Symmetry and symmetry breaking for the fractional CKN inequality

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The Caffarelli-Kohn-Nirenberg inequality

- $$\left(\int_{\mathbb{R}^n} \frac{|u|^p}{|x|^{\beta p}}\, dx\right)^{2/p} \leq (\Lambda^n_{\alpha,\beta})^{-1} \int_{\mathbb{R}^n} \frac{|\nabla u|^2}{|x|^{2\alpha}}\, dx \; (CKN)$$

  where

  $$-\infty < \alpha < \frac{n-2}{2}, \; \alpha \leq \beta \leq \alpha + 1,$$

  $$p = \frac{2n}{n-2+2(\beta-\alpha)},$$

  $$u \in D_{\alpha,\beta} = \{|x|^{-\beta} u \in L^p(\mathbb{R}^n), |x|^{-\alpha}|\nabla u| \in L^2(\mathbb{R}^n)\}$$

- Interpolation between the Sobolev inequality ($\alpha = 0, \beta = 0$) and the Hardy inequality ($\alpha = 0, \beta = 1$) or weighted Hardy inequality ($\beta = \alpha + 1$).

The extremal solution

$$S(\alpha, \beta) = \inf_{u \in D_{\alpha,\beta}} E_{\alpha,\beta}(u)$$

where

$$E_{\alpha,\beta}(u) = \frac{\int_{\mathbb{R}^n} |x|^{-2\alpha} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^n} |x|^{-\beta p} |u|^p dx\right)^{\frac{2}{p}}}$$

The extremal solution for $S(\alpha, \beta)$ are solutions of

$$-div(|x|^{-2\alpha} \nabla u) = |x|^{-\beta p} u^{p-1} \text{ in } \mathbb{R}^n.$$

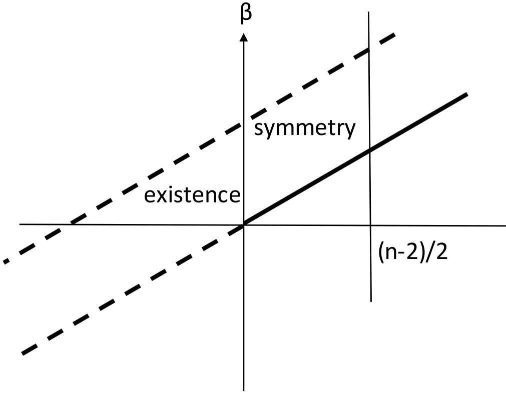
Best constants; Existence (and non-existence) of extremal solutions; Symmetry property

- For $0 \leq \alpha < \frac{n-2}{2}$, $\alpha \leq \beta \leq \alpha + 1$

  Aubin and Talenti, Lieb, Chou and Chu, Lions, Wang and Willem

- For $\alpha < 0$, $\alpha \leq \beta \leq \alpha + 1$

  Horiuchi 1997, Catrina and Wang 2001 , Willem 2002, Dolbeault, Esteban and Loss 2016


β
symmetry
existence
(n-2)/2

Symmetry breaking and the Felli-Schneider curve

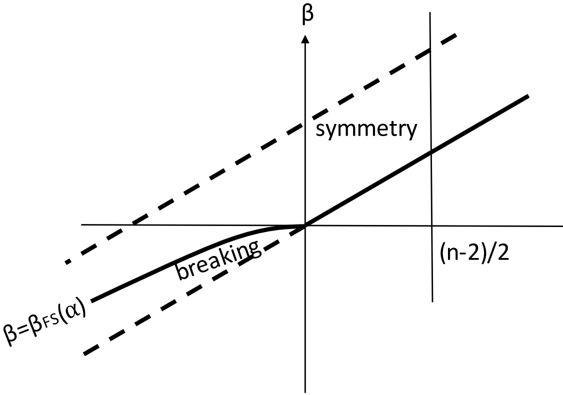
Symmetry breaking by perturbation. One expands the functional

$$F(v(r)) = (\Lambda_{\alpha,\beta})^{-1} \int_{\mathbb{R}^n} \frac{|\nabla v|^2}{|x|^{2\alpha}}\, dx - \left(\int_{\mathbb{R}^n} \frac{|v|^p}{|x|^{\beta p}}\, dx \right)^{\frac{2}{p}}$$

near the critical point $v_*(r)$ to second order by computing

$$Q(w) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \Big(F(v_* + \varepsilon w) - F(v_*) \Big)$$

The spectrum of Q determines the local stability of v_*.



