Existence and nonexistence of nontrivial solutions for critical biharmonic equations

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Based on a joint work with Zongyan Lv

Problem

$$
\begin{cases} \Delta^2 u = \mu \Delta u + \lambda u + |u|^{2^{**}-2} u, & x \in \Omega, \\ u|_{\partial\Omega} = \frac{\partial u}{\partial n}\big|_{\partial\Omega} = 0, \end{cases} \tag{1}
$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $\Delta^2 = \Delta\Delta$ denotes the iterated N-dimensional Laplacian, $2^{**} = \frac{2N}{N-4}$ $(N > 4)$ is the critical Sobolev exponent for the embedding $H_0^2(\Omega) \hookrightarrow L^{2^{**}}(\Omega)$ and $H_0^2(\Omega)$ is the closure of $C_0^\infty(\Omega)$ under the norm $||\Delta u||_{L^2(\Omega)}$.
The functional corresponding to (1) is

$$
I(u) = \frac{1}{2} \int_\Omega (|\Delta u|^2 + \mu |\nabla u|^2 - \lambda u^2) dx - \frac{1}{2^{**}} \int_\Omega |u|^{2^{**}} dx, \ u \in H_0^2(\Omega).
$$

Outline of Talk

- **Some related results**
- Main results
- Idea of proof of our main results

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$$\begin{cases} -\Delta u = \lambda u + u^{2^*-1}, & x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{2}$$

- H. Brézis, L. Nirenberg, Comm. Pure Appl. Math. 1983;
- Surprisingly, they discovered that the cases of $N = 3$ and $N \geq 4$ are quite different.
- Namely (denoting here by $\lambda_1(\Omega)$, the first eigenvalue of $-\Delta$ with zero Dirichlet boundary condition on Ω), when $N \geq 4$ nontrivial positive solutions exist if and only if $\lambda \in (0, \lambda_1(\Omega))$, while, when $N = 3$ and Ω is a ball, nontrivial positive solutions exist only for $\lambda \in (\frac{1}{4}\lambda_1(\Omega), \lambda_1(\Omega))$, which implies that $N = 3$ is a critical dimension of positive solution for (2).
- They also showed that if $\lambda \leq 0$ and Ω is a starshaped domain, then there is no solutions for (2).

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- Gu, Deng and Wang (Systems Sci. Math. Sci., 1994)

$$\begin{cases} \Delta^2 u = \lambda u + |u|^{2^{**}-2}u, & x \in \Omega, \\ u|_{\partial\Omega} = \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0, & \lambda > 0, \end{cases} \tag{3}$$

where $\delta_1(\Omega)$ denotes the first eigenvalue of $-\Delta^2$ with homogeneous Dirichlet boundary condition on Ω.

- (1) For $N \geq 8$, problem (3) possesses at least one nontrivial weak solutions if $\lambda \in (0, \delta_1(\Omega))$;
- (2) For $N = 5, 6, 7$ and $\Omega = B_R(0) \subset \mathbb{R}^N$, there exist two positive constants $\lambda^{**}(N) < \lambda^*(N) < \delta_1(\Omega)$ such that problem (3) has at least one nontrivial weak solutions if $\lambda \in (\lambda^*(N), \delta_1(\Omega))$, and problem (3) has no nontrivial solutions if $\lambda < \lambda^{**}(N)$;
- What has shown above implies that $N = 5, 6, 7$ are the critical dimensions of nontrivial solutions for (3).

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Main results

Theorem 1

- Problem (1) possesses at least one nontrivial weak solution, provided one of the following assumptions holds:

- (1) $N \geq 5$, $\mu = 0$ and $\lambda \in (\lambda^*(N), \delta_1(\Omega))$;

- (2) $N \geq 6$, $\mu \in (-\beta(\Omega), 0)$ and $\lambda < \frac{(\mu+\beta(\Omega))\delta_1(\Omega)}{\beta(\Omega)}$ (See Figure 1.(a));

- (3) $N = 5$, $(\lambda, \mu) \in A \cap B$, where $A := \{(\lambda,\mu) | \lambda \in (-\infty, \delta_1(\Omega)), \max\{-\beta(\Omega), \frac{\beta(\Omega)}{\delta_1(\Omega)}\lambda - \beta(\Omega)\} < \mu\}$, $B := \{(\lambda,\mu) | \mu < 0.0317\lambda - 11.8681\}$ (See Figure 1.(b) or 1.(c)).

