

第七届偏微分方程青年学术论坛

Harmonic analysis tools for Schrödinger operator with potential and applications to PDEs

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2020年11月27日-12月1日

Motivation

Consider the Cauchy problem for Schrödinger equation and wave equation with **electromagnetic** potential

$$\begin{cases} i\partial_t u + (i\nabla + \mathbf{A}(\mathbf{x}))^2 u + \mathbf{V}(\mathbf{x})u + \lambda|u|^{p-1}u = 0, \\ u(0, \mathbf{x}) = u_0(\mathbf{x}) \end{cases} \quad (1.1)$$

$$\begin{cases} \partial_t^2 u + (i\nabla + \mathbf{A}(\mathbf{x}))^2 u + \mathbf{V}(\mathbf{x})u + \lambda|u|^{p-1}u = 0, \\ (u, u_t)(0, \mathbf{x}) = (u_0, u_1)(\mathbf{x}), \end{cases} \quad (1.2)$$

with $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^n$ and $\lambda \in \{\pm 1\}$.

Magnetic field: $\mathbf{B} = \operatorname{curl} \mathbf{A}$.

magnetic Schrödinger operator: $(i\nabla + \mathbf{A}(\mathbf{x}))^2 + \mathbf{V}(\mathbf{x})$.

Background

Consider the following initial-value problem for the wave equation

$$\begin{cases} \partial_{tt} u + \mathcal{L}_{\mathbf{A}, a} u = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^2, \\ u(0, x) = f(x), & \partial_t u(0, x) = g(x). \end{cases} \quad (1.3)$$

Here, the operator $\mathcal{L}_{\mathbf{A}, a}$ is defined by

$$\mathcal{L}_{\mathbf{A}, a} = \left(i\nabla + \frac{\mathbf{A}(\hat{x})}{|x|} \right)^2 + \frac{a(\hat{x})}{|x|^2}, \quad (1.4)$$

where $\hat{x} = \frac{x}{|x|} \in \mathbb{S}^1$, $a \in W^{1,\infty}(\mathbb{S}^1, \mathbb{R})$ and $\mathbf{A} \in W^{1,\infty}(\mathbb{S}^1; \mathbb{R}^2)$ satisfies the transversality condition

$$\mathbf{A}(\hat{x}) \cdot \hat{x} = 0, \quad \text{for all } x \in \mathbb{R}^2. \quad (1.5)$$

Two typical examples

Two typical examples are the following:

- the *Aharonov-Bohm*(PRL,59') potential

$$\mathbf{a} \equiv 0, \quad \mathbf{A}(\hat{x}) = \alpha \left(-\frac{x_2}{|x|}, \frac{x_1}{|x|} \right), \quad \alpha \in \mathbb{R}, \quad (1.6)$$

- the *inverse-square* potential

$$\mathbf{A} \equiv 0, \quad a(\hat{x}) \equiv a > 0. \quad (1.7)$$

Friedrichs extension: self-adjoint(Reed-Simon's book)

The symmetrical operator $\mathcal{L}_{\mathbf{A}, \mathbf{a}}^0$

$$\mathcal{L}_{\mathbf{A}, \mathbf{a}}^0 = \left(i\nabla + \frac{\mathbf{A}(\hat{x})}{|x|} \right)^2 + \frac{a(\hat{x})}{|x|^2}, \quad D(\mathcal{L}_{\mathbf{A}, \mathbf{a}}^0) = C_c^\infty(\mathbb{R}^2 \setminus \{0\}).$$

Then, Hardy's inequality

$$\min_{k \in \mathbb{Z}} \{|k - \Phi_{\mathbf{A}}|^2\} \int_{\mathbb{R}^2} \frac{|f|^2}{|x|^2} dx \leq \int_{\mathbb{R}^2} \frac{|(i\partial_\theta + \alpha(\theta))f|^2}{|x|^2} dx \leq \int_{\mathbb{R}^2} |\nabla_A f|^2 dx$$

which shows the positive definite of quadratic form $Q(f)$, i.e.

$$Q(f) = \langle f, \mathcal{L}_{\mathbf{A}, \mathbf{a}}^0 f \rangle = \int_{\mathbb{R}^2} \left(|\nabla_A f|^2 + \frac{a(\hat{x})}{|x|^2} |f|^2 \right) dx \geq 0, \quad \nabla_A = i\nabla + \frac{\mathbf{A}(\hat{x})}{|x|},$$

for $a(\hat{x}) > -\min_{k \in \mathbb{Z}} \{|k - \Phi_{\mathbf{A}}|^2\}$.

The theory of **self-adjoint extensions** now guarantees that there is a unique self-adjoint extension $\mathcal{L}_{\mathbf{A}, \mathbf{a}}$ of $\mathcal{L}_{\mathbf{A}, \mathbf{a}}^0$, whose form domain

$$\mathcal{D}(\mathcal{L}_{\mathbf{A}, \mathbf{a}}) \simeq H_{\mathbf{A}, \mathbf{a}}^1 \subsetneq H^1(\mathbb{R}^2).$$

Hardy's inequality

- Hardy's inequality:

$$\frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{|f|^2}{|x|^2} dx \leq \int_{\mathbb{R}^n} |\partial_r f|^2 dx \leq \int_{\mathbb{R}^n} |\nabla f|^2 dx; \quad (1.8)$$

$$\min_{k \in \mathbb{Z}} \left\{ |k - \Phi_A|^2 \right\} \int_{\mathbb{R}^2} \frac{|f|^2}{|x|^2} dx \leq \int_{\mathbb{R}^2} \frac{|(i\partial_\theta + \alpha(\theta))f|^2}{|x|^2} dx \leq \int_{\mathbb{R}^2} |\nabla_A f|^2 dx,$$

where

$$\Phi_A := \frac{1}{2\pi} \int_0^{2\pi} \alpha(\theta) d\theta, \quad \alpha(\theta) = \mathbf{A}(\theta) \cdot (-\sin \theta, \cos \theta).$$

By using polar coordinates, we can write

$$-\Delta = -\partial_r^2 - \frac{1}{r}\partial_r + \frac{\partial_\theta^2}{r^2}$$

$$\mathcal{L}_{\mathbf{A}, a} = -\partial_r^2 - \frac{1}{r}\partial_r + \frac{(i\partial_\theta + \alpha(\theta))^2 + a(\theta)}{r^2}.$$

For $\theta \in [0, 2\pi]$, define

$$\varphi_k(\theta) = \frac{1}{\sqrt{2\pi}} e^{i[(k - \Phi_A)\theta + \int_0^\theta \alpha(y) dy]}. \quad (1.9)$$

