Asymptotic stability of shocks and rarefaction waves under space-periodic perturbations

Qian YUAN

Chinese Academy of Sciences

28 November 2020

The 7th Young Researchers PDE Forum,
South China Normal University

Joint work with Feimin Huang, Zhouping Xin, Lingda Xu & Yuan Yuan

Outline

Outline

Cauchy problem for a 1-d viscous conservation law

$$\partial_t u + \partial_x f(u) = \varepsilon \partial_x \big(B(u)\partial_x u\big), \quad x \in \mathbb{R}, t > 0, \tag{1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}. \tag{2}$$

- $u \in \mathbb{R}^n$,
- $f = (f_1, \cdots, f_n) : \mathbb{R} \to \mathbb{R}^n$ smooth, $Df(u)$ has n distinct eigenvalues,
- $B(u) \geq 0$: smooth matrix,
- $\varepsilon \geq 0$: viscosity.

Our concern:

$u_0(x) =$ Riemann solution $+$ Periodic perturbation.

i.e. $u_0(x)$ oscillates around different constants at initifinty $x = \pm\infty$.

Riemann solution

Self-similar solution $u = U(\frac{x}{t})$ to hyperbolic system $(\varepsilon = 0)$:

$$\partial_t u + \partial_x f(u) = 0. \tag{3}$$

Riemann initial data:

$$u_0(x) = \begin{cases} \overline{u}_l, & x < 0, \\ \overline{u}_r, & x > 0. \end{cases} \tag{4}$$

Riemann 1860, P. D. Lax 1957

- Basic waves: Shock, Rarefaction Wave & Contact Discontinuity.
- Riemann solution is the superposition of these basic waves.

Breakthrough: J. Glimm 1965

- General $N \times N$ system, $\mathrm{BV}(\mathbb{R})$ solution.

For inviscid conservation laws, Riemann solutions govern both large time behavior if initial data tends to constant states at infinity,

$$u_0(x) \to \begin{cases} \overline{u}_l & \text{as } x \to -\infty, \\ \overline{u}_r & \text{as } x \to +\infty; \end{cases}$$

and local structure (Building block, Riemann solver).

- E.Hopf 1950, Ilin-Oleĭnik 1960, T.P. Liu 1977/1978, $\cdots$
- J. Glimm 1965, A. Bressan 1992, Bressan-LeFloch 1999, $\cdots$

For viscous conservation laws, the viscous version of Riemann solutions characterize the large time behavior.

e.g. A viscous shock $\phi^\varepsilon(x-st)$ is a traveling wave solution to viscous conservation law,

$$\begin{cases} \big(-s\phi^\varepsilon + f(\phi^\varepsilon)\big)'(x) = \varepsilon\big(B(\phi^\varepsilon)(\phi^\varepsilon)'\big)'(x) \\ \lim\limits_{x\to-\infty} \phi^\varepsilon(x) = \overline{u}_l, \ \lim\limits_{x\to+\infty} \phi^\varepsilon(x) = \overline{u}_r, \end{cases}$$

where $\overline{u}_l > \overline{u}_r$ and s is the shock speed.

- Stability of viscous shock
  - Ilin-Oleinik 1960, Freistuhler - Serre 1998 (scalar, L^∞ and L^1)
  - Matsumura-Nishihara 1985; J. Goodman 1986 (zero mass),
  - T.P. Liu, M.AMS 1986, CPAM 1988, CPAM 1997
  - Xin-Sezepessy, ARMA 1993
  - Liu-Zeng, CMP 2010, M. AMS 2013
  - S.H.Yu, J.AMS 2011 (Boltzmann)
  - Zumbrun et al, $\cdots$
- Stability of rarefaction wave
  - Matsumura-Nishihara 1986; Liu-Xin 1988 (NS)
  - Matsumura-Nishihara 1992; Nishihara-Yang-Zhao 2004 (large perturbation)
- Stability of viscous contact discontinuity
  - Xin 1994,
  - Liu-Xin 1997,
  - Huang-Matsumura-Shi 2004, free boundary problem (NS)
  - Huang-Matsumura-Xin 2005 (NS, zero mass)
  - Huang-Xin-Yang 2008

Periodic solution to hyperbolic equations

Global existence with decay rate t^{-1} :

- P. Lax 1957 (scalar): Lax-formula.
- Glimm - Lax 1970 (2×2): Glimm scheme, generalized characteristics (characteristics or shocks).
- C. Dafermos 1995 (2×2): generalized characteristics (Filippov's sense; divides).

Global existence is still open:

- Non-isentropic Euler (3×3)
  - Majda-Rosales 1984: resonance phenomenon (sound waves reflect resonantly off entropy wave)
  - Temple-Young 1996; Qu-Xin 2015: almost global existence
  - $\cdots$

Uniform decay for scalar equation

$$\begin{aligned} \partial_t u^\varepsilon + \partial_x f(u^\varepsilon) &= \varepsilon \partial_x^2 u^\varepsilon, \quad x \in \mathbb{R}, t > 0, \\ u^\varepsilon(x,0) &= u_0(x), \end{aligned} \tag{5}$$

where $u_0 \in L^\infty(\mathbb{R})$ is periodic with period $p > 0$ and average $\overline{u}$.

Theorem 1.1 (Z. Xin - Lecture notes)

$\exists C > 0$ *independent of either p or $\varepsilon \geq 0$, s.t. for $\varepsilon \geq 0$,*

$$\|u^\varepsilon(\cdot,t) - \overline{u}\|_{L^\infty(\mathbb{R})} \leq TV_{[0,p]}(u^\varepsilon(\cdot,t)) \leq \frac{Cp}{t}, \quad t > 0.$$

Remark 1.2

If $\varepsilon > 0$,

$$\|u^\varepsilon(\cdot,t) - \overline{u}\|_{L^\infty(\mathbb{R})} \lesssim_\varepsilon e^{-\alpha_\varepsilon t}, \quad t \geq 0.$$

Our concern

? Asymptotic behavior of solution when initial data tends to different periodic functions at infinity:

$$u_0(x) \to \begin{cases} \overline{u}_l + \text{periodic function} & \text{as } x \to -\infty, \\ \overline{u}_r + \text{periodic function} & \text{as } x \to +\infty. \end{cases}$$

i.e. Do the periodic oscillations at infinity influence stability of shock and rarefaction wave?

Difficulties:

- Perturbation is not integrable on $\mathbb{R}$.
- Problem cannot be studied on a bounded period, and the perturbation is not periodic any more for $t > 0$.

