A priori estimate for the critical parameters of $SU(3)$ Toda system with arbitrary singularities and related existence result

Yang, Wen

WIPM, Innovation Academy for Precision Measurement Science and Technology, Chinese Academy of Sciences

Nov 30th, 2020
South China Normal University
Joint With Y. Lee, C.S. Lin
J.C. Wei, L. Zhang

$SU(3)$ Toda system

$$\Delta u_i + \sum_{j=1}^{2} a_{ij}\rho_j \left(\frac{h_j e^{u_j}}{\int_M h_j e^{u_j} dV_g} - 1 \right) = 4\pi \sum_{p\in\mathcal{S}} \alpha_p^i (\delta_p - 1), \quad i = 1, 2, \tag{1}$$

where

$$SU(3)\ (A_2) = (a_{ij})_{2\times 2} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

and

$\Delta g,\ M$: Laplace-Beltrami operator, $|M| = 1$,

h_1, h_2: positive and smooth functions on M,

$\delta_p,\ \mathcal{S}$: Dirac measure at p, and $\mathcal{S}$ finite set of M,

α_p^i: $\alpha_p^i > -1, \forall i = 1, 2$,

u_1, u_2: $\mathring{H}^1(M) := \{f \in H^1(M) \mid \int_M f dV_g = 0\}$.

$SU(3)$ Toda system

$$\Delta u_i + \sum_{j=1}^{2} a_{ij}\rho_j \left(\frac{h_j e^{u_j}}{\int_M h_j e^{u_j} dV_g} - 1 \right) = 4\pi \sum_{p\in\mathcal{S}} \alpha_p^i(\delta_p - 1), \quad i = 1, 2, \tag{1}$$

where

$$SU(3)\ (A_2) = (a_{ij})_{2\times 2} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

and

$\Delta g,\ M$: Laplace-Beltrami operator, $|M| = 1$,

h_1, h_2: positive and smooth functions on M,

$\delta_p,\ \mathcal{S}$: Dirac measure at p, and $\mathcal{S}$ finite set of M,

α_p^i: $\alpha_p^i > -1, \forall i = 1, 2$,

u_1, u_2: $\mathring{H}^1(M) := \{f \in H^1(M) \mid \int_M f dV_g = 0\}$.

Background

(a) Nirenberg problem: given two metrics $g_u := e^{2u} g_0$, then we have $\Delta_0 u + K_{g_u} e^{2u} = K_{g_0}$.

(b) Related to the Plücker formula, which is associated to a holomorphic curve f from $\mathbb{S}^2$ into $\mathbb{CP}^n$.

(c) Onsager theory of $2D$ Turbulence, given the density ρ with restricted total vorticity ($m(\rho) = \int_\Omega \rho$) and energy ($E(\rho) = \frac{1}{2} \int_\Omega \int_\Omega G(x, y)\rho(x)\rho(y)$), solving the extremal value of the entropy ($S(\rho) = -\int_\Omega \rho \log \rho$) is related to the Mean field equation with Dirichlet boundary condition.

(d) Non-abelian Chern-Simons-Higgs system.

Normalized equation

Let $G(x,p)$ be the Green function of Δ_g on M:

$$-\Delta_g G(x,p) = (\delta_p - 1), \quad \int_M G(x,p)dV_g = 0.$$

We make a decomposition of u_i in the following way

$$u_i = \tilde{u}_i - \sum_{p\in\mathcal{S}} 4\pi\alpha_p^i G(x,p), \quad i = 1,2.$$

Then (1) can be written as

$$\Delta\tilde{u}_i + \sum_{j=1}^{2} a_{ij}\rho_j \left(\frac{\tilde{h}_j e^{\tilde{u}_j}}{\int_M \tilde{h}_j e^{\tilde{u}_j} dV_g} - 1 \right) = 0, \tag{2}$$

where $\tilde{h}_i = h_i \exp(-\sum_{p\in\mathcal{S}} 4\pi\alpha_p^i G(x,p)), \ i = 1,2.$

Mean field equation

For the following single equation

$$\Delta \tilde{u} + \rho \left(\frac{\tilde{h} e^{\tilde{u}}}{\int_M \tilde{h} e^{\tilde{u}} dV_g} - 1 \right) = 0, \tag{3}$$

where $\tilde{h} = h \exp(-\sum_{p \in \mathcal{S}} 4\pi\alpha_p G(x,p))$ for some positive smooth function h on M.