Then, $\{\varphi_k(\theta)\}_{k \in \mathbb{Z}}$ is an **orthonormal system** in $L^2([0, 2\pi])$, and

$$(i\partial_\theta + \alpha(\theta))\varphi_k(\theta) = -(k - \Phi_A)\varphi_k(\theta). \quad (1.10)$$

For $f \in L^2([0, 2\pi])$, $f(\theta) = \sum_{k \in \mathbb{Z}} f_k \varphi_k(\theta)$.

$$\begin{aligned} \int_0^{2\pi} |(i\partial_\theta + \alpha(\theta))f(\theta)|^2 d\theta &= \int_0^{2\pi} \left| \sum_{k \in \mathbb{Z}} (k - \Phi_A) f_k \varphi_k(\theta) \right|^2 d\theta \\ &= \sum_{k \in \mathbb{Z}} \int_0^{2\pi} |(k - \Phi_A) f_k \varphi_k(\theta)|^2 d\theta \\ &\geq \min_{k \in \mathbb{Z}} \{|k - \Phi_A|\}^2 \sum_{k \in \mathbb{Z}} \int_0^{2\pi} |f_k \varphi_k(\theta)|^2 d\theta \\ &= \min_{k \in \mathbb{Z}} \{|k - \Phi_A|\}^2 \int_0^{2\pi} |f(\theta)|^2 d\theta. \end{aligned}$$

引理 1.1

Let $a, \mathbf{A} \in W^{1,\infty}(\mathbb{S}^1, \mathbb{R}^2)$, and $a(\hat{x}) > -\min_{k \in \mathbb{Z}} \{|k - \Phi_{\mathbf{A}}|\}^2$. Then

$$\left\| \mathcal{L}_{\mathbf{A},0}^{\frac{s}{2}} f \right\|_{L^2(\mathbb{R}^2)} := \|f\|_{\dot{H}_{\mathbf{A},0}^s(\mathbb{R}^2)} \simeq \|f\|_{\dot{H}_{\mathbf{A},a}^s(\mathbb{R}^2)} =: \left\| \mathcal{L}_{\mathbf{A},a}^{\frac{s}{2}} f \right\|_{L^2(\mathbb{R}^2)}, \quad \forall s \in [-1, 1]. \quad (1.11)$$

Proof: By duality and interpolation, $s = 1$. By definition,

$$\|f\|_{\dot{H}_{\mathbf{A},a}^1(\mathbb{R}^2)}^2 = \int_{\mathbb{R}^2} \left(|\nabla_{\mathbf{A}} f|^2 + \frac{a(\hat{x})}{|x|^2} |f|^2 \right) dx \quad (1.12)$$

then

$$\int_{\mathbb{R}^2} \left(|\nabla_{\mathbf{A}} f|^2 - \frac{a_-(\hat{x})}{|x|^2} |f|^2 \right) dx \leq \|f\|_{\dot{H}_{\mathbf{A},a}^1(\mathbb{R}^2)}^2 \leq \int_{\mathbb{R}^2} \left(|\nabla_{\mathbf{A}} f|^2 + \frac{a_+(\hat{x})}{|x|^2} |f|^2 \right) dx \quad (1.13)$$

where $a_- := \max\{0, -a\}$ and $a_+ := \max\{0, a\}$.

$$\int_{\mathbb{R}^2} \left(|\nabla_{\mathbf{A}} f|^2 - \frac{a_-(\hat{x})}{|x|^2} |f|^2 \right) dx \geq \left(1 - \frac{\|a_-\|_{L^\infty(\mathbb{S}^1)}}{\min_{k \in \mathbb{Z}} \{|k - \Phi_{\mathbf{A}}|\}^2} \right) \int_{\mathbb{R}^2} |\nabla_{\mathbf{A}} f|^2 dx \geq c \int_{\mathbb{R}^2} |\nabla_{\mathbf{A}} f|^2 dx$$

and

$$\int_{\mathbb{R}^2} \left(|\nabla_{\mathbf{A}} f|^2 + \frac{a_+(\hat{x})}{|x|^2} |f|^2 \right) dx \leq \left(1 + \frac{\|a_+\|_{L^\infty(\mathbb{S}^1)}}{\min_{k \in \mathbb{Z}} \{|k - \Phi_{\mathbf{A}}|\}^2} \right) \int_{\mathbb{R}^2} |\nabla_{\mathbf{A}} f|^2 dx \leq C \int_{\mathbb{R}^2} |\nabla_{\mathbf{A}} f|^2 dx.$$

Representation of kernel: $F(\mathcal{L}_{A,0})$

For $f \in L^2(\mathbb{R}^2)$,

$$f(r, \theta) = \sum_{k \in \mathbb{Z}} f_k(r) \varphi_k(\theta), \quad \varphi_k(\theta) = \frac{1}{\sqrt{2\pi}} e^{i[(k - \Phi_A)\theta + \int_0^\theta \alpha(y) dy]}$$

Then

$$\begin{aligned} \mathcal{L}_{A,0} f(r, \theta) &= \sum_{k \in \mathbb{Z}} \left(-\partial_r^2 - \frac{1}{r} \partial_r + \frac{(i\partial_\theta + \alpha(\theta))^2}{r^2} \right) (f_k(r) \varphi_k(\theta)) \\ &= \sum_{k \in \mathbb{Z}} \varphi_k(\theta) \left(-\partial_r^2 - \frac{1}{r} \partial_r + \frac{(k - \Phi_A)^2}{r^2} \right) f_k(r) \\ &:= \sum_{k \in \mathbb{Z}} \varphi_k(\theta) A_{\nu(k)} f_k(r), \quad \nu(k) := |k - \Phi_A| \\ &= \sum_{k \in \mathbb{Z}} \varphi_k(\theta) \mathcal{H}_{\nu(k)} \left[\rho^2 \mathcal{H}_{\nu(k)} (f_k(r))(\rho) \right] (r), \end{aligned}$$

where the Hankel transform of order ν is defined by

$$(\mathcal{H}_\nu f)(\rho, \theta) = \int_0^\infty J_\nu(r\rho) f(r, \theta) r dr; \quad \mathcal{H}_\nu = \mathcal{H}_\nu^{-1}, \quad \mathcal{H}_\nu(A_\nu \phi)(\xi) = |\xi|^2 (\mathcal{H}_\nu \phi)(\xi)$$