其中, $\beta(\Omega) := \inf\limits_{u \in H_0^2(\Omega)\setminus\{0\}} \frac{\int_\Omega |\Delta u|^2 dx}{\int_\Omega |\nabla u|^2 dx}$.

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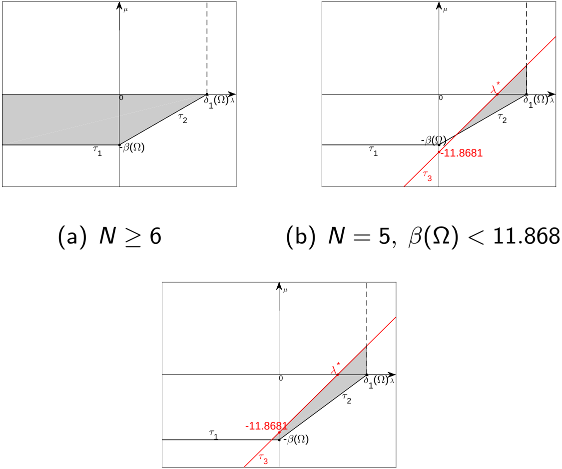
There is at least one nontrivial solution of (1) for λ, μ in dash area.

$\tau_1 : \mu = -\beta(\Omega)$, $\tau_2 : \mu = \frac{\beta(\Omega)}{\delta_1(\Omega)}\lambda - \beta(\Omega)$, $\tau_3 : \mu = 0.0317\lambda - 11.8681$.



(a) $N \geq 6$ (b) $N = 5$, $\beta(\Omega) < 11.8681$

(c) $N = 5$, $\beta(\Omega) > 11.8681$

- Different from the case $\mu = 0$, $N = 6, 7$ are not the critical dimensions of nontrivial solutions when $\mu \in (-\beta(\Omega), 0)$.

Theorem 2

- There are no nontrivial solutions of (1) in $H_0^2(\Omega) \bigcap C^4(\Omega)$ for $\mu > \max\{0, \frac{2}{\lambda_1(\Omega)}\lambda\}$ if Ω is a starshaped domain.

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Idea of proof of Theorem 1

- We define

$$\|u\|_1^2 = \int_\Omega (|\Delta u|^2 + \mu|\nabla u|^2 - \lambda|u|^2)dx,$$

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$$\beta(\Omega) := \inf_{u\in H_0^2(\Omega)\setminus\{0\}} \frac{\|u\|_2^2}{\int_\Omega |\nabla u|^2 dx}.$$

- Step 1: Prove that the norm $||u||_1$ is equivalent to $||u||_2$ in $H_0^2(\Omega)$, provided λ and μ satisfy one of the following two conditions:
(1) $\lambda \leq 0,\ \mu > -\beta(\Omega)$;
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- Step 2: Prove that the functional $I(u)$ satisfies Mountain-Pass geometry structure and the $(PS)_c$ sequence $\{u_n\}$ is bounded in $H_0^2(\Omega)$.
- Step 3: Prove that if $c < \frac{2}{N} S^{\frac{N}{4}}$, then $I(u)$ satisfies the $(PS)_c$ condition, where $S :\triangleq \inf\{|\Delta u|_2^2 : u \in H^2(R^N), |u|_{2^{**}} = 1\}$ is the best Sobolev embedding constant of the embedding $H^2(\mathbb{R}^N) \hookrightarrow L^{2^{**}}(\mathbb{R}^N)$.
- Step 4:Prove that there exists a function $u_0 \in H_0^2(\Omega)\backslash\{0\}$ such that

$$\sup_{t\geq 0} I(tu_0) < \frac{2}{N} S^{\frac{N}{4}}, \tag{4}$$

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We introduce a well-known fact that the minimization problem

$$S = \inf\{\frac{|\Delta u|_2^2}{|u|_{2^{**}}^2} : u \in H^2(\mathbb{R}^N) \setminus \{0\}\} \tag{5}$$

is attained only by the functions lu_{ε,x_0}, where $l \in \mathbb{R} \setminus \{0\}$ and u_{ε,x_0} is defined by

$$u_{\varepsilon,x_0}(x) = \frac{[N(N-4)(N^2-4)\varepsilon^2]^{\frac{(N-4)}{8}}}{(\varepsilon + |x-x_0|^2)^{\frac{(N-4)}{2}}}, \quad \forall x_0 \in \mathbb{R}^N, \forall \varepsilon > 0. \tag{6}$$