To study the compact issue, we analyze the local form of this equation

$$\Delta u + e^u = 4\pi\alpha\delta_q \text{ in } \Omega, \quad \int_\Omega e^u dx < +\infty. \tag{4}$$

Blow up analysis

When $\alpha = 0$, Brezis and Merle studied the compactness of equation (4). Suppose u_k is a sequence of solutions to (4). Then after passing to a subsequence, we have

- $\{u_k\}$ is bounded in $L^\infty_{\mathrm{loc}}(\Omega)$,
- $u_k \to -\infty$ uniformly on compact subsets of Ω,
- there exists a finite set $\mathcal{B} \subset \Omega$, for any $p \in \mathcal{B}$, there exists $\{p^k\} \subset \Omega$ such that $p^k \to p$, $u_k(p^k) \to \infty$ and $u_k(x) \to -\infty$ uniformly on compact subsets of $\Omega \setminus \mathcal{B}$. Moreover $e^{u_k} \rightharpoonup \sum_{p\in\mathcal{B}} n_p \delta_p$ with $n_p \geq 4\pi, \forall p \in \mathcal{B}$.

To compute the quantity n_p, we set

$$\sigma_p = \lim_{r\to 0} \lim_{k\to+\infty} \frac{1}{2\pi} \int_{B_r(p)} e^{u_k} dx.$$

Blow up analysis

It is obvious that $n_p = 2\pi\sigma_p$.

Li-Shafrir proved $\sigma_p = 4$ for $p \in \Omega \setminus \{q\}$ and

Bartolucci-Tarantello proved $\sigma_p = 4(1+\alpha)$ when $p = q$.

Key point: u_k blows up then $e^{u_k} \to 0$ on any compact subset of $\Omega \setminus \mathcal{B}$.

Returning back to equation (3), if blow up happens, then

$$\rho \frac{\tilde{h}e^{\tilde{u}}}{\int_M \tilde{h}e^{\tilde{u}} dV_g} \to \sum_{p\in\mathcal{B}} 8\pi\alpha_p\delta_p, \tag{5}$$

where $\mathcal{B}$ is the blow up set of $\tilde{u}$, $\alpha_p = 1$ if $p \notin \mathcal{S}$ and $\alpha_p = (1+\alpha_q)$ if $p = q \in \mathcal{S}$.

Blow up analysis

It is obvious that $n_p = 2\pi\sigma_p$.

Li-Shafrir proved $\sigma_p = 4$ for $p \in \Omega \setminus \{q\}$ and

Bartolucci-Tarantello proved $\sigma_p = 4(1+\alpha)$ when $p = q$.

Key point: u_k blows up then $e^{u_k} \to 0$ on any compact subset of $\Omega \setminus \mathcal{B}$.

Returning back to equation (3), if blow up happens, then

$$\rho \frac{\tilde{h} e^{\tilde{u}}}{\int_M \tilde{h} e^{\tilde{u}} dV_g} \to \sum_{p \in \mathcal{B}} 8\pi\alpha_p \delta_p, \tag{5}$$

where $\mathcal{B}$ is the blow up set of $\tilde{u}$, $\alpha_p = 1$ if $p \notin \mathcal{S}$ and $\alpha_p = (1+\alpha_q)$ if $p = q \in \mathcal{S}$.

Topological degree

Let

$$\Sigma = \left\{ 8m\pi + \sum_{q\in A} 8\pi(1+\alpha_q) \mid A \subset \mathcal{S},\ m \in \mathbb{N}\cup\{0\} \right\} \setminus \{0\}.$$

Based on (5) we have that if $\rho \notin \Sigma$, then the a priori bound holds. Thus one can define the Topological degree of (3).