β
symmetry
breaking
β=βFS(α)
(n-2)/2

- For $\alpha < 0$, there exists $\beta = \beta_{FS}(\alpha)$, such that the lowest eigenvalue of Q is zero.
- So for $\alpha < \beta < \beta_{FS}(\alpha)$, the extremal is non-radial.
- The FS-curve is found by Felli-Schneider 2003;
- Radial symmetry of extremal solutions for

$$\beta_{FS}(\alpha) < \beta < \alpha + 1$$

  by Dolbeault, Esteban and Loss 2016 using nonlinear flow method.

The fractional CKN inequality

Consider $\gamma \in (0,1)$, $n > 2\gamma$, and $\alpha, \beta \in \mathbb{R}$ satisfy

$$\alpha \leq \beta \leq \alpha + \gamma, \ -2\gamma < \alpha < \frac{n-2\gamma}{2},$$

and

$$p = \frac{2n}{n - 2\gamma + 2(\beta - \alpha)}.$$

The fractional Caffarelli-Kohn-Nirenberg inequality is

$$\Lambda \left(\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^{\beta p}} \, dx \right)^{\frac{2}{p}} \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x-y|^{n+2\gamma} |x|^{\alpha} |y|^{\alpha}} \, dy \, dx \ (FCKN)$$

Known results by Abdellaoui and Bentifour, Musina and Nazarou, Nguyen and Squassina

Energy functional

$$E_{\alpha,\beta}(u) = \frac{\|u\|_{\gamma,\alpha}^2}{\left(\int_{\mathbb{R}^n} |x|^{-\beta p} |u|^p \, dx \right)^{2/p}}.$$

The best constant in this inequality is defined by

$$S(\alpha,\beta) = \inf_{u \in D^{\gamma}_{\alpha,\beta}(\mathbb{R}^n) \setminus \{0\}} E_{\alpha,\beta}(u).$$

Note that extremal solutions satisfy the following equation:

$$\mathcal{L}_{\gamma,\alpha}(u) = c \frac{|u(x)|^{p-2} u(x)}{|x|^{\beta p}}, \tag{1}$$

where

$$\mathcal{L}_{\gamma,\alpha}(u) := \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2\gamma} |x|^{\alpha} |y|^{\alpha}} \, dy.$$

Best constants, existence and non-existence of extremal solutions

Theorem (1)

i. *$S(\alpha, \beta)$ is continuous in the full parameter domain .*

ii. *For $\beta = \alpha + \gamma$, we have $S(\alpha, \alpha + \gamma) = 2\kappa^n_{\alpha,\gamma}$, and it is not achieved.*

iii. *For $\alpha = \beta$, $0 \leq \alpha < \frac{n-2\gamma}{2}$, $S(\alpha, \alpha)$ is achieved and the extremal solution is radially symmetric and decreasing in the radial variable.*

iv. *For $\alpha = \beta$, $-2\gamma < \alpha < 0$, $S(\alpha, \alpha) = S(0, 0)$ and it is not achieved.*

v. *For $\alpha < \beta < \alpha + \gamma$, $S(\alpha, \beta)$ is always achieved.*

Modified inversion symmetry

Theorem (2)

For $\alpha \leq \beta < \alpha + \gamma$, any bounded solution of (6) in $D^{\gamma}_{\alpha,\beta}(\mathbb{R}^n)$ satisfying $u(x) > 0$ in $\mathbb{R}^n \setminus \{0\}$, satisfies the modified inversion symmetry

$$u\left(\frac{x}{|x|^2}\right) = |x|^{n-2\gamma-2\alpha} u(x)$$

after a dilation $u(x) = \lambda^{\frac{n-2\gamma-2\alpha}{2}} u(\lambda x), \lambda > 0$ if necessary. Moreover, writing

$$t = -\ln|x| \quad \text{and} \quad \theta = \frac{x}{|x|}, \tag{2}$$

we have that the function

$$v(t,\theta) = e^{-\frac{n-2\gamma-2\alpha}{2}t} u(e^{-t}\theta) \tag{3}$$

is even in t and monotonically decreasing for $t > 0$.