- Step 4: verification of (4)

- Case 1: $N \geq 8$
  we let $\varphi \in C_0^\infty(\mathbb{R}^N, [0,1])$ be a radial cut-off function, such that
$$\begin{cases} \varphi(|x|) = 1, & |x| \leq \rho, , \\ \varphi(|x|) \in (0,1), & \rho < |x| < 2\rho, \\ \varphi(|x|) = 0, & |x| \geq 2\rho. \end{cases}$$
  Set
$$\psi_\varepsilon(x) = \varphi(x) u_{\varepsilon,0}(x). \tag{7}$$

- Lemma 2: Assume that $N \geq 8$, $\mu < 0$ and $\lambda \in \mathbb{R}$ or $\mu = 0$ and $\lambda > 0$. Then, as $\varepsilon \to 0^+$, ψ_ε defined in (7) satisfies the following estimates:

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$$|\Delta\psi_\varepsilon|_2^2 = S^{\frac{N}{4}} + \mathrm{O}(\varepsilon^{\frac{N-4}{2}}),$$

$$|\nabla\psi_\varepsilon|_2^2 = C_N K_1 \varepsilon + \mathrm{O}(\varepsilon^{\frac{N-4}{2}}),$$

$$|\psi_\varepsilon|_{2^{**}}^{2^{**}} = S^{\frac{N}{4}} + \mathrm{O}(\varepsilon^{\frac{N}{2}})$$

and

$$|\psi_\varepsilon|_2^2 = \begin{cases} c_N K_2 \varepsilon^2 + O(\varepsilon^{\frac{N-4}{2}}), & for \;\; N > 8, \\ -\frac{1}{2} c_8 \omega_8 \varepsilon^2 \ln \varepsilon + O(\varepsilon^2), & for \;\; N = 8, \end{cases}$$

where $c_N = (N(N-4)(N^2-4))^{\frac{N-4}{4}}$, $C_N = c_N(N-4)^2$, $K_1 = \int_{\mathbb{R}^N} \frac{|z|^2}{(1+|z|^2)^{N-2}} dz$, $K_2 = \int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^{N-4}} dz$ and ω_8 denotes the volume of unit ball in $\mathbb{R}^8$. Moreover, there exists a function $u_0 \in H_0^2(\Omega) \backslash \{0\}$ such that

$$\sup_{t \geq 0} I(tu_0) < \frac{2}{N} S^{\frac{N}{4}}.$$

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- We set $\psi_\varepsilon(x) = \varphi(x)u_{\varepsilon,0}(x)$, where $\varphi(x)$ is some given function with
$\varphi(x) = \varphi(|x|) \in C^2(\bar{\Omega}, \mathbb{R})$, $\varphi(0) = 1, \varphi(1) = \varphi'(1) = 0.$

- $\varphi(x) = 1 - |x|^a \sin(\frac{\pi}{2}|x|) - \frac{2a}{\pi}|x|^b \cos(\frac{\pi}{2}|x|)$

- Case 2: $N = 5$
$\varphi(r)$ satisfies $|\varphi^2(r) - 1| \leq Cr^{1+\delta}$ and $|\varphi^{10}(r) - 1| \leq Cr^{1+\delta}$, where $\delta \in (0, 1]$ is any given constant.

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- Lemma 3: For the case $N = 5$, if $\varphi(x) = 1 - |x|^{1.81}\sin(\frac{\pi}{2}|x|) - \frac{3.62}{\pi}|x|^{2.11}\cos(\frac{\pi}{2}|x|)$, then, as $\varepsilon \to 0^+$,

$$\int_\Omega |\Delta\psi_\varepsilon|^2 dx = (105)^{\frac{1}{4}}\omega_5\varepsilon^{\frac{1}{2}}(15.8854) + S^{\frac{N}{4}} + \mathrm{O}(\varepsilon^{\frac{1+\delta}{2}}),$$

$$\int_\Omega |\nabla\psi_\varepsilon|^2 dx = (105)^{\frac{1}{4}}\omega_5\varepsilon^{\frac{1}{2}}(1.3385) + \mathrm{O}(\varepsilon),$$

$$|\psi_\varepsilon|_2^2 = (105)^{\frac{1}{4}}\omega_5\varepsilon^{\frac{1}{2}}(0.0424) + \mathrm{O}(\varepsilon)$$

and

$$|\psi_\varepsilon|_{2^{**}}^{2^{**}} = S^{\frac{N}{4}} + \mathrm{O}(\varepsilon^{\frac{1+\delta}{2}}),$$

where $\delta = 0.81$.

