Thm [C - Lou - Wang, 2018]

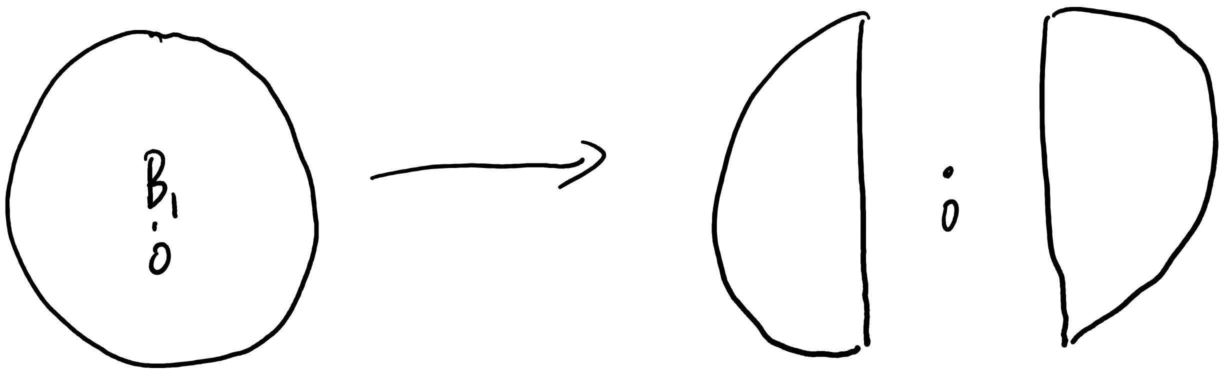
Suppose Ω, Ω^* are convex, $C^{1,1-\delta}$, $f, g \in C^{\alpha}(C^0)$

Then $u \in C^{2,\alpha}(\overline{\Omega})$ $(W^{2,p}(\overline{\Omega}))$

Rmk: when $n=2$, Ω, Ω^* are convex, Savin - Yu proved $W^{2,p}$ estimate for constant densities.

Convexity of target is necessary.

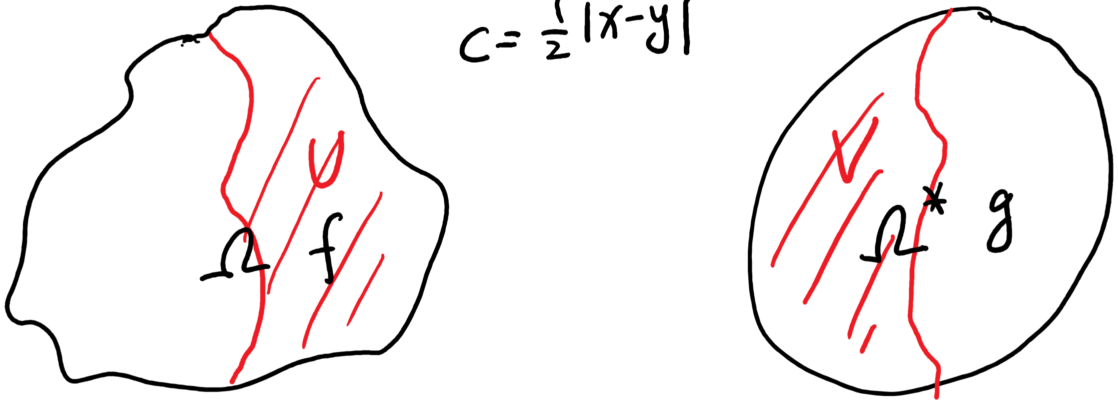
Caffarelli's example:



$u = \frac{1}{2}|x|^2 + |x_1|$. is singular at 0.

optimal partial transport:

$c=\frac{1}{2}|x-y|^2$



mass transported $m < \max\{\int_{\Omega} f, \int_{\Omega^*} g\}$

$$\min_{\substack{(U,V,T) \\ \int_U f = \int_V g = m \\ T_\# f\chi_U = g\chi_V}} \int_{\Omega} \frac{1}{2}|x-Tx|^2 f(x)\, dx$$

Thm. Caffarelli–McCann (Annals, 2010):

suppose Ω, Ω^* are strictly convex and disjoint.

suppose $\frac{1}{c} < f, g < c$. Then the free boundary

$\partial U \cap \Omega$ is $C^{1,\alpha}$

open problem: Higher regularity?

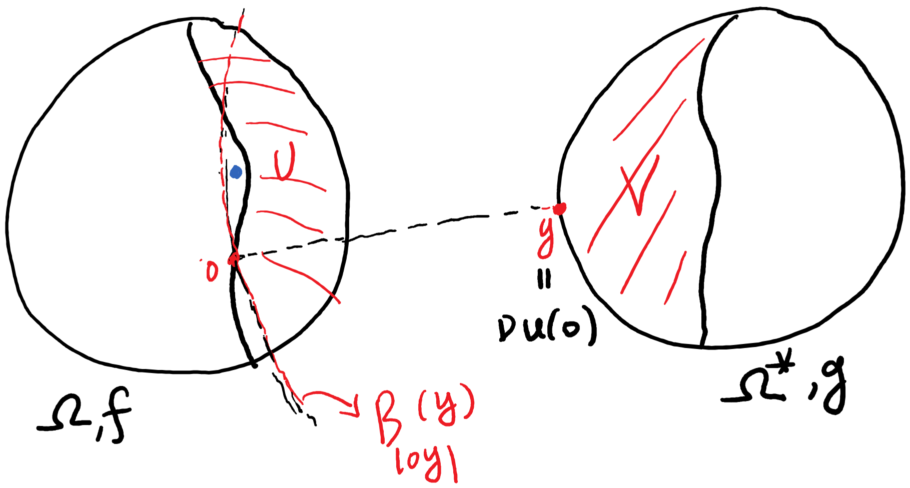
Thm (C–Liu–Wang, 2020)

suppose Ω, Ω^* are uniformly convex, C^2, disjoint.