The problem in cylindrical variable

If $u(x)$ is the extremal and be a solution of

$$\int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2\gamma}|x|^{\alpha}|y|^{\alpha}}\, dy = c\frac{|u(x)|^{p-2}u(x)}{|x|^{\beta p}},$$

then using the relation

$$v(t,\theta) = e^{-\frac{n-2\gamma-2\alpha}{2}t}u(e^{-t}\theta) = |x|^{\frac{n-2\gamma}{2}-\alpha}u(x)$$

One can get the equation for $v(t,\theta)$

$$P_\gamma v + C(\alpha)v = v^{p-1}, \quad t \in \mathbb{R}, \theta \in \mathbb{S}^{n-1}.$$

Here P_γ is the conformal fractional Laplacian on the cylinder, which is a conformally covariant non-local operator of order 2γ.

Uniqueness and Non-degeneracy of radial solution

If $\bar{v}(t)$ is a radially symmetric minimizer, consider the linearized operator given by

$$\bar{L}w = P_\gamma w + C(\alpha)w - (p-1)\bar{v}^{p-2}w, \quad t \in \mathbb{R}, \theta \in \mathbb{S}^{n-1}.$$

Theorem (Non-degeneracy)

Assume that we are in the symmetry range. Let $\bar{u}$ be a radially symmetric minimizer, and set $\bar{v}$ in cylindrical coordinates as above. Then in the radial symmetry class

$$\ker(\bar{L}) = \langle \bar{v}_t \rangle.$$

Theorem (Uniqueness)

Let $n > 2$. Then, in the symmetry range, minimizers for $E_{\alpha,\beta}$ are unique.

Symmetry breaking

Theorem (Symmetry breaking)

For $-2\gamma < \alpha < 0$, there exists an open subset H inside this region containing the set $\{(\alpha, \alpha) \in \mathbb{R}^2,\ \alpha \in (-2\gamma, 0)\}$ such that for any $(\alpha, \beta) \in H$ with $\alpha < \beta$, the extremal solution to $S(\alpha, \beta)$ is non radial.

Remark: Note that the most general theorem on existence of a Felli-Schneider type curve separating the symmetry and symmetry breaking regions should be possible, but it is a complicated question since the proof in the local case relies on the explicit knowledge of the spectrum of the linearized operator.

The inequality in cylindrical variables

Proposition

Let

$$\nu := \frac{n-2\gamma}{2} - \alpha.$$

Then the function $u(x) = |x|^{-\nu}$ *is a solution of Euler-Lagrange equation*

$$\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2\gamma}|x|^{\alpha}|y|^{\alpha}}\, dy = c\frac{|u(x)|^{p-2}u(x)}{|x|^{\beta p}},$$

with the constant normalized as $c = \kappa^n_{\alpha,\gamma}$ *with* $0 < \kappa^n_{\alpha,\gamma} < \infty$ *defined as*

$$\kappa^n_{\alpha,\gamma} = \text{P.V.} \int_{\mathbb{R}^n} \frac{(1-|\zeta|^{-\nu})}{|e_1-\zeta|^{n+2\gamma}|\zeta|^{\alpha}}\, d\zeta = \int_{\mathbb{S}^{n-1}} J(\theta)\, d\theta,$$

where we have defined

$$J(\theta) = \text{P.V.} \int_0^\infty \frac{(1-\varrho^{-\nu})\varrho^{n-1-\alpha}}{(1+\varrho^2-2\varrho\langle\sigma,\theta\rangle)^{\frac{n+2\gamma}{2}}}\, d\varrho.$$

Remark: Here we need the condition that $\alpha > -2\gamma$. Note that the best possible result should be for the whole range $\alpha < \frac{n-2\gamma}{2}$. As it happens with the standard fractional Laplacian operator, one can admit more singular distributions by giving a more general definition for $(-\Delta)^\gamma$, for instance, by means of Fourier transform.