Outline

§2.1. Inviscid scalar equation

$$\partial_t u + \partial_x f(u) = 0, \quad x \in \mathbb{R}, t > 0$$

$$u(x,0) = u_0(x) = \begin{cases} \overline{u}_l + w_{0l}(x), & x < -N, \\ \overline{u}_r + w_{0r}(x), & x > N, \end{cases}$$

where $\overline{u}_l, \overline{u}_r$ and $N > 0$ are constants, and $w_{0i}(x) \in L^\infty,\ i = l, r$ are periodic with period $p_i > 0$, satisfying

$$\frac{1}{p_i}\int_0^{p_i} w_{0i}(x)dx = 0 \quad \text{(w.l.o.g.)}.$$

Remark 2.1 (T.P. Liu 1978)

If $w_{0l} = w_{0r} = 0$, then $u \to u^S(x - x_0, t)$ (resp. $u^R(x,t)$ or $\overline{u}_l$) in $L^\infty(\mathbb{R})$ if $\overline{u}_l > \overline{u}_r$ (resp. $\overline{u}_l < \overline{u}_r$ or $\overline{u}_l = \overline{u}_r$) as $t \to +\infty$. Here x_0 is given by

$$x_0 = \frac{\int_{\mathbb{R}}(u_0 - u^S(x,0))dx}{\overline{u}_l - \overline{u}_r}.$$

Theorem 2.2 (Xin-Y.-Yuan SIAM; Y.-Yuan JDE)

- *If* $\bar{u}_l > \bar{u}_r$, *then* $\exists X(t) \in Lip[0, +\infty)$, *which is unique after a finite time, s.t.*

$$|X(t) - st - X_\infty| \lesssim t^{-1},$$

$$\sup_{x<X(t)} |u(x,t) - \bar{u}_l| + \sup_{x>X(t)} |u(x,t) - \bar{u}_r| \lesssim t^{-1},$$

with $X_\infty = \frac{1}{\bar{u}_l - \bar{u}_r}\left(X_{\infty,1} + X_{\infty,2}\right)$ *given by*

$$X_{\infty,1} = \int_{-\infty}^{0} (u_0 - \bar{u}_l - w_{0l})dx + \int_{0}^{+\infty} (u_0 - \bar{u}_r - w_{0r})dx,$$

$$X_{\infty,2} = -\min_{x\in\mathbb{R}} \int_0^x w_{0l}(y)dy + \min_{x\in\mathbb{R}} \int_0^x w_{0r}(y)dy.$$

Remark 2.3 (Comparison with integrable perturbations)

New shift $X_{\infty,2}$ is generated due to the periodic oscillations at infinity, which vanishes as periods tend to zero.

Theorem 2.4 (Xin-Y.-Yuan SIAM; Y.-Yuan JDE)

- *If* $\overline{u}_l < \overline{u}_r$, *then*

$$\left\|u(\cdot,t) - u^R(\cdot,t)\right\|_{L^\infty(\mathbb{R})} \lesssim t^{-\frac{1}{2}}.$$

In particular, without localized perturbations, i.e.

$$u_0(x) = u^R(x,0) + w_0(x)$$

with $w_0 \in L^\infty$ *being periodic with zero average, it holds that*

$$\left\|u(\cdot,t) - u^R(\cdot,t)\right\|_{L^\infty(\mathbb{R})} \lesssim t^{-1}.$$

- *If* $\overline{u}_l = \overline{u}_r$, *then*

$$\left\|u(\cdot,t) - \overline{u}_l\right\|_{L^\infty(\mathbb{R})} \lesssim t^{-\frac{1}{2}}.$$

Sketch of proof (Generalized characteristics)

Definition 2.5 (C. Dafermos 1989)

A generalized characteristic of an entropy solution $u(x,t)$ on the time interval $[\sigma,\tau] \subset [0,+\infty)$ is a Lipschitz function $\xi : [\sigma,\tau] \to \mathbb{R}$, satisfying the differential inclusion

$$\xi'(t) \in [\, f'(u(\xi(t)+,t)), \quad f'(u(\xi(t)-,t))\,] \quad \text{a.e. } [\sigma,\tau].$$

Remark 2.6

A generalized characteristic composes of either classical characteristics or shocks (same with Glimm-Lax 1970).

Lemma 2.7 (Divides, C. Dafermos 1995)

Let $u(x,t)$ be the periodic entropy solution satisfying a periodic initial data $u_0(x)$ with period p and average $\overline{u}$.
Then $\exists z \in [0,p)$ s.t.

$$u(z + f'(\overline{u})t, t) \equiv \overline{u} \quad \forall t > 0, \tag{6}$$

if and only if

$$\int_0^z [u_0(y) - \overline{u}]dy = \min_{x\in\mathbb{R}} \int_0^x [u_0(y) - \overline{u}]dy$$

$$\left(\Leftrightarrow \int_z^x [u_0(y) - \overline{u}]dy \geq 0 \quad \forall x \in \mathbb{R}. \right)$$

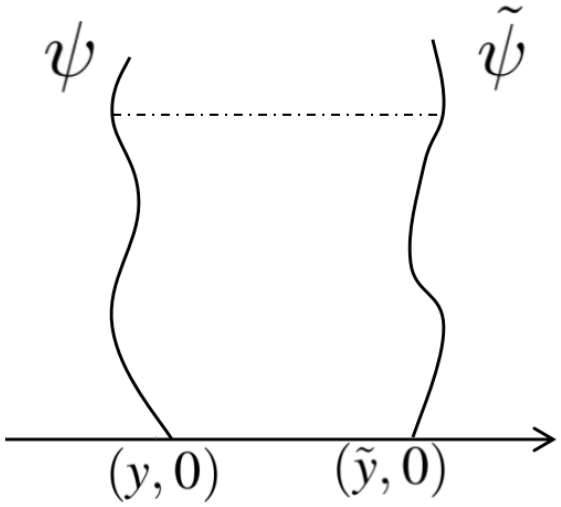
Lemma 2.8 (Comparison principle, Dafermos' book)

Let u and $\tilde{u}$ be two entropy solutions. Assume that

$$u(x,0) \leq \tilde{u}(x,0) \quad \textit{for a.e. } x \in (y, \tilde{y}).$$

Let $\psi(t)$ and $\tilde{\psi}(t)$ be the generalized characteristic curves of u and $\tilde{u}$ issuing from $(y,0)$ and $(\tilde{y},0)$, respectively. Then for any $t > 0$, if $\psi(t) < \tilde{\psi}(t)$, then

$$u(x,t) \leq \tilde{u}(x,t) \quad \textit{for a.e. } x \in (\psi(t), \tilde{\psi}(t)).$$