$f \in C^\alpha(\overline{\Omega}), g \in C^\alpha(\overline{\Omega^*})$, then $\partial U \cap \Omega$, $\partial V \cap \Omega^*$

are $C^{2,\alpha}$

Interior ball property.



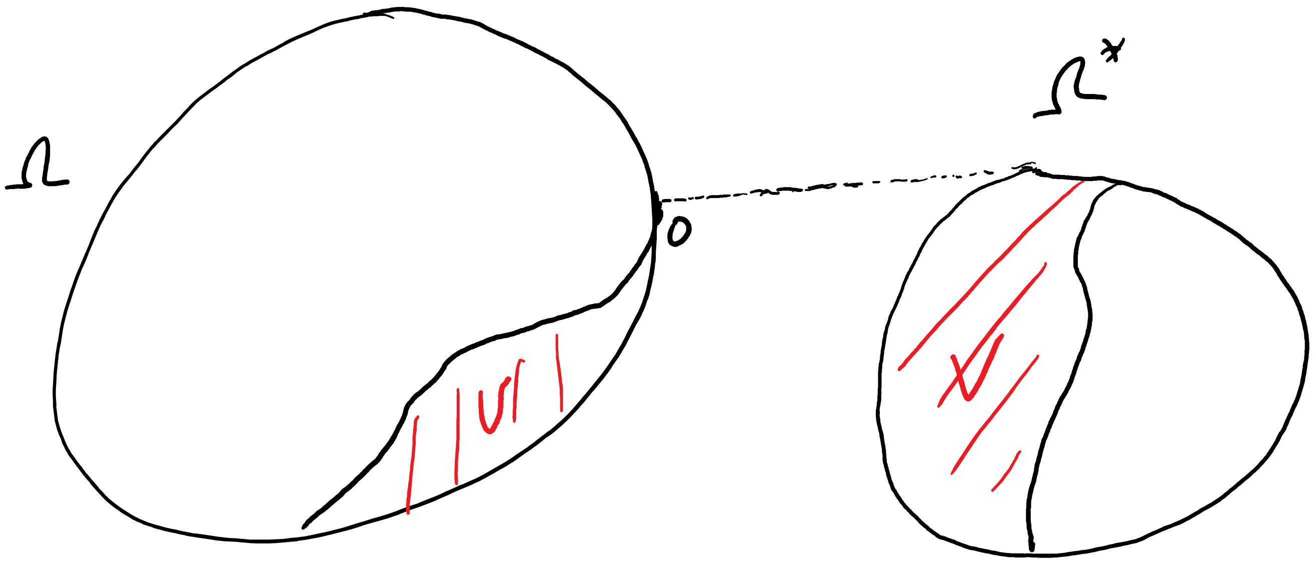
when Ω, Ω^* are strictly convex, Caffarelli–McCann proved that $\partial U \cap \Omega$ is C^1.

Lemma: $B_{|0y|}(y) \cap \Omega \subset U$. $\Rightarrow$ for any $x \in \partial U \cap \Omega$, the unit normal of $\partial U \cap \Omega$ at x is given by

$$\gamma_x = \frac{Du(x) - x}{|Du(x) - x|}$$

u is C^1 ($C^{1,\alpha}$, $C^{2,\alpha}$) up to $\partial U \cap \Omega \Rightarrow \partial U \cap \Omega$ is C^1 ($C^{1,\alpha}$, $C^{2,\alpha}$) !

What happens when free bdry meets fixed bdry



Is this picture possible? 0 is a cusp of the active region U.

- Caffarelli – McCann :

denote $\partial_{nt}\Omega := \{ x \in \partial\Omega \cap \overline{\Omega \cap \partial U} \mid \langle Du(x) - x, z \rangle \le 0 \text{ for } \forall z \in \Omega \}$