Proof.

Using polar coordinates with $\varrho = \frac{|y|}{|x|}$ and $\theta, \sigma \in \mathbb{S}^{n-1}$ for x, y

$$\int_{\mathbb{R}^n} \frac{|x|^{-\nu} - |y|^{-\nu}}{|x-y|^{n+2\gamma}|y|^{\alpha}}\, dy = |x|^{-\nu-2\gamma-\alpha} \int_{\mathbb{S}^n} \int_0^\infty \frac{(1-\varrho^{-\nu})\varrho^{n-1-\alpha}}{(1+\varrho^2 - 2\varrho\langle\sigma,\theta\rangle)^{\frac{n+2\gamma}{2}}}\, d\varrho$$

$$= |x|^{-\nu-2\gamma-\alpha} \kappa^n_{\alpha,\gamma},$$

$\square$

Equivalent problem on $\mathbb{R} \times S^{n-1}$

Recall the conformal fractional Laplacian on cylinder:

$$\begin{aligned} P_\gamma v &= r^{\frac{n+2\gamma}{2}}(-\Delta)^\gamma (r^{-\frac{n-2\gamma}{2}} v) \\ &= \varsigma_{n,\gamma} P.V. \int_{\mathcal{C}} K(t,\tilde{t},\theta,\tilde{\theta})(v(t,\theta) - v(\tilde{t},\tilde{\theta}))\, d\tilde{\mu} + c_{n,\gamma} v(t), \end{aligned}$$

where

$$K(t,\tilde{t},\theta,\tilde{\theta}) = \frac{e^{-\frac{n+2\gamma}{2}|t-\tilde{t}|}}{(1 + e^{-2|t-\tilde{t}|} - 2e^{-|t-\tilde{t}|}\langle\theta,\tilde{\theta}\rangle)^{\frac{n+2\gamma}{2}}}, \quad t,\tilde{t} \in \mathbb{R}, \theta,\tilde{\theta} \in \mathbb{S}^{n-1}.$$

$$\|u\|^2_{\gamma,\alpha} = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+2\gamma}|x|^{\alpha}|y|^{\alpha}}\,dy\,dx$$
$$= 2\kappa^n_{\alpha,\gamma}\int_{\mathcal{C}} v^2(t,\theta)\,d\mu + \int_{\mathcal{C}}\int_{\mathcal{C}} K(t,\tilde{t},\theta,\tilde{\theta})(v(t,\theta)-v(\tilde{t},\tilde{\theta}))^2\,d\mu\,d$$

$$\int_{\mathbb{R}^n} \frac{|u|^p}{|x|^{\beta p}}\,dx = \int_{\mathcal{C}} |v(t,\theta)|^p\,d\mu.$$

Thus we define an energy functional on the cylinder $\mathcal{C}$:

$$F_{\alpha,\beta}(v) = \frac{\int_{\mathcal{C}}\int_{\mathcal{C}} K(t,\tilde{t},\theta,\tilde{\theta})(v(t,\theta)-v(\tilde{t},\tilde{\theta}))^2\,d\mu d\tilde{\mu} + 2\kappa^n_{\alpha,\gamma}\int_{\mathcal{C}} v^2\,d\mu}{\left(\int_{\mathcal{C}}|v|^p\,d\mu\right)^{2/p}}.$$

The extremal will satisfy

$$P_{\gamma}(v) + C(\alpha)v = cv^{p-1} \text{ in } \mathcal{C}.$$

Proof of modified inversion symmetry

Let v be a positive solution of

$$P_\gamma v(t,\theta) + C(\alpha)v(t,\theta) = cv(t,\theta)^{p-1} \text{ in } \mathcal{C}. \tag{4}$$

We use the moving plane method to show that for $\alpha \le \beta < \alpha + \gamma$, v must be symmetric with respect to some $t = const$.