Representation of kernel II: $F(\mathcal{L}_{A,0})$

The functional calculus shows for well-behaved functions F (see Taylor's book)

$$\begin{aligned} F(\mathcal{L}_{A,0})f(r, \theta) &= \sum_{k \in \mathbb{Z}} \varphi_k(\theta) \int_0^\infty F(\rho^2) J_{\nu_k}(r\rho) \mathcal{H}_{\nu(k)}(f_k(r))(\rho) \rho d\rho \\ &= \sum_{k \in \mathbb{Z}} \varphi_k(\theta) \int_0^\infty a_k(r_2) r_2 dr_2 \int_0^\infty F(\rho^2) J_{\nu_k}(r\rho) J_{\nu_k}(r_2\rho) \rho d\rho \\ &= \int_0^\infty \int_0^{2\pi} \left(\sum_{k \in \mathbb{Z}} a_k(r_2) \varphi_k(\theta_2) \right) \cdot \left(\sum_{k \in \mathbb{Z}} \varphi_k(\theta) \overline{\varphi_k(\theta_2)} K_{\nu_k}(r, r_2) \right) d\theta_2 r_2 d\theta \\ &= \int_0^\infty \int_0^{2\pi} f(r_2, \theta_2) \left(\sum_{k \in \mathbb{Z}} \varphi_k(\theta) \overline{\varphi_k(\theta_2)} K_{\nu_k}(r, r_2) \right) r_2 dr_2 d\theta_2 \\ &= \int_0^\infty \int_0^{2\pi} f(r_2, \theta_2) K(r, \theta, r_2, \theta_2) r_2 dr_2 d\theta_2 \end{aligned} \tag{1.14}$$

where

$$K(r, \theta, r_2, \theta_2) = \sum_{k \in \mathbb{Z}} \varphi_k(\theta) \overline{\varphi_k(\theta_2)} K_{\nu_k}(r, r_2),$$

and

$$K_{\nu_k}(r, r_2) = \int_0^\infty F(\rho^2) J_{\nu_k}(r\rho) J_{\nu_k}(r_2\rho) \rho d\rho. \tag{1.15}$$

Heat kernel estimate

Proposition 1 (Heat kernel(L.Fanelli,J. Zhang,Z.20))

Let $x = r_1(\cos \theta_1, \sin \theta_1)$ and $y = r_2(\cos \theta_2, \sin \theta_2)$, then we have the expression of heat kernel

$$\begin{aligned}
 & e^{-t\mathcal{L}_{A,0}}(x, y) \\
 &= \frac{e^{-\frac{|x-y|^2}{4t}}}{t} \frac{e^{i \int_{\theta_1}^{\theta_2} A(\theta') d\theta'}}{2\pi} \left(\chi_{[0,\pi]}(|\theta_1 - \theta_2|) + e^{-i2\pi\alpha} \chi_{[\pi,2\pi]}(|\theta_1 - \theta_2|) \right) \\
 &\quad - \frac{1}{\pi} \frac{e^{-\frac{r_1^2+r_2^2}{4t}}}{t} e^{-i\alpha(\theta_1-\theta_2)+i \int_{\theta_2}^{\theta_1} A(\theta') d\theta'} \int_0^\infty e^{-\frac{r_1 r_2}{2t} \cosh s} \left(\sin(|\alpha|\pi) e^{-|\alpha|s} \right. \\
 &\quad \left. + \sin(\alpha\pi) \frac{(e^{-s} - \cos(\theta_1 - \theta_2 + \pi)) \sinh(\alpha s) - i \sin(\theta_1 - \theta_2 + \pi) \cosh(\alpha s)}{\cosh(s) - \cos(\theta_1 - \theta_2 + \pi)} \right) ds.
 \end{aligned} \tag{1.16}$$

As a consequence, we obtain the heat kernel estimate

$$|e^{-t\mathcal{L}_{A,0}}(x, y)| \lesssim t^{-1} e^{-\frac{|x-y|^2}{4t}} \quad \forall t > 0. \tag{1.17}$$

Keypoint observation

Note that on the line, we have

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \cos(s|k + \alpha|) e^{-ik(\theta_1 - \theta_2)} &= \sum_{k \in \mathbb{Z}} \frac{e^{i(k+\alpha)s} + e^{-i(k+\alpha)s}}{2} e^{-ik(\theta_1 - \theta_2)} \\ &= \frac{1}{2} (e^{-i\alpha s} \delta(\theta_1 - \theta_2 + s) + e^{i\alpha s} \delta(\theta_1 - \theta_2 - s)). \end{aligned}$$

To get $\cos(s\sqrt{L_{A,0}})$ on $\mathbb{R}/(2\pi\mathbb{Z})$, by the method of image, we make the above periodic

$$\begin{aligned} &\cos(s\sqrt{L_{A,0}})\delta(\theta_1 - \theta_2) \\ &= \frac{1}{2} \sum_{j \in \mathbb{Z}} [e^{-is\alpha} \delta(\theta_1 + 2j\pi - \theta_2 + s) + e^{is\alpha} \delta(\theta_1 + 2j\pi - \theta_2 - s)]. \quad (1.18) \end{aligned}$$

Mikhlin Multipliers, Littlewood-Paley theory

定理 1.2 (Mikhlin Multipliers)

Suppose $m : [0, \infty) \rightarrow \mathbb{C}$ satisfies $|\partial^j m(\lambda)| \lesssim \lambda^{-j}$ for all $j \geq 0$ and that $1 < p < \infty$. Then $m(\sqrt{\mathcal{L}_{A,0}})$ extends to a bounded operator on $L^p(\mathbb{R}^2)$.

Let $\phi : [0, \infty) \rightarrow [0, 1]$ be a smooth function such that

$$\phi(\lambda) = 1 \quad \text{for } 0 \leq \lambda \leq 1 \quad \text{and} \quad \phi(\lambda) = 0 \quad \text{for } \lambda \geq 2.$$

For each integer $j \in \mathbb{Z}$, we define

$$\phi_j(\lambda) := \phi(\lambda/2^j) \quad \text{and} \quad \varphi_j(\lambda) := \phi_j(\lambda) - \phi_{j-1}(\lambda).$$

Clearly, $\{\varphi_j(\lambda)\}_{j \in \mathbb{Z}}$ forms a partition of unity for $\lambda \in (0, \infty)$. We define the Littlewood-Paley projections as follows:

$$P_{\leq j} := \phi_j(\sqrt{\mathcal{L}_{A,0}}), \quad P_j := \varphi_j(\sqrt{\mathcal{L}_{A,0}}), \quad \text{and} \quad P_{>j} := I - P_{\leq j}. \quad (1.19)$$

Bernstein inequality, Square function inequality

Proposition 2 (Bernstein inequality)