Proposition 2.9

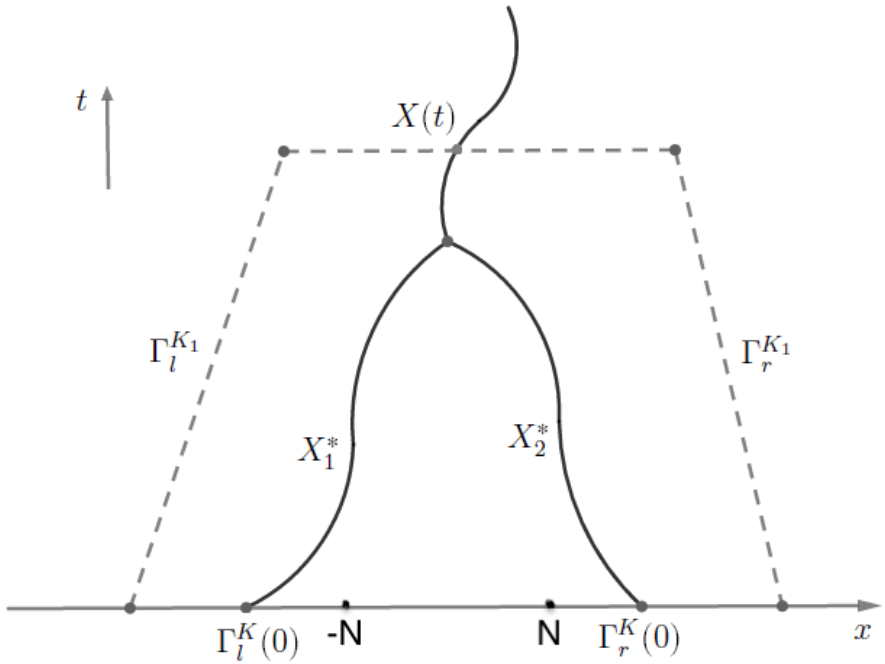
There exist two Lipschitz continuous curves X_1^* and X_2^*, s.t.

$$u(x,t) = \begin{cases} u_l(x,t), & x < X_1^*(t), \\ u_r(x,t), & x > X_2^*(t). \end{cases} \tag{7}$$

Moreover, if $\overline{u}_l > \overline{u}_r$, then X_1^* and X_2^* coincide after a finite time.

Here $u_{l,r}(x,t)$ is the periodic solution with $u_{l,r}(x,0) = \overline{u}_{l,r} + w_{0l,0r}(x)$.

Proof of Shock



- Γ_l^K and Γ_r^K : divides of u_l and u_r, respectively.
- Integrating $\partial_t u + \partial_x f(u) = 0$ over trapezium

$$\Rightarrow \quad |X(t) - st - X_\infty| \lesssim t^{-1}, \quad t \geq T.$$

Proof of Rarefaction wave and Constant

Set

$$
\begin{aligned}
P(t) &:= \lim_{k\to+\infty} \min_{x\in\mathbb{R}} \int_{\Gamma_l^k(t)}^{x} (u(y,t)-\overline{u}_l)\,dy \le 0,\\
Q(t) &:= \lim_{k\to+\infty} \max_{x\in\mathbb{R}} \int_{x}^{\Gamma_r^k(t)} (u(y,t)-\overline{u}_r)\,dy \ge 0.
\end{aligned}
\tag{8}
$$

With the aid of divides, we can prove that

Proposition 2.10 (Two time-invariants)

If $\overline{u}_l \le \overline{u}_r$, *then* $\exists\, x_P, x_Q \in [\Gamma_l^K(0), \Gamma_r^K(0)]$ *s.t.* $P(t) \equiv P(0)$ *and* $Q(t) \equiv Q(0)$.

$$
\Rightarrow \quad X_1^*(t) \ge f'(\overline{u}_l)t - Ct^{\frac{1}{2}} \quad \text{and} \quad X_2^*(t) \le f'(\overline{u}_r)t + Ct^{\frac{1}{2}}, \quad t \ge 0.
$$

Then similar to [T. P. Liu, 1978], one can finish the proof. ■

§2.2. Viscous scalar equation

$$\partial_t u^\varepsilon + \partial_x f(u^\varepsilon) = \varepsilon \partial_x^2 u^\varepsilon, \quad x \in \mathbb{R}, t > 0$$
$$u^\varepsilon(x,0) = u_0(x) \in L^\infty(\mathbb{R}),$$

$$\text{where } u_0(x) \to \begin{cases} \overline{u}_l + w_{0l}(x) & \text{as } x \to -\infty, \\ \overline{u}_r + w_{0r}(x) & \text{as } x \to +\infty, \end{cases} \quad \text{exponentially fast,}$$

where $\overline{u}_l, \overline{u}_r$ are constants, and for $i = l, r$, $w_{0i} \in L^\infty$ is periodic with period $p_i > 0$ and zero average.

Remark 2.11 (Ilin-Oleĭnik 1960)

If $w_{0l} = w_{0r} = 0$, then $u^\varepsilon \to \phi^\varepsilon(x - st - x_0)$ (or $u^R(x,t)$) if $\overline{u}_l > \overline{u}_r$ (or $\overline{u}_l < \overline{u}_r$) in $L^\infty(\mathbb{R})$ as $t \to +\infty$. Here x_0 is determined by the initial excessive mass

$$x_0 = \frac{\int_{\mathbb{R}} (u_0 - \phi^\varepsilon)(x)dx}{\overline{u}_l - \overline{u}_r}. \tag{9}$$

Theorem 2.12 (Xin-Y. -Yuan, preprint in Indiana. Univ. Math. J.)

- *If* $\overline{u}_l < \overline{u}_r$, *then* $\lim_{t\to+\infty} \left\|u^\varepsilon(\cdot,t) - u^R(\cdot,t)\right\|_{L^\infty(\mathbb{R})} = 0.$
- *If* $\overline{u}_l > \overline{u}_r$, *then* $\exists \alpha_\varepsilon > 0$, *s.t.*

$$\|u^\varepsilon(\cdot,t) - \phi^\varepsilon(\cdot - st - X^\varepsilon_\infty, t)\|_{L^\infty(\mathbb{R})} \lesssim_\varepsilon e^{-\alpha_\varepsilon t}, \quad t > 0, \tag{10}$$

where the shift $X^\varepsilon_\infty = \frac{1}{\overline{u}_l - \overline{u}_r}\left(X^\varepsilon_{\infty,1} + X^\varepsilon_{\infty,2}\right)$ *is given by*

$$X^\varepsilon_{\infty,1} = \int_{-\infty}^{0} (u_0 - \phi^\varepsilon - w_{0l})\,dx + \int_{0}^{+\infty} (u_0 - \phi^\varepsilon - w_{0r})\,dx,$$

$\cdots$ *(to continue)*

Theorem (Continuing)

$\cdots$ *(continuing)*

$$X^{\varepsilon}_{\infty,2} = \int_0^{+\infty} \frac{1}{p_l} \int_0^{p_l} [f(u^{\varepsilon}_l(x,t)) - f(\overline{u}_l)]\, dxdt$$
$$- \int_0^{+\infty} \frac{1}{p_r} \int_0^{p_r} [f(u^{\varepsilon}_r(x,t)) - f(\overline{u}_r)]\, dxdt$$
$$- \frac{1}{p_l} \int_0^{p_l} \int_0^{x} w_{0l}(y) dydx + \frac{1}{p_r} \int_0^{p_r} \int_0^{x} w_{0r}(y) dydx,$$

where $u^{\varepsilon}_{l,r}(x,t)$ *: periodic solution with* $u^{\varepsilon}_{l,r}(x,0) = \overline{u}_{l,r} + w_{0l,0r}(x)$.