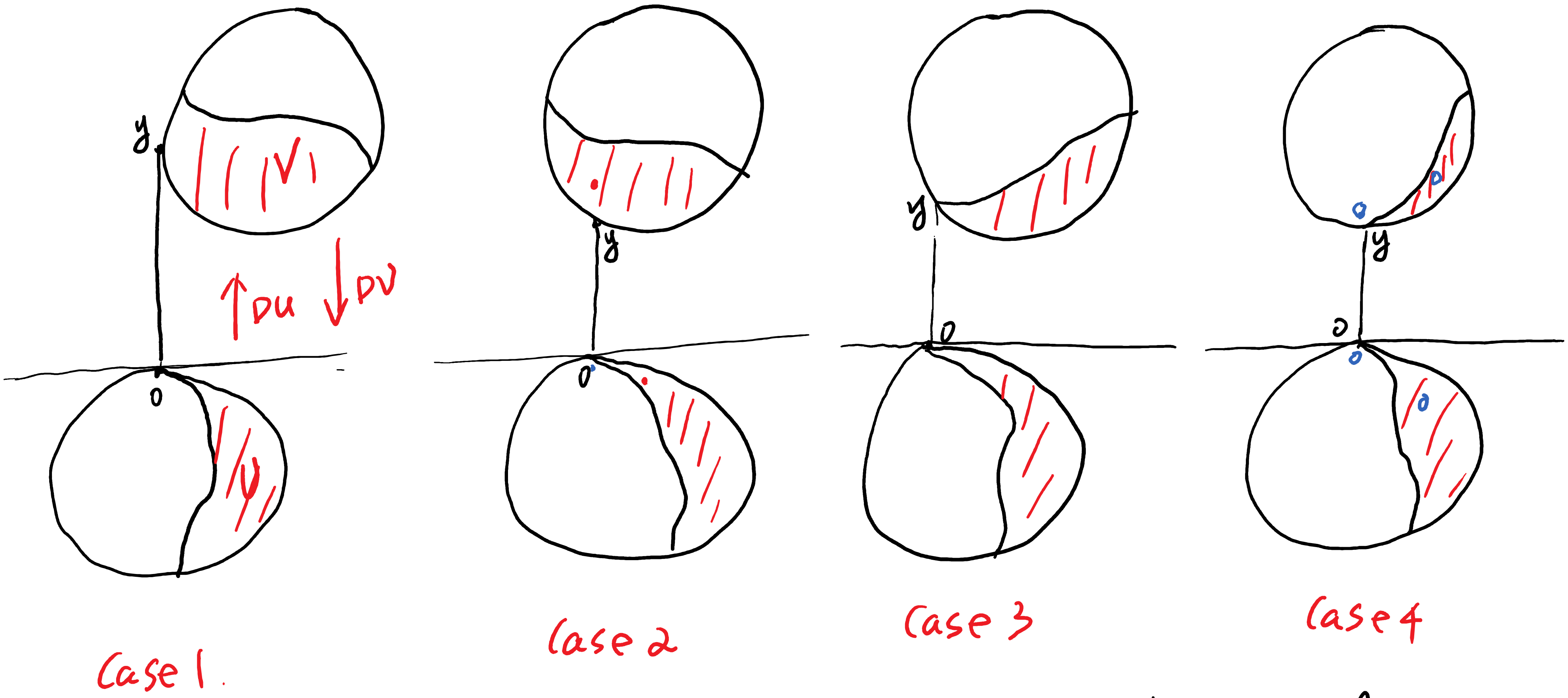
It was not clear whether such cusp exists or not.

- E. Indrei (JFA, 2013) :

$\partial_{nt}(\Omega)$ has Hausdorff dimension $n-2$, assuming $\Omega, \Omega^* \in C^{1,1}$ and uniformly convex.

- (C – Liu – ···) $\partial_{nt}(\Omega) = \phi$. i.e There is no cusp !

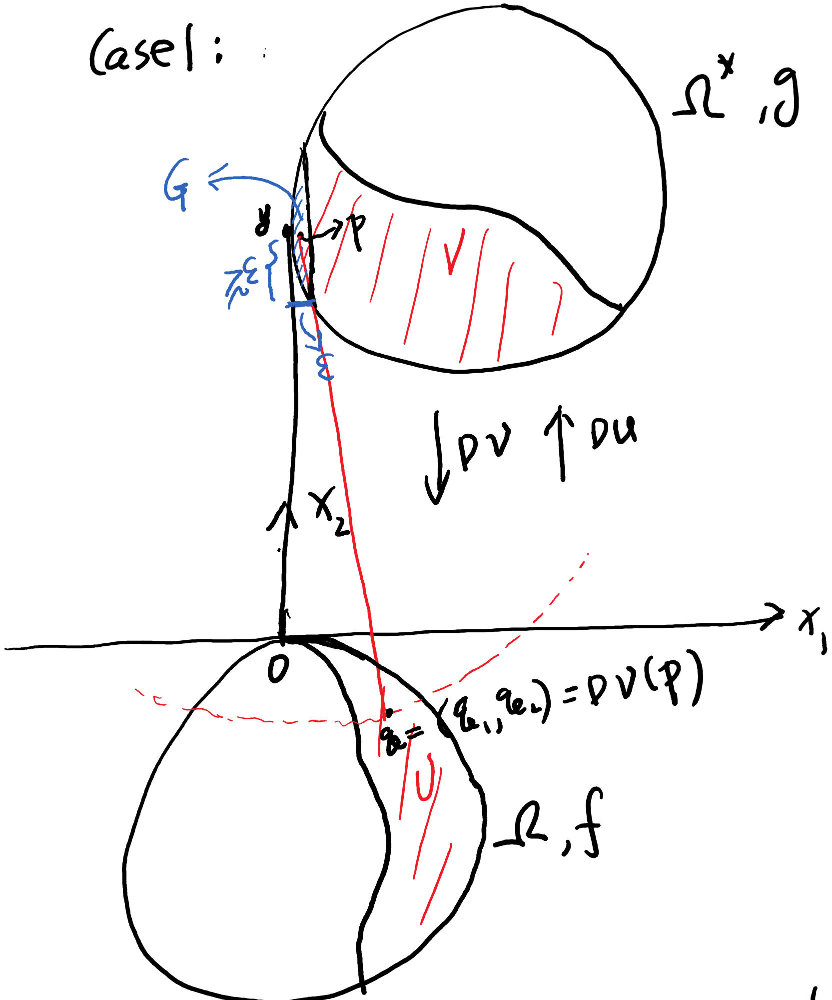
Ω, Ω^* are assumed to be C^1, strictly convex proof: There are four cases



case 2, 4 ruled out by constructing cheaper plan!

case 1 and 3 are delicate!

case 1:



$G := \{ x = (x_1, x_2) \in V \mid x_1 \le \varepsilon \}$

$|G| \gtrsim \varepsilon^2$.