Let $\{\varphi_j\}_{j \in \mathbb{Z}}$ be a Littlewood-Paley sequence, let $\mathbf{A} \in W^{1,\infty}(\mathbb{S}^1)$ and assume (1.5). Then for $1 < p \leq q < \infty$ or $1 \leq p < q \leq \infty$, there exists constant $C_{p,q}$ depending on p, q such that

$$\left\| \varphi_j(\sqrt{\mathcal{L}_{\mathbf{A},0}})f \right\|_{L^q(\mathbb{R}^2)} \leq C_{p,q} 2^{2j(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^2)}, \quad (1.20)$$

and for $1 \leq p \leq \infty$

$$\left\| \nabla_{\mathbf{A}} \varphi_j(\sqrt{\mathcal{L}_{\mathbf{A},0}})f \right\|_{L^p(\mathbb{R}^2)} \leq C_{p,q} 2^j \|f\|_{L^p(\mathbb{R}^2)}. \quad (1.21)$$

Proposition 3 (Littlewood-Paley square function inequality)

Let $\{\varphi_j\}_{j \in \mathbb{Z}}$ be a Littlewood-Paley sequence. Then for $1 < p < \infty$, there exist constants c_p and C_p depending on p such that

$$c_p \|f\|_{L^p(\mathbb{R}^2)} \leq \left\| \left(\sum_{j \in \mathbb{Z}} |\varphi_j(\sqrt{\mathcal{L}_{\mathbf{A},0}})f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} \leq C_p \|f\|_{L^p(\mathbb{R}^2)}. \quad (1.22)$$



Dispersive/Strichartz estimate(Propagation of regularity)

From the explicit formula

$$e^{it\Delta} f(x) = \frac{1}{(4i\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{i\frac{|x-y|^2}{4t}} f(y) dy,$$

we get the standard **dispersive inequality**

$$\|e^{it\Delta} f\|_{L_x^\infty(\mathbb{R}^n)} \lesssim |t|^{-\frac{n}{2}} \|f\|_{L_x^1(\mathbb{R}^n)}, \quad \forall t \neq 0. \quad (1.23)$$

This will yield Strichartz estimate ($\frac{2}{q} + \frac{d}{r} = \frac{n}{2}$ and $(q, r, n) \neq (2, \infty, 2)$)

$$\|e^{it\Delta} f\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^n)} \leq C \|f\|_{L_x^2} \quad (1.24)$$

Indeed, by TT^* -argument, it is equivalent to

$$\left\| \int_{\mathbb{R}} e^{i(t-s)\Delta} F(s, x) ds \right\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^n)} \lesssim \left\| \int_{\mathbb{R}} |t-s|^{-\frac{n}{2}} \|F(s, x)\|_{L_x^{r'}(\mathbb{R}^n)} ds \right\|_{L_t^q(\mathbb{R})} \lesssim \|F(t, x)\|_{L_t^{q'} L_x^{r'}(\mathbb{R} \times \mathbb{R}^n)}.$$

Remark: $f(x) \in L_x^2(\mathbb{R}^n) \implies e^{it\Delta} f(x) \in L_x^r(\mathbb{R}^n)$ ($2 \leq r \leq \frac{2n}{n-2}$), a.e. $t \in \mathbb{R}$.

Strichartz estimates for wave

wave equation:

$$\partial_{tt} u - \Delta u = 0, \quad u(0) = u_0, \quad \partial_t u(0) = u_1.$$

- Strichartz, Ginibre-Velo, Smith-Sogge, Keel-Tao ⋯

$$\|\psi(\sqrt{-\Delta})e^{it\sqrt{-\Delta}}\|_{L_x^1(\mathbb{R}^n) \rightarrow L_x^\infty(\mathbb{R}^n)} \leq c|t|^{-\frac{n-1}{2}}. \quad (1.25)$$

- Strichartz estimates: for any $(q, r) \in \Lambda_s$

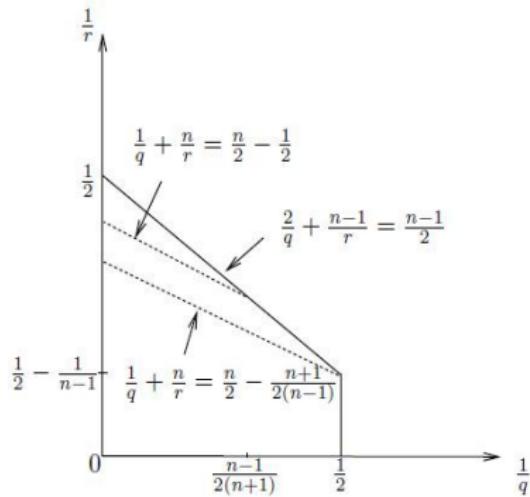
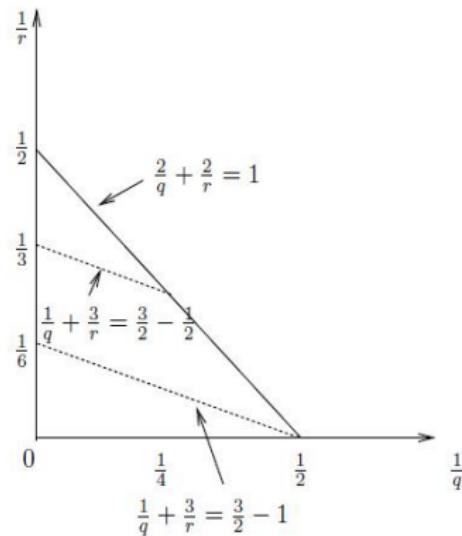
$$\|u(t, x)\|_{L_t^q L_x^r(I \times \mathbb{R}^n)} \leq C(\|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}}),$$

For $s \in \mathbb{R}$, we say the pair $(q, r) \in \Lambda_s$ if $(q, r) \in [2, \infty]$ satisfies

$$\frac{2}{q} + \frac{n-1}{r} \leq \frac{n-1}{2}, \quad (q, r, n) \neq (2, \infty, 3). \quad (1.26)$$

and

$$\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - s = \frac{1}{\tilde{q}'} + \frac{n}{\tilde{r}'} - 2. \quad (1.27)$$

wave-admissible pair in \mathbb{R}^n FIGURE 1. $n \geq 4$.FIGURE 2. $n = 3$.

Abstract Strichartz estimates I

Let $(X, d\mu)$ be a measure space and H a Hilbert space.
 Suppose that for each time $t \in \mathbb{R}$, $U(t) : H \rightarrow L^2(X)$ which
 satisfies the energy estimate

$$\|U(t)\|_{H \rightarrow L^2} \leq C, \quad t \in \mathbb{R} \quad (1.28)$$

and that for some $\sigma > 0$ either

$$\|U(t)U(s)^*f\|_{L^\infty} \leq C|t-s|^{-\sigma}\|f\|_{L^1}, \quad t \neq s. \quad (1.29)$$

or

$$\|U(t)U(s)^*f\|_{L^\infty} \leq C(1 + |t-s|)^{-\sigma}\|f\|_{L^1}. \quad (1.30)$$

Abstract Strichartz estimates II

We say that the exponent pair (q, r) is σ -admissible, if $(q, r) \in [2, \infty]^2$ such that $(q, r, \sigma) \neq (2, \infty, 1)$ and

$$\frac{1}{q} + \frac{\sigma}{r} \leq \frac{\sigma}{2}, \quad (1.31)$$

If the equality holds, (q, r) is said to be sharp σ -admissible, otherwise (q, r) is non-sharp σ -admissible. Note when $\sigma > 1$, the endpoint

$$P = \left(2, \frac{2\sigma}{\sigma - 1}\right) \quad (1.32)$$

is sharp σ -admissible. For wave $\sigma = \frac{n-1}{2}$; for Schrödinger $\sigma = \frac{n}{2}$.