- What's more, there exists a function $u_0 \in H_0^2(\Omega)\backslash\{0\}$ such that $\sup\limits_{t\geq 0} I(tu_0) < \frac{2}{N} S^{\frac{N}{4}}$, if one of the following assumptions holds:
  (i) $\mu = 0$, $\lambda > \lambda^*(5)$,
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  Let φ satisfies $|\varphi^2(r) - 1| \leq Cr^{2+\delta}$, $|\varphi^6(r) - 1| \leq Cr^{2+\delta}$, $\frac{(\varphi')^2}{r} \leq C$, where $0 < \delta < 1$ is any fixed constant.

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- Lemma 4: For the case $N = 6$, if $\varphi(x) = 1 - |x|^{2.02}\sin(\frac{\pi}{2}|x|) - \frac{4.04}{\pi}|x|^{3.07}\cos(\frac{\pi}{2}|x|)$, then, as $\varepsilon \to 0^+$,

$$\int_\Omega |\Delta\psi_\varepsilon|^2 dx = (384)^{\frac{1}{2}}\omega_6\varepsilon(37.9823) + S^{\frac{N}{4}} + \mathrm{O}(\varepsilon^{1+\frac{\delta}{2}}),$$

$$\begin{aligned}\int_\Omega |\nabla\psi_\varepsilon|^2 dx &= (384)^{\frac{1}{2}}\omega_6\varepsilon[-0.1242 - \tfrac{2}{3}(1+\varepsilon)^{-3} - (1+\varepsilon)^{-2} \\ &\quad -2(1+\varepsilon)^{-1} + 2\ln(1+\varepsilon) - 2\ln\varepsilon] + \mathrm{O}(\varepsilon^{\frac{3}{2}}),\end{aligned}$$

$$|\psi_\varepsilon|_2^2 = (384)^{\frac{1}{2}}\omega_6\varepsilon(0.1417) + \mathrm{O}(\varepsilon^{\frac{3}{2}}),$$

and

$$|\psi_\varepsilon|_{2^{**}}^{2^{**}} = S^{\frac{N}{4}} + \mathrm{O}(\varepsilon^{\frac{2+\delta}{2}}),$$

where $\delta = 0.6$.

Idea of proof of Theorem 1

- Moreover, there exists a function $u_0 \in H_0^2(\Omega)\backslash\{0\}$ such that $\sup_{t\geq 0} I(tu_0) < \frac{2}{N}S^{\frac{N}{4}}$, provided one of the following assumptions holds:
  (i) $\mu = 0,\ \lambda > \lambda^*(6),$
  (ii) $\mu < 0, \lambda \in \mathbb{R}.$

- Case 4: $N = 7$
  Let $\varphi(x)$ satisfies $\frac{(\varphi')^2}{r^2} \leq C,\ |\varphi^2(r) - 1| \leq Cr^{3+\delta}$ and $||\varphi(r)|^{\frac{14}{3}} - 1| \leq Cr^{3+\delta}$, where $0 < \delta < 1$ is any fixed constant.

- Moreover, there exists a function $u_0 \in H_0^2(\Omega)\backslash\{0\}$ such that $\sup\limits_{t\geq 0} I(tu_0) < \frac{2}{N} S^{\frac{N}{4}}$, provided one of the following assumptions holds:
  $(i)\quad \mu = 0,\ \lambda > \lambda^*(6),$
  $(ii)\ \ \mu < 0, \lambda \in \mathbb{R}.$
- Case 4: $\quad N = 7$
  Let $\varphi(x)$ satisfies $\frac{(\varphi')^2}{r^2} \leq C$, $|\varphi^2(r) - 1| \leq Cr^{3+\delta}$ and $||\varphi(r)|^{\frac{14}{3}} - 1| \leq Cr^{3+\delta}$, where $0 < \delta < 1$ is any fixed constant.