For any $p = (p_1, p_2) \in G$, $|p_1| \le \varepsilon$, $p_2 \gtrsim 1$

By interior ball $\Rightarrow |pq| \le |p0|$

$\Rightarrow |q_2| \le |p0| - p_2 = |p_1^2 + p_2^2|^{\frac{1}{2}} - p_2$

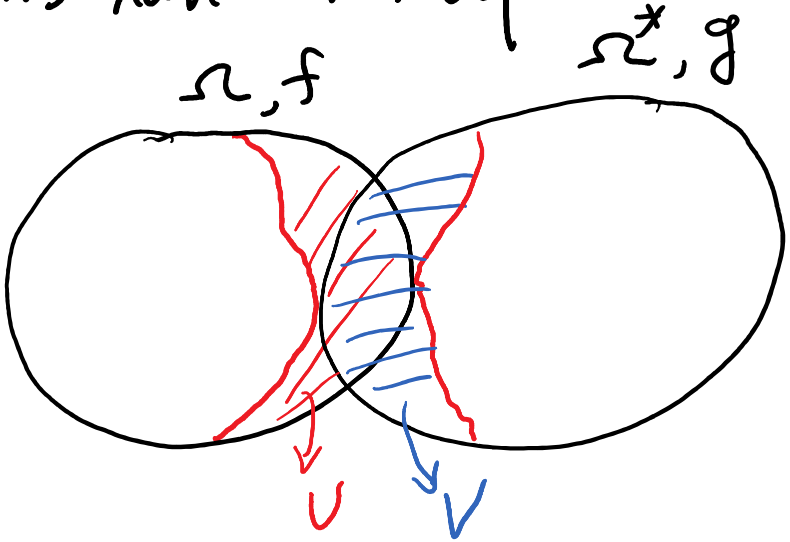
$\approx \dfrac{p_1^2}{2p_2} \lesssim \varepsilon^2$

$q_2 = o(1)$

$\Rightarrow q = Dv(p) \in$ rectangle with sides $\varepsilon^2, o(1)$, centre 0, Hence $|Dv(G)| = o(\varepsilon^2) \ll |G|$

On the other hand $\det D^2 v \approx 1 \Rightarrow |Dv(G)| \approx |G|$ #

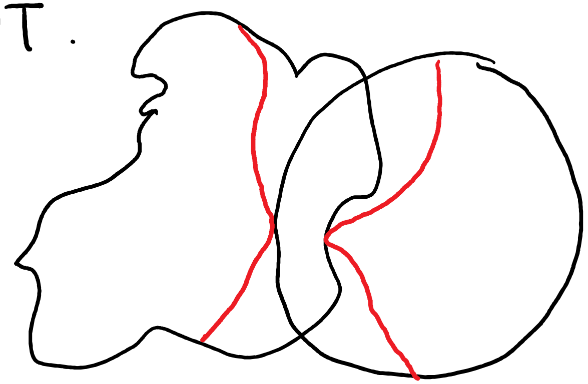
when domains have overlap



Thm (Figalli, 2010). suppose Ω, Ω^* are strictly convex $\frac{1}{C} < f, g < C$. Then $\partial U \cap \Omega \setminus (\overline{\Omega \cap \Omega^*})$ is C^1.

Rmk. Under same conditions, Indrei improves it to $C^{1,\alpha}$

We develop a regularity theory can be seen as a countepart of caffarello's regularity theory of classical OT.



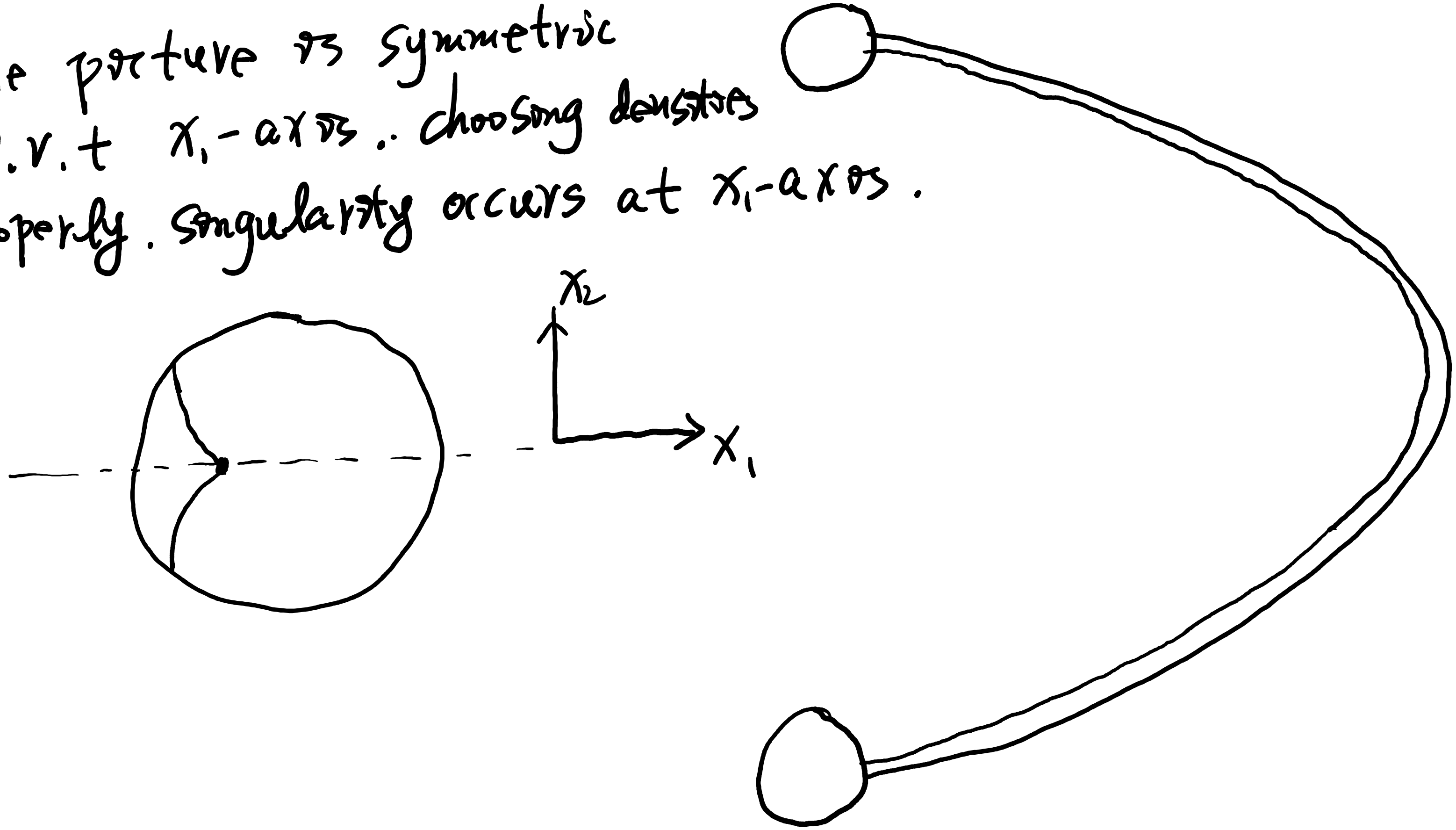
Thm (C-Lou, forthcoming)