Denote by $z^\lambda = (2\lambda - t, \theta) \in \mathcal{C}$ the reflection of z relative to the plane $t = \lambda$. We let

$$w_\lambda(z) := v(z^\lambda) - v(z), \tag{5}$$

which is defined on

$$\Sigma_\lambda := \{(t,\theta) \in \mathcal{C}, t < \lambda\}$$

Two key points: the Maximum principle and the Hopf Lemma.

Lemma (Maximum Principle)

Let v a solution to

$$P_\gamma v = f(v) \text{ in } \Sigma_\lambda,$$

satisfying $v \geq 0$ in Σ_λ, $f(v) \geq 0$ for $v \geq 0$ and v is anti-symmetric with respect to $\partial\Sigma_\lambda$, i.e. $v(z^\lambda) = -v(z)$. Then $v \equiv 0$ or $v > 0$ in Σ_λ.

Proof.

Let us assume that there exists $(t_0, \theta_0) \in \Sigma_\lambda$ with $v(t_0, \theta_0) = 0$. Then, as above,

$$\begin{aligned} P_\gamma v(t_0, \theta_0) &= \varsigma_{n,\gamma} \int_{\mathcal{C}} K(t_0, \tilde{t}, \theta_0, \theta)(v(t_0, \theta_0) - v(\tilde{t}, \tilde{\theta}))\, d\tilde{\mu} + c_{n,\gamma} v(t_0, \theta_0) \\ &= -\varsigma_{n,\gamma} \int_{\Sigma_\lambda} [K(t_0, \tilde{t}, \theta_0, \tilde{\theta}) - K(t_0, -\tilde{t}, \theta_0, \tilde{\theta})] v(\tilde{t}, \tilde{\theta})\, d\tilde{\mu} \\ &\leq 0. \end{aligned} \tag{6}$$

Thus $P_\gamma v = f(v)$ is satisfied if and only if $v \equiv 0$. □

Lemma (Hopf Lemma for anti-symmetric functions)

Assume that $w \in C^3_{loc}(\Sigma)$,

$$\limsup_{z \to \partial\Sigma} c(z) = o\Big(\frac{1}{[\text{dist}(z, \partial\Sigma)]^2}\Big),$$

and

$$\begin{cases} P_\gamma w + c(z)w = 0 \quad \text{in } \Sigma, \\ w(z) > 0 \quad \text{in } \Sigma, \\ w(z^\lambda) = -w(z) \quad \text{in } \Sigma. \end{cases}$$

Then

$$\frac{\partial w}{\partial \nu} < 0, \quad \text{for every } z \in \partial\Sigma.$$

Symmetry breaking

Let $\bar{v}(t)$ be energy minimizer of $F_{\alpha,\beta}$ in radial symmetry class,

$$P_\gamma \bar{v} + C(\alpha)\bar{v} = \bar{v}^{p-1}.$$

and consider the eigenvalue problem of the linearized operator:

$$P_\gamma w + C(\alpha)w - (p-1)\bar{v}^{p-2}w = \lambda w.$$

Projecting over spherical harmonics, eigenvalues are given by pairs $(\lambda_m, \phi_m(t))$, that solve

$$P_\gamma^{(m)}\phi_m + C(\alpha)\phi_m - (p-1)\bar{v}^{p-2}\phi_m = \lambda_m\phi_m, \quad m = 0, 1, \dots,$$

The first eigenvalue λ_0 is simple and assume the eigenfunctions corresponds to λ_0, λ_1 are w_0, w_1, choose

$$\bar{v} + \delta w_0 + s w_1$$

as a test function,

$$F_{\alpha,\beta}(\bar{v} + \delta w_0 + s w_1) = F_{\alpha,\beta}(\bar{v}) + 2\lambda_1 s^2 \int_{\mathcal{C}} w_1^2 d\,\mu + o(s^2)$$

Uniqueness and Non-degeneracy

Uniqueness and non-degeneracy for a non-local ODE with fractional Laplacian

$$(-\Delta)^\gamma v + v = v^{p_0 - 1} \quad \text{in } \mathbb{R},$$

where first developed in Frank-Lenzmann 2013. The idea can be applied to deal with

$$P_\gamma^{(0)} \bar{v} + C(\alpha)\bar{v} = \bar{v}^{p-1}, \quad \bar{v} = \bar{v}(t).$$

We will do a continuation argument in γ in order to use the known uniqueness results in the local case $\gamma = 1$.