Consider the following function

$$\mathcal{G}(x) = (1 + x + x^2 + \cdots)^{-\chi(M)+|\mathcal{S}|} \prod_{p\in\mathcal{S}} (1 - x^{1+\alpha_p}), \qquad (6)$$

where $\chi(M) = 2 - 2g$ is Euler characteristic number of M. Writing

$$\Sigma = \{8\pi n_k \mid n_1 < n_2 < \cdots\},$$

and represent $\mathcal{G}(x)$ in the powers of x:

$$\mathcal{G}(x) = 1 + b_1 x^{n_1} + b_2 x^{n_2} + \cdots + b_k x^{n_k} + \cdots . \qquad (7)$$

Topological degree

Let

$$\Sigma = \left\{ 8m\pi + \sum_{q\in A} 8\pi(1+\alpha_q) \mid A \subset \mathcal{S},\ \ m \in \mathbb{N} \cup \{0\} \right\} \setminus \{0\}.$$

Based on (5) we have that if $\rho \notin \Sigma$, then the a priori bound holds. Thus one can define the Topological degree of (3).

Consider the following function

$$\mathcal{G}(x) = (1 + x + x^2 + \cdots)^{-\chi(M)+|\mathcal{S}|} \prod_{p\in\mathcal{S}} (1 - x^{1+\alpha_p}), \qquad (6)$$

where $\chi(M) = 2 - 2g$ is Euler characteristic number of M. Writing

$$\Sigma = \{8\pi n_k \mid n_1 < n_2 < \cdots\},$$

and represent $\mathcal{G}(x)$ in the powers of x:

$$\mathcal{G}(x) = 1 + b_1 x^{n_1} + b_2 x^{n_2} + \cdots + b_k x^{n_k} + \cdots . \qquad (7)$$

Topological degree

Chen-Lin proved that the degree d_ρ can be expressed in terms of b_i.

Theorem (Chen-Lin)

Let d_ρ be the Leray-Schauder degree for (3). Suppose that $8\pi n_k < \rho < 8\pi n_{k+1}$. Then

$$d_\rho = \sum_{j=0}^{k} b_j,$$

where $b_0 = 1$.

Remark: If $\mathcal{S} = \emptyset$ and $\rho \in (8m\pi, 8(m+1)\pi)$, we have

$$d_\rho = b_m = \binom{m - \chi(M)}{m}.$$

Local form of Toda system

$$\Delta u_i^k + \sum_{j=1}^{2} k_{ij} e^{u_j^k} = 4\pi\alpha_i\delta_0 \text{ in } B(0,1), \quad i = 1, 2. \tag{8}$$

For solutions $u^k = (u_1^k, u_2^k)$ of (8) we assume:

$$\begin{cases} (i): 0 \text{ is the only blow up point of } u^k \text{ in } B(0,1), \\ (ii): |u_i^k(x) - u_i^k(y)| \leq C, \ \forall x, y \text{ on } \partial B(0,1), \ i = 1, 2, \\ (iii): \int_{B_1(0)} e^{u_i^k} dx \leq C, \ i = 1, 2. \end{cases} \tag{9}$$

We set $\sigma_i = \frac{1}{2\pi} \lim_{r\to 0} \lim_{k\to\infty} \int_{B_r(0)} e^{u_i^k} dx$ and call it the local mass of u_i^k, $i = 1, 2$.

Without singular source

By Pohozaev identity, (σ_1, σ_2) must belong to

$$\Gamma = \left\{ (\sigma_1, \sigma_2) \mid \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = 2\mu_1\sigma_1 + 2\mu_2\sigma_2 \right\}, \tag{10}$$

where $\mu_i = 1 + \alpha_i,\ \ i = 1, 2.$

When $\alpha_1 = \alpha_2 = 0$. Jost-Lin-Wang proved there are only five types for (σ_1, σ_2), i.e., (σ_1, σ_2) could be one of

$$\{(2,0), (0,2), (2,4), (4,2), (4,4)\}. \tag{11}$$

The proof of Jost-Lin-Wang uses holonomy theory. An alternative proof is given Lin-Wei-Zhang.