Suppose Ω^* is convex, $\frac{1}{C} < f, g < C$, Then $(\partial U \cap \Omega) \setminus (\Omega \cap \Omega^*)$ is $C^{1,\alpha}$.

The conditions of the theorem is optimal

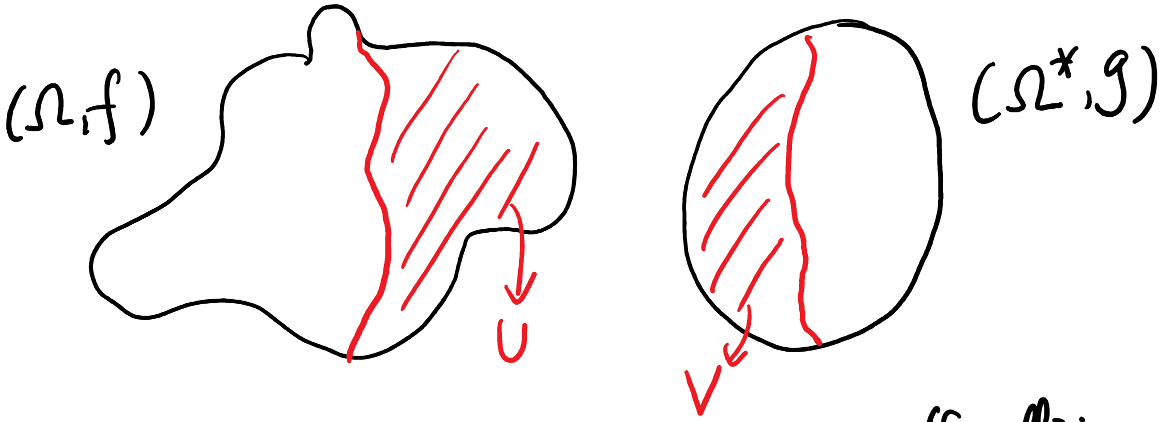
If Ω^* is not convex, we have the following example

The picture is symmetric w.r.t x_1-axis. choosing densities properly. singularity occurs at x_1-axis.



Sketch of proof.

for simplicity, only consider the case $\overline{\Omega} \cap \overline{\Omega^*} = \phi$.

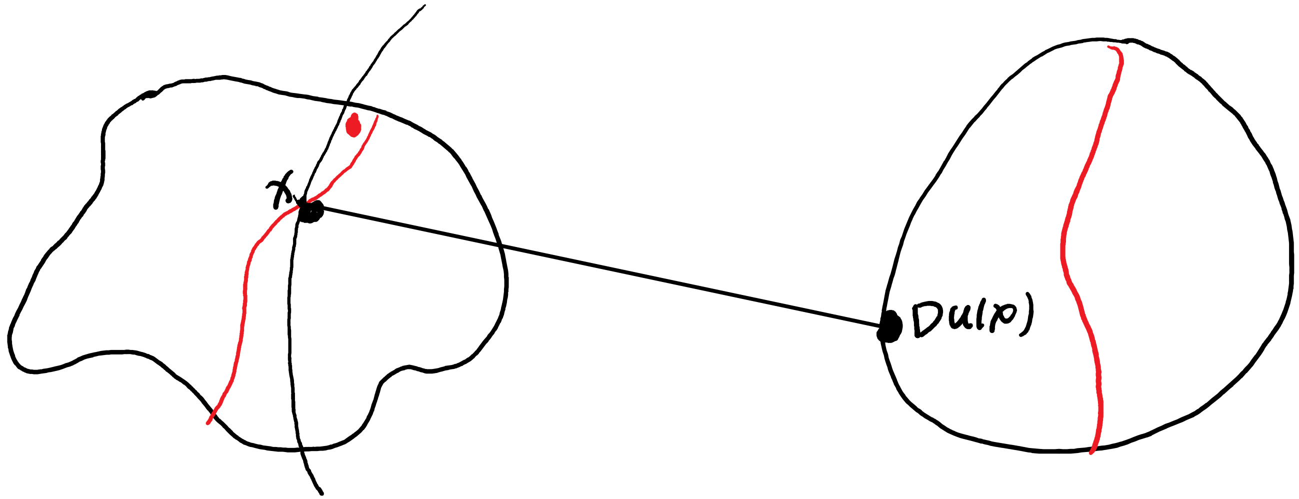


Caffarelli:

$$(Du)_\# (f \chi_U + g \chi_{\Omega^* \setminus V}) = g \chi_{\Omega^*} \Longrightarrow$$ u is strictly convex, $C^{1,\alpha}$ in U.