Remark 2.13

1) Compared to localized perturbations: $X^{\varepsilon}_{\infty,2}$ is new, due to the periodic oscillations at infinity.
2) Compared to inviscid case: $X^{\varepsilon}_{\infty,2}$ depends more on the flux and viscosity (and time maybe), which also vanishes as periods tend to zero.

$X^{\varepsilon}_{\infty,2} \neq 0$ vs. $X_{\infty,2} = 0$

When oscillations at $x = \pm\infty$ coincide, i.e. $w_{0l} = w_{0r}$,

- Recall that for inviscid case $\varepsilon = 0$, $X_{\infty,2} = 0$.
  e.g. $u_0(x) = u^S(x) + w_0(x) \quad \Rightarrow \quad u(x,t) \to u^S(x - st)$.

- However, for viscous case $\varepsilon > 0$, $X^{\varepsilon}_{\infty,2} \neq 0$ in general.
  e.g. $u_0(x) = \phi^{\varepsilon}(x) + w_0(x) \quad \Rightarrow \quad u^{\varepsilon}(x,t) \to \phi^{\varepsilon}(x - st - X^{\varepsilon}_{\infty,2})$.

Theorem 2.14 (Xin-Y.-Yuan)

Assume that $w_{0l} = w_{0r} := w_0$. Then

1) *if $f(u) = u^2/2$, then $X^{\varepsilon}_{\infty,2} = 0$;*
2) *$\forall$ periodic perturbation $w_0 \in L^{\infty}(\mathbb{R})$ with zero average, if $0 < \|w_0\|_{L^{\infty}} < (\bar{u}_l - \bar{u}_r)/2$, then $\exists$ a smooth and strictly convex flux $f(u)$, s.t. $X^{\varepsilon}_{\infty,2} \neq 0$.*

Proof. 1) By Hopf's formula.

2) Note that $\sup_x u_r^\varepsilon(x,t) < \inf_x u_l^\varepsilon(x,t) \quad \forall t \geq 0.$

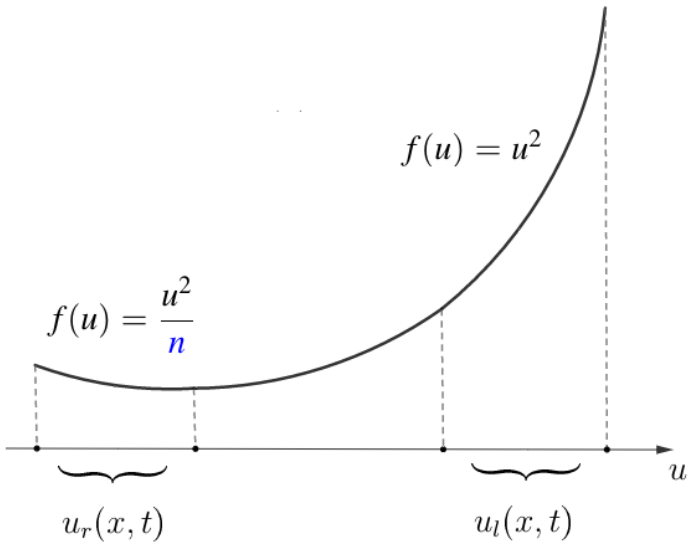


Figure: Construction of flux f

$$X_{\infty,2}^\varepsilon = \int_0^{+\infty} \frac{1}{p} \int_0^p (u_l^\varepsilon - \overline{u}_l)^2 dxdt - \frac{1}{n} \int_0^{+\infty} \frac{1}{p} \int_0^p (u_r^\varepsilon - \overline{u}_r)^2 dxdt$$
$$\neq 0, \qquad \text{if } n \text{ is large enough?}$$

- Method 1: Energy estimates $\Rightarrow X^{\varepsilon}_{\infty,2} > 0$ if $n > 0$ is large enough.
- Method 2: For $f(u) = \frac{1}{n}u^2$, Hopf-Cole transformation yields that

Lemma 2.15 (Xin-Y.-Yuan)

Let u^{ε} be the periodic solution with periodic initial data $u_0(x)$, which has period $p > 0$ and average $\overline{u}$, then

$$\int_0^{+\infty} \frac{1}{p} \int_0^p \left(f(u^{\varepsilon}) - f(\overline{u})\right) dxdt = \frac{1}{p} \int_0^p \int_0^x (u_0(y) - \overline{u}) dydx + \varepsilon n \log \left(\frac{1}{p} \int_0^p \exp \left\{ -\frac{1}{\varepsilon n} \int_0^x (u_0(y) - \overline{u}) dy \right\} dx \right). \tag{11}$$

Then L'Hopital rule $\Rightarrow$ RHS of (11) tends to zero as $n \to +\infty$.

Vanishing viscosity of $X^{\varepsilon}_{\infty,2}$

Theorem 2.16 (Xin-Y.-Yuan)

As $\varepsilon \to 0+$,

$$X^{\varepsilon}_{\infty,2} \to X_{\infty,2} = -\min_{x\in\mathbb{R}} \int_0^x w_{0l}(y)dy + \min_{x\in\mathbb{R}} \int_0^x w_{0r}(y)dy.$$

Additionally, if both $TV_{[0,p_l]}(w_{0l})$ *and* $TV_{[0,p_r]}(w_{0r})$ *are finite, then*

$$\left|X^{\varepsilon}_{\infty,2} - X_{\infty,2}\right| \lesssim \varepsilon^{\frac{1}{5}}.$$

Proof. Ingredients of proof:

- uniform decay rate t^{-1} of periodic, solutions
- divides of inviscid periodic solutions,
- Kruzhkov's theory.

Proof of viscous shock

For brevity, we omit $\varepsilon > 0$ in the proof.

Ingredients of proof:

- Construction of an ansatz to make anti-derivative method available
- Comparison principle

Construction of ansatz

For the viscous shock $\phi(x - st)$, inspired by

$$\phi(x) = \overline{u}_l\, g(x) + \overline{u}_r(1 - g(x)), \qquad \text{where } g(x) \triangleq \frac{\phi(x) - \overline{u}_r}{\overline{u}_l - \overline{u}_r}.$$

the **ansatz** is constructed as

$$\psi_\xi(x,t) \triangleq u_l(x,t)\, g(x - \xi(t)) + u_r(x,t)\,[1 - g(x - \xi(t))]\,,$$

where $\xi(t)$ is a C^1 curve to be determined.