Non-degeneracy of radial solutions

We use the ODE method for fractional operators we have developed to deal with this problem. Let $\bar{v}$ be radial solution of

$$P_\gamma \bar{v} + C(\alpha)\bar{v} = c\bar{v}^{p-1} \quad \text{in } \mathcal{C},$$

The linearized problem

$$P_\gamma \bar{w} + C(\alpha)\bar{w} = c(p-1)\bar{v}^{p-2} w \quad \text{in } \mathcal{C},$$

First one knows that $w_* = \partial_t v(t)$ belongs to the kernel.

$$P_\gamma^{(0)} w_0 + C(\alpha) w_0 = h_0$$

Take Fourier transform

$$(\Theta_\gamma^{(0)}(\xi) + C(\alpha))\hat{w}_0 = \hat{h}_0,$$

The behavior of the equation depends on the zeroes of the symbol $\Theta_\gamma^{(0)}(\xi) + C(\alpha)$. Formally we write

$$w_0(t) = \int_{\mathbb{R}} \frac{1}{\Theta_\gamma^{(0)}(\xi) + C(\alpha)} \hat{h}_0(\xi) e^{i\xi t}\, d\xi = \int_{\mathbb{R}} \mathcal{G}_0(t - t') h_0(t')\, dt',$$

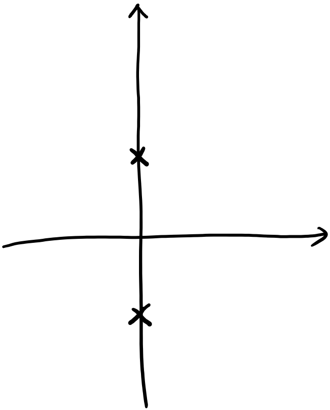
where the Green's function for the problem is given by

$$\mathcal{G}_m(t) = \int_{\mathbb{R}} e^{i\xi t} \frac{1}{\Theta_\gamma^{(0)}(\xi) + C(\alpha)}\, d\xi.$$

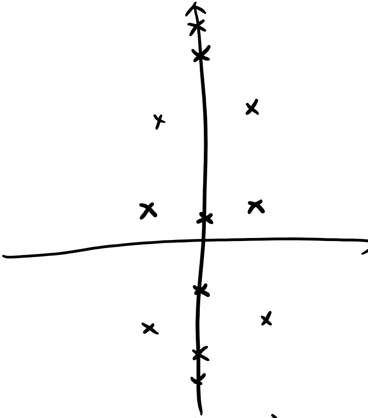
Lemma

The zeroes are of the form $\{\tau_j \pm i\sigma_j\}$, $\{-\tau_j \pm i\sigma_j\}$, *for some* $\tau_j, \sigma_j > 0$, $j = 0, 1, \ldots$, *satisfying in addition that* $\sigma_j > 0$ *is an increasing sequence with no accumulation points. Moreover,* $\tau_j = 0$ *for large* j *and the first zero lies on the imaginary axis away from the origin* ($\tau_0 = 0$, $\sigma_0 > 0$).

In particular, $\Theta_\gamma^{(0)}(\xi) + C(\alpha)$ *is bounded from below for* $\xi \in \mathbb{R}$.



$\gamma = 1$, two indicial roots



$\gamma \in (0,1)$
infinitely many

Greens function for homogeneous equation

All solutions of the homogeneous problem $P^0_\gamma w + C(\alpha)w = 0$ are of the form

$$w_h(t) = C_0^- e^{-\sigma_0 t} + C_0^+ e^{\sigma_0 t} + \sum_{j=1}^{\infty} e^{-\sigma_j t}[C_j^- \cos(\tau_j t) + C_j'^- \sin(\tau_j t)]$$
$$+ \sum_{j=1}^{\infty} e^{+\sigma_j t}[C_j^+ \cos(\tau_j t) + C_j'^+ \sin(\tau_j t)]$$

for some real constants $C_j^-, C_j^+, C_j'^-, C_j'^+$, $j = 0, 1, \ldots$.