$$(Dv)_\# (g \chi_V + f \chi_{\Omega \setminus U}) = f \chi_\Omega .$$

$Dv = id$ a.e. in $\Omega \setminus U \Longrightarrow \left.\begin{array}{l} v^* = \frac{1}{2}|x|^2 + c \text{ in } \Omega \setminus U \\ v^* = u \text{ in } U \end{array}\right\} \Longrightarrow v^*$ is strictly convex in Ω.

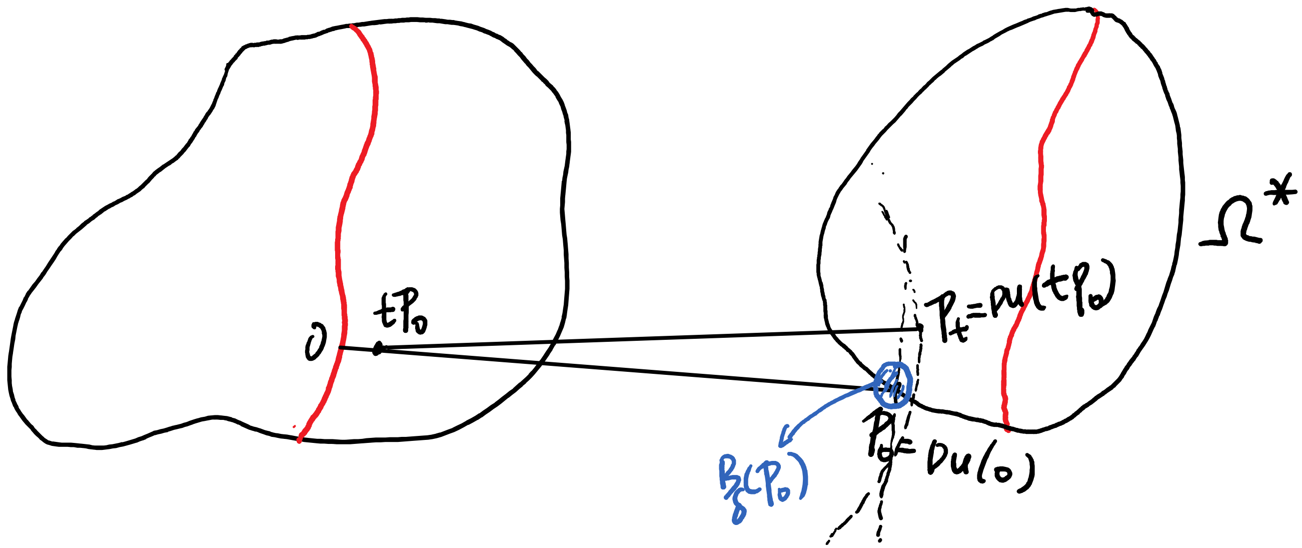


Interior ball property:

$$B^{(Du(x))}_{|x-Du(x)|} \cap \Omega \subset U$$

This elementary fact plays important role!

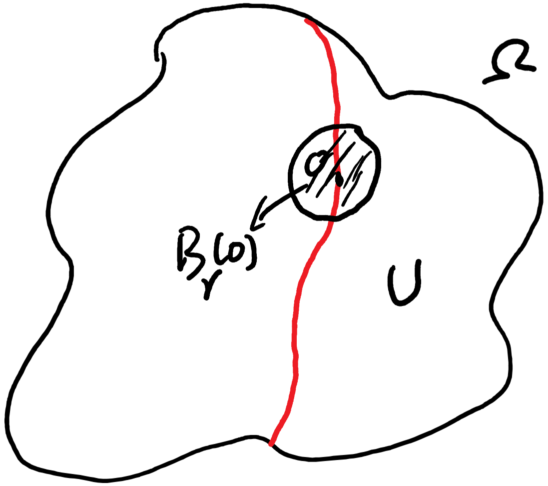
Free boundary never maps to free boundary

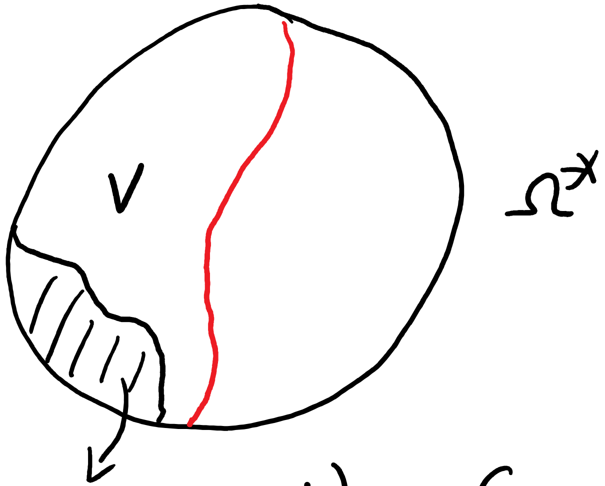


u convex $\Rightarrow tP_0 \cdot (P_t - P_0) \geq 0 \Rightarrow |P_t - tP_0| > |P_0 - tP_0|$

$\Rightarrow \exists\, \delta > 0$, s.t $B_\delta(P_0) \cap \Omega^* \subset B_{|P_t - tP_0|}(tP_0) \cap \Omega^* \subset V$.