Without singular source

By Pohozaev identity, (σ_1, σ_2) must belong to

$$\Gamma = \left\{ (\sigma_1, \sigma_2) \mid \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = 2\mu_1\sigma_1 + 2\mu_2\sigma_2 \right\}, \tag{10}$$

where $\mu_i = 1 + \alpha_i, \;\; i = 1, 2.$

When $\alpha_1 = \alpha_2 = 0$. Jost-Lin-Wang proved there are only five types for (σ_1, σ_2), i.e., (σ_1, σ_2) could be one of

$$\{(2,0), (0,2), (2,4), (4,2), (4,4)\}. \tag{11}$$

The proof of Jost-Lin-Wang uses holonomy theory. An alternative proof is given Lin-Wei-Zhang.

With singular source

Let

$$\Gamma(\mu_1,\mu_2)=\Big\{(0,0),(2\mu_1,0),(0,2\mu_2),(2\mu_1,2(\mu_1+\mu_2)),\\ (2(\mu_1+\mu_2),2\mu_2),(2(\mu_1+\mu_2),2(\mu_1+\mu_2))\Big\} \tag{12}$$

Theorem (Lin-Wei-Y.-Zhang)

Let $\sigma_i,\ i=1,2$ be the local masses of a sequence of blow up solutions of (8) and (9) hold. Then

$$(\sigma_1,\sigma_2)=(\sigma_1^*+2N_1,\sigma_2^*+2N_2),$$

where $(\sigma_1^*,\sigma_2^*)\in\Gamma(\mu_1,\mu_2)$ and $N_1,N_2\in\mathbb{Z}$.

Some Remarks

Musso-Pistoia-Wei For symmetric domains (like balls) and for $SU(3)$ Toda system without singular source, there are five types of bubbling solutions with masses, which are listed in (6).

Ao-Wang Another type bubbling solution with local mass: $(2, 4)$.

Battaglia-Pistoia For symmetric domains (like balls) and for $SU(3)$ Toda system with singular source, they established five types of bubbling solutions with masses, which are listed in (8).

For $SU(3)$ Toda system, the strong concentration property is lost. Indeed, D'Aprile-Pistoia-Ruiz constructed an example such that both u_1 and u_2 blow up and $(\sigma_1, \sigma_2) = (4, 2)$, but e^{u_2} has a residual mass outside the blow up set B.

Outline of the proof

Selection process: we pick out a set Σ_k of all the "bad points" $\{0, x_1^k, \cdots, x_m^k\}$ and it holds that

$$u_i^k + 2\log \mathrm{dist}(x, \Sigma_k) \le C, \quad \forall x \in B(0,1),\ i = 1, 2, \tag{13}$$

Definition

Suppose that (u_1^k, u_2^k) satisfies the Harnack inequality (13) in $B(x_k, 2r_k) \setminus B(x_k, \frac{1}{2}r_k)$. Then we say that u_i^k has fast decay on $\partial B(x_k, r_k)$ if after passing a subsequence,

$$u_i^k(x) + 2\log|x - x_k| \le -L_k, \quad \forall x \in \partial B(x_k, r_k)$$

for some $L_k \to +\infty$ and u_i^k is said to have slow decay on $\partial B(x_k, r_k)$ if there exists a constant C independent of k and

$$u_i^k(x) + 2\log|x - x_k| \ge -C, \quad \forall x \in \partial B(x_k, r_k).$$

Let (u_1^k, u_2^k) be a sequence of solutions to (8). Then there exists a finite set Σ_k such that the following hold

1) There exists $C > 0$ independent of k such that (13) holds.
2) $u_{i_0}^k(x_j^k) = \max_i \max_{B(x_j^k, l_j^k)} u_i^k(x) \to +\infty$ as $k \to \infty$. Setting

$$v_i^k(y) = u_i^k(x_j^k + e^{-\frac{1}{2}u_{i_0}^k(x_j^k)}y) - u_{i_0}^k(x_j^k),$$

then after passing to a subsequence (v_1^k, v_2^k) converges to (v_1, v_2) which satisfies either

$$\Delta v_i + \sum_{j=1}^{2} a_{ij} e^{v_j} = 0 \ \text{ in } \ \mathbb{R}^2, \quad i = 1, 2,$$

or $v_{i_0}^k$ converges to v_{i_0} with v_{i_0} satisfying

$$\Delta v_{i_0} + 2e^{v_{i_0}} = 0 \quad \text{in} \quad \mathbb{R}^2,$$

and $v_j^k \to -\infty$ for $j \neq i_0$.