Abstract Strichartz estimates III

Theorem 1 (Keel-Tao)

If $U(t)$ satisfies (1.28) and (1.29), then the estimates

$$\|U(t)f\|_{L_t^q L_z^r} \lesssim \|f\|_H, \quad (1.33)$$

$$\left\| \int_{\mathbb{R}} (U(s))^* F(s) ds \right\|_H \lesssim \|F\|_{L_t^{q'} L_z^{r'}}, \quad (1.34)$$

$$\left\| \int_{s < t} U(t) (U(s))^* F(s) ds \right\|_{L_t^q L_z^r} \lesssim \|F\|_{L_t^{\tilde{q}'} L_z^{\tilde{r}'}} \quad (1.35)$$

hold for all sharp σ -admissible pairs $(q, r), (\tilde{q}, \tilde{r})$. Moreover, if the decay is strengthened to (1.30), then the above estimates hold for all σ -admissible pairs $(q, r), (\tilde{q}, \tilde{r})$.

Dispersive/Strichartz estimate perturbated by potential

$$iu_t - \Delta u + V(x)u = 0.$$

smoothness and decay of V has a great influence.

- $H = -\Delta + V(x)$, $V(x)$ less singular, Kato class; Schlag(07), Schlag-Soffer-Staubach(10), etc.
- $V(x) = a|x|^{-2}$ and $a < 0$, the classical **dispersive estimate** for the wave equation does not hold, see Goldberg- Vega-Visciglia, Duyckaerts. Fanelli-Felli-Fontelos-Primo (13, $d = 3$)

$$e^{it\mathcal{L}_a} f(x) = \frac{e^{\frac{i|x|^2}{4t}}}{i(2t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{\frac{i|y|^2}{4t}} f(y) K\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right) dy, \quad (1.36)$$

$$K(x, y) = (|x| \cdot |y|)^{-\frac{d-2}{2}} \sum_{k=0}^{\infty} \sum_{m=1}^{m(k)} e^{-\frac{1}{2}\nu(k)\pi i} J_{\nu_k}(|x| \cdot |y|) Y_{k,m}\left(\frac{x}{|x|}\right) Y_{k,m}\left(\frac{y}{|y|}\right).$$

- $V(x) = a|x|^{-2}$: Burq, Planchon, Visciglia ... (perturbation method)

Strichartz estimate, Burq-Planchon-Stalker-Zadeh, 03

$$\|e^{-it\mathcal{L}_a} f\|_{L_t^q L_x^r} \leq C \|f\|_{L_x^2}. \quad (1.37)$$

Since $u(t, x) := e^{-it\mathcal{L}_a} f$ solves

$$i\partial_t u + \Delta u = \frac{a}{|x|^2} u, \quad u(0, x) = f(x),$$

we have

$$u(t, x) = e^{it\Delta} f + ia \int_0^t e^{i(t-s)\Delta} (|x|^{-2} u)(s) ds.$$

Then,

$$\begin{aligned} \left\| \int_0^t e^{i(t-s)\Delta} (|x|^{-2} u)(s) ds \right\|_{L_t^2 L_x^{\frac{2n}{n-2}}} &\leq \left\| \int_0^t e^{i(t-s)\Delta} (|x|^{-2} u)(s) ds \right\|_{L_t^2 L_x^{\frac{2n}{n-2}, 2}} \\ &\leq C \| |x|^{-2} u \|_{L_t^2 L_x^{\frac{2n}{n+2}, 2}} \leq C \| |x|^{-1} \|_{L_x^{n, \infty}} \| |x|^{-1} u \|_{L_{t,x}^2} \leq C \| f \|_{L_x^2}, \end{aligned}$$

where we use local smoothing estimate ($n \geq 3$)

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} \frac{|u(t, x)|^2}{|x|^2} dx dt \leq C \|f\|_{L_x^2}^2.$$



Main result I: $a = 0$ Dispersive estimate

$$u_{tt} + \left(i\nabla + \frac{\mathbf{A}(\hat{x})}{|x|} \right)^2 u = 0.$$

The unique solution to (1.3) with $a = 0$ can be represented by

$$u(t, \cdot) = \cos(t \sqrt{\mathcal{L}_{\mathbf{A},0}}) f(\cdot) + \frac{\sin(t \sqrt{\mathcal{L}_{\mathbf{A},0}})}{(\sqrt{\mathcal{L}_{\mathbf{A},0}})} g(\cdot). \quad (1.38)$$

定理 1.3 (Dispersive estimate: $a = 0$, Fanelli-Zhang-Z,20)

Let $\mathbf{A} \in W^{1,\infty}(\mathbb{S}^1, \mathbb{R}^2)$. There exists a constant $C > 0$ such that, for any

$$\left\| \frac{\sin(t \sqrt{\mathcal{L}_{\mathbf{A},0}})}{\sqrt{\mathcal{L}_{\mathbf{A},0}}} f \right\|_{L^\infty(\mathbb{R}^2)} \leq C |t|^{-\frac{1}{2}} \|f\|_{\dot{\mathcal{B}}_{1,1,\mathbf{A}}^{1/2}(\mathbb{R}^2)}. \quad (1.39)$$

Main result II: $a \neq 0$, Strichartz estimate

The unique solution to (1.3) can be represented by

$$u(t, \cdot) = \cos(t \sqrt{\mathcal{L}_{\mathbf{A},a}})f(\cdot) + \frac{\sin(t \sqrt{\mathcal{L}_{\mathbf{A},a}})}{(\sqrt{\mathcal{L}_{\mathbf{A},a}})}g(\cdot). \quad (1.40)$$

定理 1.4 (Strichartz estimate, Fanelli-Zhang-Z, 20)