Localize the problem



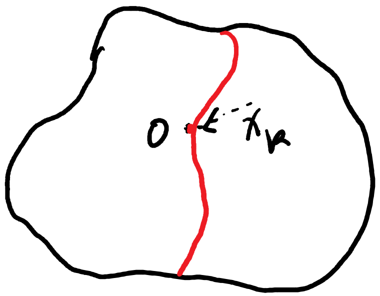


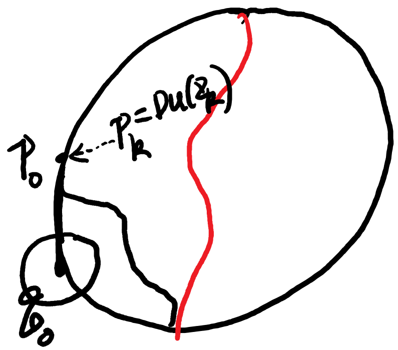
$Du(B_r(0) \cap U) := G$

$$(Dv)_\# \, g\chi_G + f\chi_{B_r(0)\setminus U} = f\chi_{B_r(0)}$$, target is $B_r(0)$!

Extend v to $\mathbb{R}^n$. s.t $\tilde{v} = v$ in $G \cup (B_r(0)\setminus U)$

$$\frac{1}{C}\chi_{G\cup(B_r(0)\setminus U)} < \det D^2\tilde{v} < C\,\chi_{G\cup(B_r(0)\setminus U)} .$$



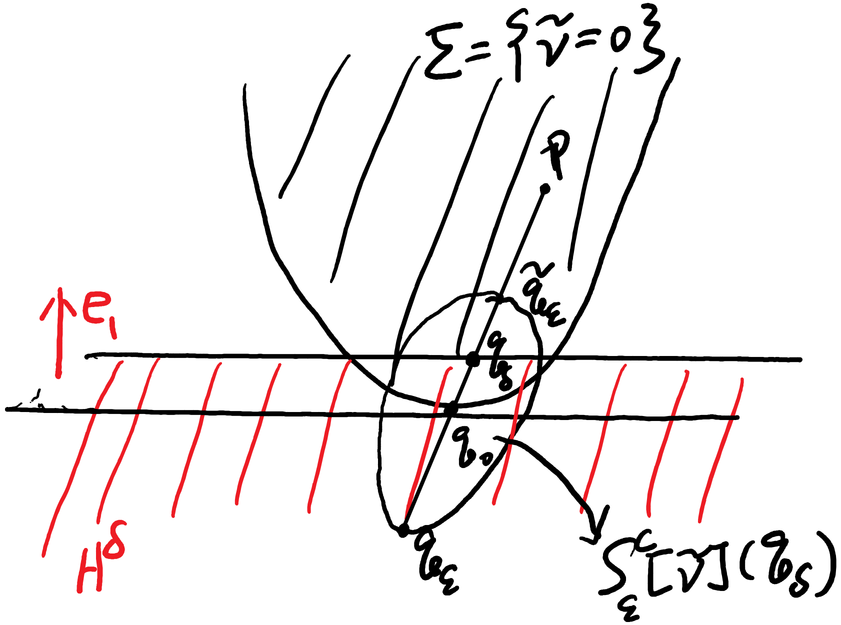


Up to a constant, we may assume $\tilde{v}(p_0)=0$

Denote $\Sigma = \{\tilde{v}=0\}$, is convex, let q_0 be an expose point of Σ.

Claim: $\exists\ \delta_1>0$ s.t $B_{\delta_1}(q_0)\cap\Omega^* \underset{a.e}{\subset} G \subset V$.

Follows from strict convexity of v^* in Ω!