Remark 2.17

$$\|\psi_\xi(\cdot,t) - \phi(\cdot - \xi(t))\|_{L^\infty(\mathbb{R})} \lesssim e^{-\alpha t},$$
$$|u(x,t) - \psi_\xi(x,t)| \lesssim_t e^{-\beta|x-\xi(t)|}.$$

- Error of ψ_ξ :

$$
\begin{aligned}
h_\xi &\triangleq \partial_t \psi_\xi + \partial_x f(\psi_\xi) - \partial_x^2 \psi_\xi \\
&= \partial_x \Big[(f(\psi_\xi) - f(u_l))\, g_\xi + (f(\psi_\xi) - f(u_r))\, (1 - g_\xi) - 2(u_l - u_r)\, g_\xi' \Big] \\
&\qquad + (f(u_l) - f(u_r))\, g_\xi' - (u_l - u_r)\, g_\xi'\, \xi'(t) + (u_l - u_r)\, g_\xi'',
\end{aligned}
$$

where red terms vanish as $|x| \to \infty$, blue terms are integrable on $\mathbb{R}$.

- Choose $\xi = X(t)$, s.t.

$$
\begin{cases}
\frac{d}{dt} \int_{\mathbb{R}} (u - \psi_{X(t)})(x,t)dx = -\int_{\mathbb{R}} h_X(x,t)dx = 0, \\
\int_{\mathbb{R}} (u - \psi_{X(0)})(x,0)dx = 0,
\end{cases}
\tag{12}
$$

$$\Rightarrow \int_{\mathbb{R}} (u - \psi_{X(t)})(x,t)dx \equiv 0,$$

$\Rightarrow$ anti-derivative method works.

Determine shift curve $X(t)$

$\int_{\mathbb{R}}$ (blue terms above) $dx = 0$

$$\Rightarrow \quad X'(t) = \frac{\int_{\mathbb{R}} [(u_l - u_r) g_X'' + (f(u_l) - f(u_r))\, g_X']\, dx}{\int_{\mathbb{R}} (u_l - u_r)\, g_X'\, dx}, \quad t > 0, \tag{13}$$

The initial data $X(0) = X_0$ is the unique point satisfying

$$\int_{\mathbb{R}} (u_0 - \psi_{X_0})(x, 0) dx$$
$$= \int_{\mathbb{R}} [u_0 - (\overline{u}_l + w_{0l}) g_{X_0} - (\overline{u}_r + w_{0r})(1 - g_{X_0})]\, dx = 0.$$

- $X(t) \in C^1(0, +\infty)$ exists and is unique (Cauchy-Lipschitz theorem)
- $|X'(t) - s| \lesssim e^{-\alpha t},\ t > 0.$

Anti-derivative variables

$$U(x,t) \triangleq \int_{-\infty}^{x} (u-\psi_X)(y,t)dy, \quad H(x,t) \triangleq -\int_{-\infty}^{x} h_X(y,t)dy.$$

satisfy

$$\partial_t U - \partial_x^2 U + a(u,\psi_X)\partial_x U = H(x,t), \quad x \in \mathbb{R}, t > 0,$$

where $a(v,w) = \int_0^1 f'(w+\theta(v-w))\, d\theta$.

Lemma 2.18

- $|H(x,t)| \lesssim e^{-\alpha t} e^{-\beta|x-X(t)|} \quad \forall x \in \mathbb{R}$;
- $\exists \delta_0 > 0, T_1 > 0,$ *and* $N_1 > 0$, *s.t. for all* $t > T_1$,

$$a(u,\psi_X) - X'(t) > \delta_0 \quad \forall x < X(t) - N_1,$$
$$a(u,\psi_X) - X'(t) < -\delta_0 \quad \forall x > X(t) + N_1.$$

Comparison principle with auxiliary functions in exponential forms (similar to [A. M. Ilin - O. A. Oleĭnik 1960])

$$\begin{aligned}
&\Rightarrow \quad \|U(x,t)\|_{L^\infty(\mathbb{R})} \lesssim_\varepsilon e^{-\mu t},\\
&\Rightarrow \quad \|u(x,t) - \psi_X(x,t)\|_{L^\infty(\mathbb{R})} \lesssim_\varepsilon e^{-\mu t},\\
&\Rightarrow \quad \|u(x,t) - \phi(x - st - X_\infty)\|_{L^\infty(\mathbb{R})} \lesssim_\varepsilon e^{-\mu t},
\end{aligned}$$

where $X_\infty = \lim_{t\to+\infty} X(t) - st$.

The proof of rarefaction waves can follow from the theorem of shock profile and comparison principle, similar to [Ilin - Oleĭnik 1960].

■

Outline

§3.1. A single shock with zero mass condition

- Isentropic Navier-Stokes equations read,

$$\begin{cases} \partial_t v - \partial_x u = 0, \\ \partial_t u + \partial_x p(v) = \partial_x \left(\frac{\mu(v)}{v} \partial_x u \right), \end{cases} \quad x \in \mathbb{R}, t > 0, \tag{14}$$

where $p(v) > 0, \mu(v) > 0$ are smooth, and $p'(v) < 0, p''(v) > 0$.

- Initial data $(v, u)(x, 0) = (v_0, u_0)(x)$ satisfies

$$(v_0, u_0)(x) \to \begin{cases} (\overline{v}_l, \overline{u}_l) + (\phi_{0l}, \psi_{0l})(x) & \text{as } x \to -\infty, \\ (\overline{v}_r, \overline{u}_r) + (\phi_{0r}, \psi_{0r})(x) & \text{as } x \to +\infty, \end{cases} \tag{15}$$

where

  - $(\overline{v}_l, \overline{u}_l)$ and $(\overline{v}_r, \overline{u}_r)$ generate a single 1-shock (or 2-shock),
  - (ϕ_{0i}, ψ_{0i}) for $i = l, r$, are periodic functions with period $\pi_i > 0$ and zero averages.

With a suitable ansatz $(\tilde{v}, \tilde{u})$, we can prove that

Theorem 3.1 (Huang-Y., preprint)

Assume that $\sum_{i=l,r} \|(\phi_{0i}, \psi_{0i})\|_{H^5([0,\pi_i])} << 1$,
$\left\| \int_{-\infty}^{x} (v_0(x) - \tilde{v}(x,0))dx, \int_{-\infty}^{x} (u_0(x) - \tilde{u}(x,0))dx \right\|_{H^3(\mathbb{R})} << 1$ *and the zero-mass condition holds, then*

$$\|(v,u)(\cdot,t) - (v^S, u^S)(\cdot - st - X_\infty)\|_{L^\infty(\mathbb{R})} \to 0 \quad \text{as } t \to +\infty.$$

Remark 3.2 (Matsumura-Nishihara, Goodman, Liu, Xin, $\cdots$)

Without the periodic perturbations, a viscous shock profile with a shift,

$$(v^S, u^S)(x - st - \delta),$$

govern the large time behavior of the solution in the $L^\infty(\mathbb{R})$ space. Note that here $\delta = 0$ under zero-mass condition.