- Lemma 5: For the case $N = 7$, if $\varphi(x) = 1 - |x|^{2.53}\sin(\frac{\pi}{2}|x|) - \frac{5.06}{\pi}|x|^{3.78}\cos(\frac{\pi}{2}|x|)$, then, as $\varepsilon \to 0^+$,

$$\int_\Omega |\Delta\psi_\varepsilon|^2 dx = (945)^{\frac{3}{4}}\omega_7\varepsilon^{\frac{3}{2}}(77.8060) + S^{\frac{N}{4}} + \mathrm{O}(\varepsilon^{\frac{3+\delta}{2}}),$$

$$\int_\Omega |\nabla\psi_\varepsilon|^2 dx = (945)^{\frac{3}{4}}\varepsilon^{\frac{3}{2}}\omega_7(-1.7550+9\int_0^1 \frac{r^8}{(\varepsilon + r^2)^5} dr)+\mathrm{O}(\varepsilon^2),$$

$$|\psi_\varepsilon|_2^2 = (945)^{\frac{3}{4}}\omega_7\varepsilon^{\frac{3}{2}}(0.5530) + \mathrm{O}(\varepsilon^2)$$

and

$$|\psi_\varepsilon|_{2^{**}}^{2^{**}} = S^{\frac{N}{4}} + \mathrm{O}(\varepsilon^{\frac{3+\delta}{2}}),$$

where $\delta = 0.53$.

Idea of proof of Theorem 1

- Moreover, there exists a function $u_0 \in H_0^2(\Omega)\backslash\{0\}$ such that $\sup\limits_{t\geq 0} I(tu_0) < \frac{2}{N}S^{\frac{N}{4}}$, if one of the following assumptions holds:
  (i) $\mu = 0$, $\lambda > \lambda^*(7)$,
  $(ii) \mu < 0, \lambda \in \mathbb{R}$.

Hence, we conclude that

$$\lambda^*(N) := \begin{cases} 374.3880, & N = 5, \\ 268.0473, & N = 6, \\ 140.6980, & N = 7, \\ 0, & N \geq 8. \end{cases}$$

Idea of proof of Theorem 2

- Assume that u is a nontrivial solution of (1) in $H_0^2(\Omega)\bigcap C^4(\Omega)$. then we obtain

$$
\begin{aligned}
\frac{1}{2}\int_{\partial\Omega}(x.\nu)|\Delta u|^2 dS &= \frac{4-n}{2}\int_\Omega u(\mu\Delta u+\lambda u+|u|^{2^{**}-2}u)dx \\
&+\int_{\partial\Omega}(x\cdot\nu)(\frac{\partial\Delta u}{\partial\nu})(\frac{\partial u}{\partial\nu})dS \\
&-\int_\Omega (x\cdot\nabla u)\Delta^2 u dx
\end{aligned}
$$

Idea of proof of Theorem 2

- A straightforward computation gives us

$$-\mu\int_\Omega |\nabla u|^2 dx + 2\int_\Omega \lambda u^2 dx = \frac{1}{2}\int_{\partial\Omega}(x\cdot\nu)|\Delta u|^2 dS. \quad (8)$$

- Since Ω is a star-shaped domain, we have $x\cdot\nu > 0$. When $\lambda \le 0$ and $\mu > 0$, we have $u = 0$.
  When $\lambda > 0$ and $\mu > \frac{2}{\lambda_1(\Omega)}\lambda > 0$, we obtain

$$\begin{aligned}\frac{1}{2}\int_{\partial\Omega}(x\cdot\nu)|\Delta u|^2 dS &= -\mu\int_\Omega |\nabla u|^2 dx + 2\int_\Omega \lambda u^2 dx\\ &\le (-\mu\lambda_1(\Omega) + 2\lambda)\int_\Omega u^2 dx.\end{aligned}$$

  So $u = 0$.

- From the discussions above, we can see that there are no nontrivial solutions of (1) in $H_0^2(\Omega)\bigcap C^4(\Omega)$ for $\mu > \max\{0, 2\frac{\lambda}{\lambda_1(\Omega)}\}$ if Ω is a star-shaped domain.

Idea of proof of Theorem 2

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- From the discussions above, we can see that there are no nontrivial solutions of (1) in $H_0^2(\Omega)\bigcap C^4(\Omega)$ for $\mu>\max\{0,2\frac{\lambda}{\lambda_1(\Omega)}\}$ if Ω is a star-shaped domain.

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Thanks for your attention !