Let $a, \mathbf{A} \in W^{1,\infty}(\mathbb{S}^1, \mathbb{R}^2)$, assume

$$a > -\min_{k \in \mathbb{Z}} \{|k - \Phi_{\mathbf{A}}|\}^2.$$

Then there exists a constant C such that

$$\|u\|_{L_t^q(\mathbb{R}; L^r(\mathbb{R}^2))} \leq C \left(\|f\|_{\dot{H}_{\mathbf{A},0}^s(\mathbb{R}^2)} + \|g\|_{\dot{H}_{\mathbf{A},0}^{s-1}(\mathbb{R}^2)} \right), \quad (1.41)$$

for any $s \in \mathbb{R}$ and $(q, r) \in \Lambda_s^W$ given by

$$\Lambda_s^W := \left\{ (q, r) \in [2, \infty] \times [2, \infty), \frac{2}{q} + \frac{1}{r} \leq \frac{1}{2}, s = 2 \left(\frac{1}{2} - \frac{1}{r} \right) - \frac{1}{a} \right\}, \quad s \in \mathbb{R}.$$

Representation of fundamental solution

First, we recall Poisson's formula

$$\frac{\sin(t\sqrt{-\Delta_{\mathbb{R}^2}})}{\sqrt{-\Delta_{\mathbb{R}^2}}} f(x) = \frac{1}{2\pi} \int_{B(x,t)} \frac{f(y)}{\sqrt{t^2 - |y-x|^2}} dy, \quad t > 0, \quad (2.1)$$

which implies

$$\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}(x, y) = \frac{1}{2\pi} \frac{\chi_{B(x,t)}(y)}{\sqrt{t^2 - |y-x|^2}}.$$

By using polar coordinates,

$$x = r_1(\cos \theta_1, \sin \theta_1), \quad y = r_2(\cos \theta_2, \sin \theta_2),$$

we get

$$|x-y|^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2).$$

Hence,

$$\begin{aligned} & \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} f(x) \\ &= \int_0^\infty \int_0^{2\pi} \frac{f(r_2, \theta_2)}{\sqrt{t^2 - (r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2))}} \chi\left\{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2) < t^2\right\} d\theta_2 r_2 dr_2 \end{aligned}$$

Decomposition: three regions

Region I: $t < |r_1 - r_2|$. In this region, note that

$$|r_1 - r_2|^2 < r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2),$$

we obtain

$$K(t, r_1, \theta_1, r_2, \theta_2) = 0.$$

Region II: $|r_1 - r_2| < t < r_1 + r_2$. Set

$$\beta_1 := \arccos \frac{r_1^2 + r_2^2 - t^2}{2r_1 r_2} \in [0, \pi].$$

Note that

$$|\theta_1 - \theta_2| \in [0, 2\pi],$$

and when $\beta_1 < |\theta_1 - \theta_2| < 2\pi - \beta_1$, we have

$$r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2) > t.$$

$$K(t, r_1, \theta_1, r_2, \theta_2) = \begin{cases} 0 & \text{if } \beta_1 < |\theta_1 - \theta_2| < 2\pi - \beta_1 \\ [t^2 - (r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2))]^{-\frac{1}{2}} & \text{if else.} \end{cases} \quad (2.2)$$

Region III: $t > r_1 + r_2$. In this region, note that

$$t^2 > (r_1 + r_2)^2 \geq r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2),$$

we easily get

$$K(t, r_1, \theta_1, r_2, \theta_2) = [t^2 - (r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2))]^{-\frac{1}{2}}.$$

Now, consider the kernel

$$\begin{aligned} K_\nu(t, r_1, r_2) &= \int_0^\infty \frac{\sin(t\rho)}{\rho} J_\nu(r_1\rho) J_\nu(r_2\rho) \rho d\rho \\ &= \begin{cases} 0, & \text{if } (t, r_1, r_2) \in I \\ \frac{1}{\pi} \int_0^{\beta_1} \frac{\cos(\nu s)}{\sqrt{t^2 - r_1^2 - r_2^2 + 2r_1 r_2 \cos(s)}} ds & \text{if } (t, r_1, r_2) \in II \\ \frac{\cos(\pi\nu)}{\pi} \int_{\beta_2}^\infty \frac{e^{-s\nu}}{\sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cosh(s) - t^2}} ds & \text{if } (t, r_1, r_2) \in III \end{cases} \end{aligned} \quad (2.3)$$

Representation of fundamental solution($a = 0$)

$$u(t, \cdot) = \cos(t \sqrt{\mathcal{L}_{A,0}})f(\cdot) + \frac{\sin(t \sqrt{\mathcal{L}_{A,0}})}{(\sqrt{\mathcal{L}_{A,0}})}g(\cdot). \quad (2.4)$$

Proposition 4

Let $K(t, x, y)$ be the Schwartz kernel of the operator $\frac{\sin(t \sqrt{\mathcal{L}_{A,0}})}{\sqrt{\mathcal{L}_{A,0}}}$. Suppose $x = r_1(\cos \theta_1, \sin \theta_1)$ and $y = r_2(\cos \theta_2, \sin \theta_2)$ and define

$$\gamma = \frac{r_1^2 + r_2^2 - t^2}{2r_1 r_2} = \frac{(r_1 + r_2)^2 - t^2}{2r_1 r_2} - 1 = \frac{(r_1 - r_2)^2 - t^2}{2r_1 r_2} + 1 \quad (2.5)$$

and

$$\beta_1 = \cos^{-1}\left(\frac{r_1^2 + r_2^2 - t^2}{2r_1 r_2}\right), \quad \beta_2 = \cosh^{-1}\left(\frac{t^2 - r_1^2 - r_2^2}{2r_1 r_2}\right). \quad (2.6)$$

Then when $t \geq 0$, the kernel can be written as

$$K(t, x, y) = K(t, r_1, \theta_1, r_2, \theta_2) = G_w(t, r_1, \theta_1, r_2, \theta_2) + D_w(t, r_1, \theta_1, r_2, \theta_2) \quad (2.7)$$