Ansatz and Zero-mass condition

Notations. Let $(v_{l,r}, u_{l,r})(x,t)$ be the periodic solution with initial data $(\overline{v}_{l,r}, \overline{u}_{l,r}) + (\phi_{0l,0r}, \psi_{0l,0r})(x)$, and let

$$g(x) := \frac{v^S(x) - \overline{v}_l}{\overline{v}_r - \overline{v}_l} \quad \Big(= \frac{u^S(x) - \overline{u}_l}{\overline{u}_r - \overline{u}_l} \Big).$$

Ansatz is constructed as

$$\begin{cases} \tilde{v}(x,t) = v_l(x,t)[1 - g(x - st - X(t))] + v_r(x,t)g(x - st - X(t)), \\ \tilde{u}(x,t) = u_l(x,t)[1 - g(x - st - Y(t))] + u_r(x,t)g(x - st - Y(t)), \end{cases}$$

where $X(t), Y(t)$ are two curves to be determined.

Errors of ansatz:

$$\begin{cases} \partial_t \tilde{v} - \partial_x \tilde{u} = -\partial_x F_1 - f_2, \\ \partial_t \tilde{u} + \partial_x p(\tilde{v}) - \partial_x \left(\frac{\mu(\tilde{v})}{\tilde{v}} \partial_x \tilde{u} \right) = -\partial_x F_3 - f_4, \end{cases}$$

- $\int_{\mathbb{R}} f_2(x,t)dx \equiv 0 \;\Rightarrow\;$ ODE of $X(t)$.
- $\int_{\mathbb{R}} f_4(x,t)dx \equiv 0 \;\Rightarrow\;$ ODE of $Y(t)$.

Given initial data $X(0) = X_0$ and $Y(0) = Y_0$, it holds that

$$\begin{aligned} X(t) &\to X_\infty = H_1(X_0), \\ Y(t) &\to Y_\infty = H_2(Y_0), \end{aligned} \qquad \text{exponentially fast as } t \to +\infty, \tag{16}$$

where $H_1', H_2' \sim 1$ if (ϕ_{0i}, ψ_{0i}) for $i = l, r$ are small.

- Zero masses of perturbations require that

$$\int_{\mathbb{R}} (v_0(x) - \tilde{v}(x,0))\, dx = 0 \quad \text{and} \quad \int_{\mathbb{R}} (u_0(x) - \tilde{u}(x,0))\, dx = 0. \tag{17}$$

- Besides, note that

$$\left\| (\tilde{v}, \tilde{u})(\cdot, t) - \Big(v^S(\cdot - st - X(t)),\ u^S(\cdot - st - Y(t)) \Big) \right\|_{L^\infty(\mathbb{R})} \to 0.$$

Here the limit $\big(v^S(\cdot - st - X_\infty), u^S(\cdot - st - Y_\infty)\big)$ is a solution if and only if

$$X_\infty = Y_\infty. \tag{18}$$

Contradiction: Two free variables, X_0 and Y_0, should satisfy three constrains.

Zero-mass condition

For convenience, we pose an additional condition on initial data, which leads to the "zero-mass condition":

$$
\begin{aligned}
& s\left\{\int_{-\infty}^{0}\left(v_0-v^S-\phi_{0l}\right)(x)dx+\int_{0}^{+\infty}\left(v_0-v^S-\phi_{0r}\right)(x)dx\right. \\
& \qquad \left.-\frac{1}{\pi_l}\int_0^{\pi_l}\int_0^x \phi_{0l}(y)dydx+\frac{1}{\pi_r}\int_0^{\pi_r}\int_0^x \phi_{0r}(y)dydx\right\} \\
= & -\int_{-\infty}^{0}\left(u_0-u^S-\psi_{0l}\right)(x)dx-\int_{0}^{+\infty}\left(u_0-u^S-\psi_{0r}\right)(x)dx \\
& +\frac{1}{\pi_l}\int_0^{\pi_l}\int_0^x \psi_{0l}(y)dydx-\int_0^{+\infty}\frac{1}{\pi_l}\int_0^{\pi_l}\left[p(v_l(x,t))-p(\bar{v}_l)\right]dxdt \\
& -\frac{1}{\pi_r}\int_0^{\pi_r}\int_0^x \psi_{0r}(y)dydx+\int_0^{+\infty}\frac{1}{\pi_r}\int_0^{\pi_r}\left[p(v_r(x,t))-p(\bar{v}_r)\right]dxdt \\
& +\sigma(\bar{v}_l)-\sigma(\bar{v}_r)-\frac{1}{\pi_l}\int_0^{\pi_l}\sigma(\bar{v}_l+\phi_{0l}(x))dx+\frac{1}{\pi_r}\int_0^{\pi_r}\sigma(\bar{v}_r+\phi_{0r}(x))dx.
\end{aligned}
$$

§3.2. The combination of two shocks

- Full Navier-Stokes equations read,

$$\begin{cases} \partial_t v - \partial_x u = 0, \\ \partial_t u + \partial_x p(v,\theta) = \mu \partial_x \left(\frac{\partial_x u}{v} \right), \\ \partial_t E + \partial_x \left(p(v,\theta) u \right) = \kappa \partial_x \left(\frac{\partial_x \theta}{v} \right) + \mu \partial_x \left(\frac{u \partial_x u}{v} \right), \end{cases} \tag{19}$$

where $p = \frac{R\theta}{v}$, $E = e + \frac{1}{2}u^2$ with $e = \frac{R}{\gamma - 1}\theta + \text{const.}$

- Initial data $(v, u, E)(x, 0) = (v_0, u_0, E_0)(x)$ satisfies

$$(v_0, u_0, E_0)(x) \to \begin{cases} (\overline{v}_l, \overline{u}_l, \overline{E}_l) + (\phi_{0l}, \psi_{0l}, w_{0l})(x) & \text{as } x \to -\infty, \\ (\overline{v}_r, \overline{u}_r, \overline{E}_r) + (\phi_{0r}, \psi_{0r}, w_{0r})(x) & \text{as } x \to +\infty, \end{cases}$$

where $(\overline{v}_l, \overline{u}_l, \overline{E}_l)$ and $(\overline{v}_r, \overline{u}_r, \overline{E}_r)$ generate a composite wave of 1-shock and 3-shock with the middle state $(\overline{v}_m, \overline{u}_m, \overline{E}_m)$. And $(\phi_{0l,0r}, \psi_{0l,0r}, w_{0l,0r})$ is periodic with period $p_{l,r} > 0$ and zero averages.