Variation of constants formula

Assume that the right hand side h satisfies

$$h(t) = \begin{cases} O(e^{-\delta t}) & \text{as } t \to +\infty, \\ O(e^{\delta_0 t}) & \text{as } t \to -\infty, \end{cases} \tag{7}$$

for some real constants $\delta, \delta_0 > -\sigma_0$. A particular solution can be written as

$$w_p(t) = \int_{\mathbb{R}} \mathcal{G}_0(t - t')h(t')\, dt', \tag{8}$$

where

$$\mathcal{G}_0(t) = c_0 e^{-\sigma_0|t|} + \sum_{j=1}^{\infty} e^{-\sigma_j|t|}[c_j \cos(\tau_j|t|) + c_j' \sin(\tau_j|t|)],$$

for some precise real constants c_j, c_j' depending on κ, n, γ.

Reformulation into infinite system of second order ODEs

Define the complex-valued functions $w_j : \mathbb{R} \to \mathbb{C}$ by

$$w_j = e^{-(\sigma_j + i\tau_j)|\cdot|} * h.$$

They satisfy the second order ODE

$$-\frac{1}{\sigma_j + i\tau_j} w_j'' + (\sigma_j + i\tau_j) w_j = 2h,$$

and the original (real-valued) function w can be still recovered by

$$w(t) = \mathrm{Re} \sum_{j=0}^{\infty} c_j w_j(t).$$

$$P_\gamma w + C(\alpha) w = (p-1)\bar{v}^{p-2} w =: h.$$

Let $w = w(t)$ be a kernel satisfying that $w = O(e^{-\alpha_0|t|})$ as $|t| \to \infty$ for some $\alpha_0 > -\sigma_0$. Then there exists a non-negative integer j such that either

$$w(t) = (a_j + o(1))e^{-\sigma_j t} \quad \text{as } t \to +\infty,$$

for some real number $a_j \neq 0$, or

$$w(t) = \left(a_j^1 \cos(\tau_j t) + a_j^2 \sin(\tau_j t) + o(1)\right) e^{-\sigma_j t},$$

for some real numbers a_j^1, a_j^2 not vanishing simultaneously.

Proof for non-degeneracy

Let $w, \tilde{w}$ be solutions of

$$P_\gamma w + C(\alpha)w - (p-1)\bar{v}^{p-2}w =: P_\gamma w - v_\infty w = 0$$

Consider the infinite system of ODEs:

$$w_j'' - \sigma_j^2 w_j = -2\sigma_j v_\infty(t) w$$

$$W_j[w, \tilde{w}](t) = w_j' \tilde{w}_j - w_j \tilde{w}_j' = const \text{ since } \frac{d}{dt} W_j[w, \tilde{w}] = 0$$

Define

$$W[w, \tilde{w}] = \sum_{j=0}^{\infty} \frac{c_j}{\sigma_j} W_j[w, \tilde{w}]$$

$$W[w, w_*](t) \equiv \lim_{t \to +\infty} W[w, w_*](t) = 0.$$

$$w(t) = a_{j_0}(1 + o(1))e^{-\sigma_{j_0} t}, \quad w_*(t) = a_{j_0^*}(1 + o(1))e^{-\sigma_{j_0^*} t},$$

$$W[w, w_*](t) = a_{j_0} a_{j_0^*}(\sigma_{j_0} - \sigma_{j_0^*} + o(1))e^{-(\sigma_{j_0} + \sigma_{j_0^*})t} \quad \text{as } t \to +\infty,$$

we obtain that $\sigma_{j_0} = \sigma_{j_0^*}$. We now look at the next order. We suppose

$$w(t) = e^{-\sigma_{j_0} t} + a_\alpha(1+o(1))e^{-\alpha t}, \quad w_*(t) = e^{-\sigma_{j_0^*} t} + a_{\alpha^*}(1+o(1))e^{-\alpha^* t},$$

A direct computation of the Wrońskian yields $\alpha = \alpha^*$, $a_\alpha = a_{\alpha^*}$. w, w^* has the same asymptotic expansion, by unique continuation, $w = w_*$.