“geometric” term

$$\begin{aligned}
 & G_w(t, r_1, \theta_1, r_2, \theta_2) \\
 &= \frac{1}{2\pi} \left(t^2 - (r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)) \right)^{-1/2} e^{i \int_{\theta_2}^{\theta_1} \alpha(\theta') d\theta'} \\
 &\quad \times \left\{ \chi_{(|r_1 - r_2|, r_1 + r_2)}(t) \left[\chi_{[0, \beta_1]}(|\theta_1 - \theta_2|) + e^{-2\alpha\pi i} \chi_{[2\pi - \beta_1, 2\pi]}(|\theta_1 - \theta_2|) \right] \right. \\
 &\quad \left. + \chi_{(r_1 + r_2, \infty)}(t) \left[\chi_{[0, \pi]}(|\theta_1 - \theta_2|) + e^{-2\alpha\pi i} \chi_{[\pi, 2\pi]}(|\theta_1 - \theta_2|) \right] \right\} \tag{2.8}
 \end{aligned}$$

and “diffractive” term

$$\begin{aligned}
 D_w(t, r_1, \theta_1, r_2, \theta_2) &= \frac{\chi_{(r_1 + r_2, \infty)}(t)}{\pi} e^{-i \left(\alpha(\theta_1 - \theta_2) - \int_{\theta_2}^{\theta_1} \alpha(\theta') d\theta' \right)} \\
 &\quad \times \int_0^{\beta_2} \left(t^2 - r_1^2 - r_2^2 - 2r_1 r_2 \cosh s \right)^{-1/2} \left(\sin(|\alpha|\pi) e^{-|\alpha|s} \right. \\
 &\quad \left. + \sin(\alpha\pi) \frac{(e^{-s} - \cos(\theta_1 - \theta_2 + \pi)) \sinh(\alpha s) + i \sin(\theta_1 - \theta_2 + \pi) \cosh(\alpha s)}{\cosh(s) - \cos(\theta_1 - \theta_2 + \pi)} \right) ds. \tag{2.9}
 \end{aligned}$$

When $t \leq 0$, the similar conclusion hold for (2.8) and (2.9) with replacing t by $-t$.

Remarks 1

If $\alpha = 0$, the Diffractive term D vanishes and the geometric term G consists with the fundamental solution of wave equation without potential

$$\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}(x, y) = \frac{1}{2\pi} H(t^2 - |x - y|^2)(t^2 - |x - y|^2)^{-1/2} \quad (2.10)$$

where H is the Heaviside step function on \mathbb{R} .

As a consequence of Proposition 5,

$$|G_w(t, r_1, \theta_1, r_2, \theta_2)| \lesssim \frac{1}{\sqrt{t^2 - |x - y|^2}}, \quad t^2 > |x - y|^2 \quad (2.11)$$

and

$$|D_w(t, r_1, \theta_1, r_2, \theta_2)| \lesssim \frac{1}{\sqrt{t^2 - (r_1 + r_2)^2}}, \quad t^2 > (r_1 + r_2)^2. \quad (2.12)$$

Localized-frequency decay estimate

Proposition 5

Let $\varphi \in C_c^\infty(\mathbb{R} \setminus \{0\})$, with $0 \leq \varphi \leq 1$, and $\text{supp } \varphi \subset [1/2, 2]$. Then for all $j \in \mathbb{Z}$, there exists a constant C independent of x, y and t such that

$$\left\| \frac{\sin(t\sqrt{\mathcal{L}_{A,0}})}{\sqrt{\mathcal{L}_{A,0}}} f \right\|_{L^\infty(\mathbb{R}^2)} \leq C 2^{\frac{j}{2}} (2^{-j} + |t|)^{-\frac{1}{2}} \|f\|_{L^1(\mathbb{R}^2)}, \quad (2.13)$$

where $f = \varphi(2^{-j}\sqrt{\mathcal{L}_{A,0}})f$.

$$\left\| \int_{|x-y|^2 \leq t^2} \frac{|f(y)|}{\sqrt{t^2 - |x-y|^2}} dy \right\|_{L^\infty(\mathbb{R}^2)} \leq C(1+|t|)^{-\frac{1}{2}} \|f\|_{L^1}, \quad (2.14)$$

and

$$\left\| \int_{(r_1+r_2)^2 \leq t^2} \int_0^{2\pi} \frac{|f(r_2, \theta_2)|}{\sqrt{t^2 - (r_1+r_2)^2}} r_2 dr_2 d\theta_2 \right\|_{L^\infty(\mathbb{R}^2)} \leq C(1+|t|)^{-\frac{1}{2}} \|f\|_{L^1}.$$

Strichartz estimates for purely magnetic waves

Proposition 6

Let $U(t) = e^{it\sqrt{\mathcal{L}_{A,0}}}$ and $f = \varphi_j(\sqrt{\mathcal{L}_{A,0}})f$ for $j \in \mathbb{Z}$, then

$$\|U(t)f\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^2)} \lesssim 2^{js} \|f\|_{L^2(\mathbb{R}^2)}, \quad (2.16)$$

where $s \in \mathbb{R}$ and

$$(q, r) \in \Lambda_s^W = \left\{ (q, r) \in [2, \infty] \times [2, \infty), \frac{2}{q} + \frac{1}{r} \leq \frac{1}{2}, s = 2\left(\frac{1}{2} - \frac{1}{r}\right) - \frac{1}{q} \right\}.$$

As a consequence,

$$\|e^{it\sqrt{\mathcal{L}_{A,0}}} f\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^2)} \lesssim \|f\|_{\dot{H}_{A,0}^s(\mathbb{R}^2)}, \quad (2.17)$$

for $s \in \mathbb{R}$, any wave-admissible pair $(q, r) \in \Lambda_s^W$.

Local smoothing for wave associated with $\mathcal{L}_{\mathbf{A},a}$

Proposition 7 (Local smoothing estimate)

There exists a constant $C > 0$ such that,

$$\|r^{-\beta} e^{it\sqrt{\mathcal{L}_{\mathbf{A},a}}} f\|_{L_t^2(\mathbb{R}; L^2(\mathbb{R}^2))} \leq C \|f\|_{\dot{H}_{\mathbf{A},a}^{\beta-\frac{1}{2}}}, \quad (2.18)$$

for any $\beta \in (\frac{1}{2}, 1 + \nu_0)$.

Strichartz estimate for electromagnetic wave

$$\begin{aligned} \|u(t, x)\|_{L^\infty(\mathbb{R}; L^r(\mathbb{R}^2))} &\lesssim \|\mathcal{L}_{\mathbf{A}, a}^{\frac{s}{2}} u(t, x)\|_{L^\infty(\mathbb{R}; L^2(\mathbb{R}^2))} \\ &\lesssim \|f\|_{\dot{H}_{\mathbf{A}, a}^s(\mathbb{R}^2)} + \|g\|_{\dot{H}_{\mathbf{A}, a}^{s-1}(\mathbb{R}^2)} \end{aligned}$$

$$u(t, \cdot) = \cos(t \sqrt{\mathcal{L}_{\mathbf{A}, 0}}) f(\cdot) + \frac{\sin(t \sqrt{\mathcal{L}_{\mathbf{A}, 0}})}{(\sqrt{\mathcal{L}_{\mathbf{A}, 0}})} g(\cdot) - \int_0^t \frac{\sin(t - \tau) \sqrt{\mathcal{L}_{\mathbf{A}, 0}}}{\sqrt{\mathcal{L}_{\mathbf{A}, 0}}} \left(\frac{a(\hat{x})}{|x|^2} u(\tau, \cdot) \right) d\tau.$$

定理 2.1 (Strichartz estimate, L. Fanelli, J. Zhang, Z.20)

There exists a constant C such that

$$\|u\|_{L_t^q(\mathbb{R}; L^r(\mathbb{R}^2))} \leq C \left(\|f\|_{\dot{H}_{\mathbf{A}, 0}^s(\mathbb{R}^2)} + \|g\|_{\dot{H}_{\mathbf{A}, 0}^{s-1}(\mathbb{R}^2)} \right), \quad (2.19)$$

for any $s \in \mathbb{R}$ and $(q, r) \in \Lambda_s^W$.