Remark 3.3 (Huang-Matsumura 2009)

Without the periodic perturbations, a composite wave of 1-viscous shock $(v_1^S, u_1^S, E_1^S)(x - s_1 t)$ and 3-viscous shock $(v_3^S, u_3^S, E_3^S)(x - s_3 t)$ with shifts,

$$\begin{aligned}\big(&v_1^S(x - s_1 t - \delta_1) + v_3^S(x - s_3 t - \delta_3) - \overline{v}_m,\\ &u_1^S(x - s_1 t - \delta_1) + u_3^S(x - s_3 t - \delta_3) - \overline{u}_m,\\ &E_1^S(x - s_1 t - \delta_1) + E_3^S(x - s_3 t - \delta_3) - \overline{E}_m\big),\end{aligned}$$

governs the large time behavior of the solution in the $L^\infty(\mathbb{R})$ space.

Back to our problem, we first let

$$g_1(x) = \frac{v_1^S(x) - \overline{v}_l}{\overline{v}_m - \overline{v}_l} = \frac{u_1^S(x) - \overline{u}_l}{\overline{u}_m - \overline{u}_l}, \qquad h_1(x) = \frac{E_1^S(x) - \overline{E}_l}{\overline{E}_m - \overline{E}_l},$$

$$g_3(x) = \frac{v_3^S(x) - \overline{v}_m}{\overline{v}_r - \overline{v}_m} = \frac{u_3^S(x) - \overline{u}_m}{\overline{u}_r - \overline{u}_m}, \qquad h_3(x) = \frac{E_3^S(x) - \overline{E}_m}{\overline{E}_r - \overline{E}_m},$$

Construction of ansatz

For three curves $X(t)$, $Y(t)$ and $Z(t)$, and a constant $\sigma \in \mathbb{R}$, let

$$
\begin{aligned}
v^\sharp &= v_l(x,t)[1 - g_1(x - s_1 t - X(t))] + v_r(x,t) g_3(x - s_3 t - X(t) - \sigma) - \overline{v}_m, \\
u^\sharp &= u_l(x,t)[1 - g_1(x - s_1 t - Y(t))] + u_r(x,t) g_3(x - s_3 t - Y(t) - \sigma) - \overline{u}_m, \\
E^\sharp &= E_l(x,t)[1 - h_1(x - s_1 t - Z(t))] + E_r(x,t) h_3(x - s_3 t - Z(t) - \sigma) - \overline{E}_m.
\end{aligned}
$$

- $\int_{\mathbb{R}}$ "Errors of $\left(v^\sharp, u^\sharp, E^\sharp\right)$" $dx \equiv 0 \Rightarrow$ ODEs of X, Y and Z, respectively.
- Given $X(0) = X_0$, $Y(0) = Y_0$ and $Z(0) = Z_0$, it holds that

$$
\begin{aligned}
X(t) &\to X_\infty = H_1(X_0, \sigma), \\
Y(t) &\to Y_\infty = H_2(Y_0, \sigma), \qquad \text{exponentially fast as } t \to +\infty, \qquad (20) \\
Z(t) &\to Z_\infty = H_3(Z_0, \sigma),
\end{aligned}
$$

where $\partial_{X_0} H_1, \partial_{Y_0} H_2, \partial_{Z_0} H_3 \sim 1$, if both periodic perturbations and two shock strengths are small.

Construction of ansatz

- Similar to the case for one single shock, it must hold

$$X_\infty = Y_\infty = Z_\infty, \qquad \text{denoted by } \xi.$$

- Besides, ansatz needs to carry a diffusion wave with a constant mass, which propagates along the 2-family of characteristics $\bar{r}_2 = r_2(\bar{v}_m, \bar{u}_m, \bar{E}_m)$.

Ansatz is constructed as

$$\left(\tilde{v}, \tilde{u}, \tilde{E}\right) = \left(v^\sharp, u^\sharp, E^\sharp\right) + \Theta(x,t)\bar{r}_2,$$

where $\partial_t \Theta = \frac{\kappa(\gamma-1)}{R\gamma\bar{v}_m}\partial_x^2\Theta$ and $\int_{\mathbb{R}} \Theta(x,t)dx \equiv \eta$.

From above, six free variables $\sigma, X_0, Y_0, Z_0, \xi$ and η satisfy six constrains

$$\xi = H_1(X_0, \sigma) = H_2(Y_0, \sigma) = H_3(Z_0, \sigma) \tag{21}$$

and

$$\int_{\mathbb{R}} \big(v_0 - \tilde{v}(x,0), u_0 - \tilde{u}(x,0), E_0 - \tilde{E}(x,0)\big) dx = 0, \tag{22}$$

where (22) are zero-mass constrains.

- The Jacobian of (21), (22) is away from zero if both the periodic perturbations and two shock strengths are small.
- Constants $\sigma, X_0, Y_0, Z_0, \xi$ and η can be uniquely determined, thus the ansatz is well defined.

Main result for two shocks

Theorem 3.4 (Y.-Yuan, preprint)

If both the periodic perturbations and two shock strengths are small, then there exist unique shifts $\delta_1, \delta_3 \in \mathbb{R}$ s.t.,

$$\left\|(v,u,E) - (V_1 + V_3 - \overline{v}_m, U_1 + U_3 - \overline{u}_m, E_1 + E_3 - \overline{E}_m)\right\|_{L^\infty(\mathbb{R})} \to 0,$$

where $V_i = v_i^S(x - s_i t - \delta_i)$, $U_i = u_i^S(x - s_i t - \delta_i)$ and $E_i = E_i^S(x - s_i t - \delta_i)$ for $i = 1, 3$.

Outline

§4.1. Scalar equation

Multi-d viscous convex conservation law read,

$$\begin{cases} \partial_t u + \sum_{i=1}^{n} \partial_{x_i} f_i(u) = \triangle u, \\ u(x,0) = u_0(x) = \tilde{u}^R(x_1, 0) + w_0(x), \end{cases} \qquad x \in \mathbb{R}^n, t > 0, \tag{23}$$

where $f_1'' > 0$ and

- $\tilde{u}^R(x_1, t)$ is a smooth rarefaction wave solving

$$\begin{cases} \partial_t \tilde{u}^R + \partial_{x_1} f_1(\tilde{u}^R) = \partial_{x_1}^2 \tilde{u}^R, \\ \tilde{u}^R(x_1, 0) = \frac{\bar{u}_l + \bar{u}_r}{2} + \frac{\bar{u}_r - \bar{u}_l}{2} \tanh x_1, \end{cases} \qquad \text{where} \quad \bar{u}_l < \bar{u}_r.$$

- The perturbation $w_0(x)$ is periodic on $\mathbb{T}^n = [0,1]^n$ with zero average,

$$\int_{\mathbb{T}^n} w_0(x) dx = 0.$$

$\because$ The solution to (23) is periodic w. r. t. $x_2, \cdots, x_n$, but not to x_1.