Uniqueness

Step 1. Local invertibility

Consider the linearized operator

$$L_\gamma w := P_\gamma^{(0)} w + c_0 w - (p_0 - 1) v^{p_0 - 2} w, \quad w \in L^2(\mathbb{R}).$$

By non-degeneracy, $\overline{L}_\gamma^{(0)}$ is invertible (with bounded inverse) in $L^2_{even}(\mathbb{R})$.

Proposition

Assume that we have a solution $\bar{v}_\gamma$ with non-degenerate kernel for $\gamma = \gamma_0$. Then, for some $\delta > 0$, there exists a map in $v \in C^1(I, \mathcal{F})$ defined on the interval $I = [\gamma_0, \gamma_0 + \delta])$ and denoted by $v_\gamma := v(\gamma)$, such that the following holds:

a. *v_γ solves the equation for all $\gamma \in I$, with $v_\gamma|_{\gamma=\gamma_0} = \bar{v}_{\gamma_0}$.*
b. *There exists $\epsilon > 0$ such that v_γ is the unique solution for $\gamma \in I$ in the neighborhood $\{v \in \mathcal{F} : \|v - \bar{v}_{\gamma_0}\| < \epsilon\}$.*

Pohozaev Identity

By extension this is equivalent to

$$\begin{cases} \partial_\rho(e_1\rho^{1-2\gamma}\partial_\rho\bar{V}) + e_2\rho^{1-2\gamma}\partial_{tt}\bar{V} = 0, \rho \in (0,\rho_0), t \in \mathbb{R}, \\ -\lim_{\rho\to 0}\rho^{1-2\gamma}\partial_\rho\bar{V}(\rho,t) + C(\alpha)\bar{v} - \bar{v}^{p-1} = 0 \text{ on } \{\rho = 0\}. \end{cases} \tag{9}$$

Proposition

If $V = V(t,\rho)$ is a solution of above, then we have the following Pohožaev identities:

$$\iint \rho^{1-2\gamma}\left\{e_1(\rho)(\partial_\rho V)^2 + e_2(\rho)(\partial_t V)^2\right\} d\rho dt + \tau\int v^2\,dt = \int v^{p_0}\,dt$$

and

$$\begin{aligned} &\frac{\tau}{2}\int v^2\,dt - \frac{1}{p_0}\int v^{p_0}\,dt \\ &= \frac{1}{2}\iint \rho^{1-2\gamma}\left\{-e_1(\rho)(\partial_\rho V)^2 + e_2(\rho)(\partial_t V)^2\right\} d\rho dt. \end{aligned}$$

Step 2. Apriori estimates

It is then natural to consider, for the branch v_γ, $\gamma \geq \gamma_0$, the energy

$$I_\gamma := \iint \rho^{1-2\gamma} \left\{ e_1(\rho)(\partial_\rho V_\gamma)^2 + e_2(\rho)(\partial_t V_\gamma)^2 \right\} \, d\rho \, dt.$$

Lemma

$$I_\gamma \sim \int v_\gamma^2 \, dt \sim \int v_\gamma^{p_0} \, dt \sim 1 \tag{10}$$

independently of γ*, for* $\gamma \in [\gamma_0, \gamma_*)$*.*

Step 3. Global continuation

Lemma

Let (v_γ) *be the maximal branch starting at* $\bar{v}_\gamma$ *with* $\gamma \in [\gamma_0, \gamma_*)$*.*
Then $\gamma_* = 1$*.*

At $\gamma_* = 1$,

$$P_1^{(0)} v + c_0 v = v^{p_0 - 1} \quad \text{in } \mathbb{R}. \tag{11}$$

Here $P_1^{(0)} = -\partial_{tt} + c_n$ on $\mathbb{R}$ for $c_n = \frac{(n-2)^2}{4}$. The corresponding solution is unique.

Thank You!