Equivalence of Sobolev norms

$$\mathcal{L}_a = -\Delta + \frac{a}{|x|^2}, \quad a > -\frac{(n-2)^2}{4}, \quad n \geq 3.$$

定理 2.2 (Equivalence of Sobolev norms, Killip-Miao-Visan-Zhang-Z,18, Math.Z)

Suppose $d \geq 3$, $a \geq -(\frac{n-2}{2})^2$, and $0 < s < 2$. If $1 < p < \infty$ satisfies $\frac{s+\sigma}{n} < \frac{1}{p} < \min\{1, \frac{n-\sigma}{n}\}$, then

$$\left\| (-\Delta)^{\frac{s}{2}} f \right\|_{L^p} \lesssim_{n,p,s} \left\| \mathcal{L}_a^{\frac{s}{2}} f \right\|_{L^p} \quad \text{for all } f \in C_c^\infty(\mathbb{R}^n \setminus \{0\}). \quad (2.20)$$

If $\max\{\frac{s}{n}, \frac{\sigma}{n}\} < \frac{1}{p} < \min\{1, \frac{n-\sigma}{n}\}$, which ensures already that $1 < p < \infty$, then

$$\left\| \mathcal{L}_a^{\frac{s}{2}} f \right\|_{L^p} \lesssim_{n,p,s} \left\| (-\Delta)^{\frac{s}{2}} f \right\|_{L^p} \quad \text{for all } f \in C_c^\infty(\mathbb{R}^n \setminus \{0\}). \quad (2.21)$$

Uniform Sobolev inequality

Kenig-Ruiz-Sogge (87') showed

$$\|(-\Delta - \sigma)^{-1} f\|_{L^{q,2}(\mathbb{R}^d)} \leq C|\sigma|^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})-1} \|f\|_{L^{p,2}(\mathbb{R}^d)}, \quad \sigma \in \mathbb{C} \setminus \mathbb{R}^+, f \in C_0^\infty(\mathbb{R}^d), \quad (2.22)$$

where $d \geq 3$ and (p, q) satisfies

$$\frac{2}{d+1} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{2}{d}, \quad \frac{2d}{d+3} < p < \frac{2d}{d+1}, \quad \frac{2d}{d-1} < q < \frac{2d}{d-3}. \quad (2.23)$$

定理 2.3 (Uniform Sobolev inequality, Mizutani-Zhang-Z,2020,JFA)

Let $\nu_0 = \sqrt{a + \frac{(d-2)^2}{4}}$, $\mu_0 = \frac{1}{2}$ when $\nu_0 \geq 1/2$ and $\mu_0 = \frac{\nu_0^2}{1-2\nu_0^2}$ for $0 < \nu_0 < 1/2$, and

$$\frac{2}{d+1} \leq \frac{1}{p} - \frac{1}{q} \leq \frac{2}{d}, \quad \frac{2d}{d+2(1+\mu_0)} < p < \frac{2d}{d+1}, \quad \frac{2d}{d-1} < q < \frac{2d}{d-2(1+\mu_0)}. \quad (2.24)$$

Then there exists a positive constant C such that

$$\|(\mathcal{L}_a - \sigma)^{-1} f\|_{L^{q,2}(\mathbb{R}^d)} \leq C|\sigma|^{\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-1} \|f\|_{L^{p,2}(\mathbb{R}^d)}, \quad \sigma \notin \mathbb{R}^+, f \in C_0^\infty(\mathbb{R}^d).$$



Strichartz estimate: metric cone (M, g)

定理 2.4 (Zhang-Z., Math. Ann., 2020)

Let $d \geq 3$, suppose $u : I \times M \rightarrow \mathbb{R}$ is a solution to $(\partial_t^2 + \mathcal{L}_a)u = F$. Then, the Strichartz estimate

$$\|u\|_{L_t^q L_x^r(\mathbb{R} \times M)} \lesssim \|u_0\|_{\dot{H}^s(\mathbb{R}^d)} + \|u_1\|_{\dot{H}^{s-1}(M)} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(I \times M)}, \quad (2.25)$$

holds for all $(q, r), (\tilde{q}, \tilde{r}) \in \Lambda_{s, \nu_0}$ where $\nu_0^2 = a + \frac{(d-2)^2}{4}$ and

$$\Lambda_{s, \nu_0} = \{(q, r) \in \Lambda_s : 1/r > 1/2 - (1 + \nu_0)/d\}. \quad (2.26)$$

Moreover, the requirement (2.26) is sharp.

The set Λ_{s, ν_0} is different from Λ_s . If $s \notin [0, 1 + \nu_0]$, the set Λ_{s, ν_0} is empty. It is easy to check that

$$\Lambda_{s, \nu_0} = \Lambda_s \quad \text{for } s \in [0, 1/2 + \nu_0)$$

and while $\Lambda_{s, \nu_0} \subset \Lambda_s$ for $s \in [1/2 + \nu_0, 1 + \nu_0)$.

Main result : Defocusing case

Let

$$c_d = \begin{cases} \frac{1}{25} & d = 3, \\ \frac{1}{9} & d = 4. \end{cases} \quad (2.27)$$

定理 2.5 (Defocusing, Miao-Murphy-Z,2020, AIH Poincaré-NA)

Let $d \in \{3, 4\}$, $a > -(\frac{d-2}{2})^2 + c_d$, and $\mu = +1$. For any $(u_0, u_1) \in \dot{H}^1 \times L^2$, the corresponding solution to

$$\begin{cases} \partial_t^2 u - \Delta u + \frac{a}{|x|^2} u + |u|^{\frac{4}{d-2}} u = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\ (u, u_t)(0, x) = (u_0, u_1)(x) \in \dot{H}^1 \times L^2(\mathbb{R}^d). \end{cases} \quad (2.28)$$

is **global** and **scatters** in the sense that there exist solutions $v_{\pm}(t)$ to $(\partial_t^2 + \mathcal{L}_a)v_{\pm} = 0$ such that

$$\lim_{t \rightarrow \pm\infty} \|\vec{u}(t) - \vec{v}_{\pm}(t)\|_{\dot{H}^1 \times L^2} = 0.$$



Thanks for your attention!