$\therefore$ We should consider the unbounded domain

$$\Omega = \mathbb{R} \times \mathbb{T}^{n-1}. \tag{24}$$

Ansatz

Let $u_{l,r}(x,t)$ be the periodic solution with initial data $\overline{u}_{l,r} + w_0(x)$, and let

$$g(x_1,t) = \frac{\tilde{u}^R(x_1,t) - \overline{u}_l}{\overline{u}_r - \overline{u}_l}. \tag{25}$$

- **Ansatz** is constructed as,

$$\tilde{u}(x,t) = u_l(x,t)(1 - g(x_1,t)) + u_r(x,t)g(x_1,t), \tag{26}$$

  which is periodic w.r.t. $x_2, \cdots, x_n$, but not to x_1.
- Perturbation $\phi = u - \tilde{u}$ satisfies

$$\begin{cases} \partial_t \phi + \sum_{i=1}^{n} \partial_{x_i} \left[f_i(\tilde{u} + \phi) - f_i(\tilde{u}) \right] = \triangle \phi - h, \\ \phi(x,0) = 0, \end{cases} \tag{27}$$

Lemma 4.1

$$\|h\|_{L^p(\mathbb{R}\times\mathbb{T}^{n-1})} \lesssim \|w_0\|_{H^{[\frac{n}{2}]+2}(\mathbb{T}^n)} e^{-\alpha t}. \tag{28}$$

$\because$ The n-d G-N inequality does not hold on the domain $\mathbb{R} \times \mathbb{T}^{n-1}$ in general.

$\therefore$ **Gagliardo-Nirenberg type inequality** on $\Omega = \mathbb{R} \times \mathbb{T}^{n-1}$:

Theorem 4.2 (Huang-Y., to appear in MAA)

There exists a decomposition $u(x) = \sum_{k=0}^{n-1} u^{(k)}(x)$ *s.t. each* $u^{(k)}$ *satisfies the* $k+1$*-d G-N inequality,*

$$\left\|\nabla^j u^{(k)}\right\|_{L^p(\Omega)} \leq C \left\|\nabla^m u\right\|_{L^r(\Omega)}^{\theta_k} \left\|u\right\|_{L^q(\Omega)}^{1-\theta_k}, \quad 0 \leq j < m, 1 \leq p \leq +\infty,$$

where $\frac{1}{p} = \frac{j}{k+1} + \left(\frac{1}{r} - \frac{m}{k+1}\right)\theta_k + \frac{1}{q}(1-\theta_k)$ *and* $\frac{j}{m} \leq \theta_k \leq 1$ *hold. Hence,*

$$\left\|\nabla^j u\right\|_{L^p(\Omega)} \leq C \sum_{k=0}^{n-1} \left\|\nabla^m u\right\|_{L^r(\Omega)}^{\theta_k} \left\|u\right\|_{L^q(\Omega)}^{1-\theta_k}.$$

Main result for scalar equation

Following [Kawashima-Nihibata-Nishikawa 2004], L^p method yields that

Theorem 4.3 (Huang-Y.)

If the initial periodic perturbation $w_0 \in H^{[\frac{n}{2}]+2}(\mathbb{T}^n)$ has zero average, then

$$\sup_{x\in\mathbb{R}^n} \left|u(x,t) - \tilde{u}^R(x_1,t)\right| \lesssim t^{-\frac{1}{2}}. \tag{29}$$

Navier-Stokes equations in 3 dimensions

- Isentropic 3-d N-S equations read,

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho \mathbf{u}) = 0, & x \in \mathbb{R}^3, t > 0, \\ \partial_t(\rho \mathbf{u}) + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) = \mu \triangle \mathbf{u} + (\mu + \lambda) \nabla \mathrm{div} \mathbf{u}, \end{cases}$$

  where $\rho-$ density, $\mathbf{u}-$ velocity, $p(\rho) = \rho^\gamma$ with $\gamma \geq 1$, and $\mu > 0, \lambda + \frac{2}{3}\mu \geq 0$.

- Initial data

$$(\rho, \rho \mathbf{u})(x, 0) = \left(\rho^R, \rho^R u_1^R, 0, 0\right)(x_1, 0) + (v_0, \mathbf{w}_0)(x), \quad x \in \mathbb{R}^3.$$

  where

  - $\left(\rho^R, u_1^R\right)(x_1, t)-$ smooth rarefaction wave connects $(\overline{\rho}^-, \overline{u}_1^-)$ and $(\overline{\rho}^+, \overline{u}_1^+)$, and solves 1-d isentropic Euler equations.
  - $v_0(x), \mathbf{w}_0(x) = (w_{0,1}, w_{0,2}, w_{0,3})(x)$ are periodic on $\mathbb{T}^3 = [0, 1]^n$ with zero averages.

Ansatz and main result

Let $(\rho^{\pm}, \mathbf{u}^{\pm})(x,t)$ be the periodic solution with initial data

$$(\rho^{\pm}, \rho^{\pm}\mathbf{u}^{\pm})(x,0) = (\overline{\rho}^{\pm}, \overline{\rho}^{\pm}\overline{\mathbf{u}}^{\pm}) + (v_0, \mathbf{w}_0)(x).$$

And set

$$g(x_1,t) = \frac{\rho^R(x_1,t) - \overline{\rho}^-}{\overline{\rho}^+ - \overline{\rho}^-}, \quad h(x_1,t) = \frac{u_1^R(x_1,t) - \overline{u}_1^-}{\overline{u}_1^+ - \overline{u}_1^-}. \tag{30}$$

Ansatz is constructed as

$$\begin{cases} \tilde{\rho}(x,t) = \rho^-(x,t)(1 - g(x_1,t)) + \rho^+(x,t)\, g(x_1,t), \\ \tilde{\mathbf{u}}(x,t) = \mathbf{u}^-(x,t)(1 - h(x_1,t)) + \mathbf{u}^+(x,t)\, h(x_1,t). \end{cases} \tag{31}$$

Theorem 4.4 (Huang-Xu-Y., preprint)

Assume that both the amplitude of rarefaction wave, $|\overline{\rho}^+ - \overline{\rho}^-| << 1$, *and the periodic perturbations,* $\|v_0, \boldsymbol{w}_0\|_{H^5(\mathbb{T}^3)} << 1$, *then*

$$\sup_{x\in\mathbb{R}^3} \left|(\rho, \boldsymbol{u})(x,t) - \left(\rho^R, u_1^R, 0, 0\right)(x_1,t)\right| \to 0 \quad \textit{as } t \to +\infty.$$

Thank You!