$H^\delta := \{ z \mid (z - q_0)\cdot e_1 \le (q_\delta - q_0)\cdot e_1 \}$

e_1 is the unit normal of support plane of Σ at q_0

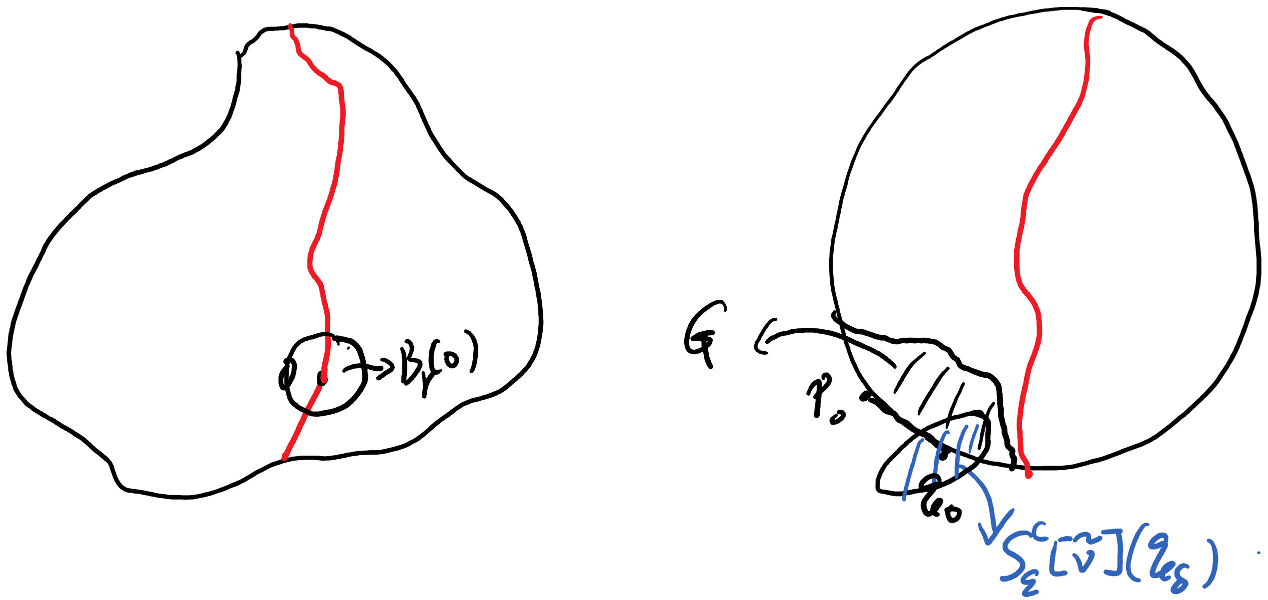
Let $S^c_\varepsilon[\tilde{v}](q_\delta)$ be the centred section of $\tilde{v}$ at q_δ.

i) $\tilde{v} \lesssim \varepsilon$ in $S^c_\varepsilon[\tilde{v}]$

ii) $S^c_\varepsilon[\tilde{v}] \cap H^\delta \xrightarrow[\varepsilon \to 0]{} \Sigma \cap H^\delta$ in Hausdorff

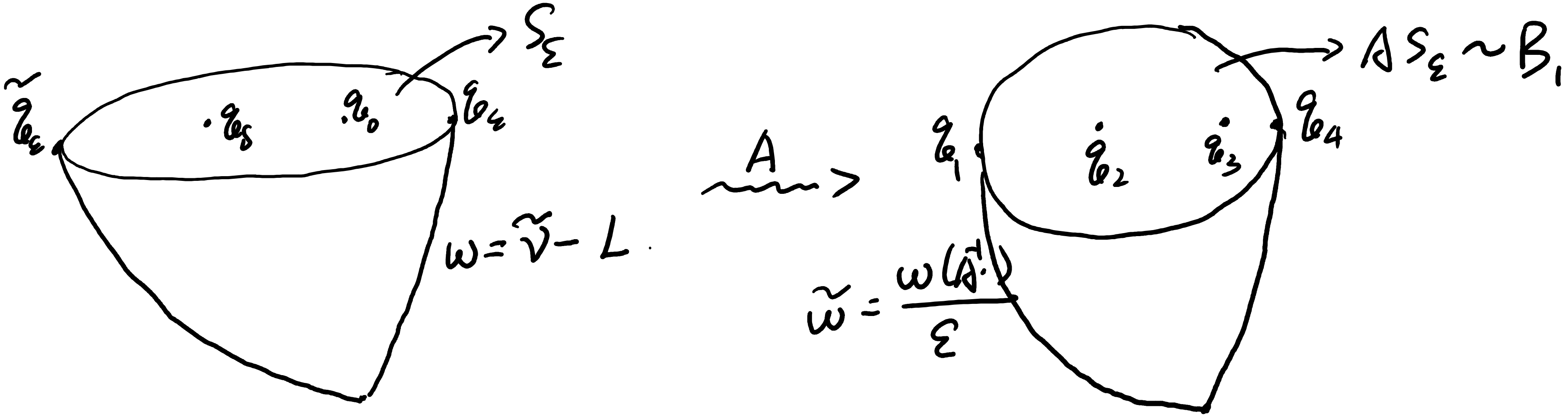
iii) $q_\varepsilon \xrightarrow[\varepsilon\to 0]{} q_0$. $\Rightarrow$ $\dfrac{|\tilde{q}_\varepsilon q_0|}{|q_\varepsilon q_0|} \to +\infty$

i) + ii) + iii) $\Rightarrow$ For δ, ε small.



$S_\varepsilon^c[\tilde{v}](q_{\varepsilon_\delta})$ is doubling : $\left|\frac{1}{2} S_\varepsilon^c[\tilde{v}](q_{\varepsilon_\delta}) \cap G\right| \gtrsim \left|S_\varepsilon^c[\tilde{v}](q_{\varepsilon_\delta}) \cap G\right|$

we can use Alexandrov estimate to make contradiction!



$M_{\tilde w} := \det D^2 \tilde w$ is doubling, i.e. $M_{\tilde w}(\tfrac{1}{2} A S_\varepsilon) \gtrsim M_{\tilde w}(A S_\varepsilon)$

On one hand $|\tilde w(q_{\varepsilon 3})|^n \lesssim |q_{\varepsilon 3} q_{\varepsilon 4}| = c \dfrac{|q_{\varepsilon 3} q_{\varepsilon 4}|}{|q_{\varepsilon 1} q_{\varepsilon 3}|} \cdot |q_{\varepsilon 1} q_{\varepsilon 3}|$

$$= c \frac{|q_{\varepsilon} q_{\varepsilon 0}|}{|\tilde q_{\varepsilon} q_{\varepsilon 0}|} \cdot |q_{\varepsilon 1} q_{\varepsilon 3}| \xrightarrow{\varepsilon \to 0} 0$$

on the other hand $\tilde w(q_{\varepsilon 3}) \approx -1$.

Contradiction !

Thanks for your attention!