3) There is a sequence of l_j^k such that both u_1^k and u_2^k have fast decay on $\partial B(x_j^k, l_j^k)$ and $l_j^k \ll |x_j^k|$ for any $x_j^k \in \Sigma \setminus \{0\}$. In addition, $B(x_j^k, l_j^k) \cap B(x_i^k, l_i^k) = \emptyset, \; i \neq j$.

4) Let

$$\sigma_{i,j} = \lim_{k \to +\infty} \frac{1}{2\pi} \int_{B(x_j^k, l_j^k)} h_i^k e^{u_i^k} \, dx, \quad i = 1, 2, \quad j = 1, \cdots, m.$$

Then $(\sigma_{1,j}, \sigma_{2,j})$ satisfies

$$\sigma_{1,j}^2 - \sigma_{1,j}\sigma_{2,j} + \sigma_{2,j}^2 = 2\sigma_{1,j} + 2\sigma_{2,j}, \quad j = 1, \cdots, m,$$

and

$$\sigma_{i,j} \in 2\mathbb{N} \cup \{0\}, \quad i = 1, 2, \quad j = 1, \cdots, m.$$

We could decompose $\Sigma_k = \{0\} \cup S_1 \cup \cdots \cup S_l$, where each S_i is the maximal collection of x_i^k in the following sense:

- $0 \notin S_i$ and there is $x_i^k \in S_i$ such that $\mathrm{dist}(x, y) \le d(S_i) \doteqdot \max_{x^k \in S_i} \mathrm{dist}(x_i^k, x^k), \ \forall x, y \in S_i$.
  Furthermore, we have $\mathrm{dist}(x_i^k, x^k) \ll |x_i^k|, \ \forall x^k \in S_i$,
- $\mathrm{dist}(x_i^k, x_j^k) \ge C \max\{|x_i^k|, |x_j^k|\}$ for some constant $C > 0$,
- The local mass $\hat{\sigma}_i(B(x_j^k, \frac{1}{2}\tau_{S_j}^k)) \in 2\mathbb{N} \cup \{0\}, \ i = 1, 2$, where $\tau_{S_j}^k = \frac{1}{2}\mathrm{dist}(S_j, \Sigma_k \setminus S_j)$.

Important intermediate step

During the combination, we shall meet the following problem

$$\Delta u + 2e^u = 4\pi\alpha_0\delta_{p_0} + \sum_{j=1}^{N} 4\pi\alpha_j\delta_{p_j} \text{ in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^u dx < +\infty, \quad (14)$$

where $p_0 = 0$ and $p_0, p_1, \cdots, p_N$ are distinct points in $\mathbb{R}^2$.

Theorem (Lin-Wei-Y.-Zhang)

Suppose u is a solution to (14) and $\alpha_1, \cdots, \alpha_N$ are positive integers, $\alpha_0 > -1$. Then $\frac{1}{2\pi}\int_{\mathbb{R}^2} e^u dx$ is equal to $2(\alpha_0 + 1) + 2k$ for some $k \in \mathbb{Z}$ or $2k_1$ for some $k_1 \in \mathbb{N}$.

Eremenko-Gabrielov-Tarasov's result

Concerning equation (14), Eremenko-Gabrielov-Tarasov study this equation and obtain the following result

Theorem (Eremenko-Gabrielov-Tarasov)

Let u be a solution of (14), then we have

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} e^u dx = \sum_{i=0}^{N+1} \alpha_i + 2 \tag{15}$$

for some $\alpha_{N+1} > -1$ such that either $\alpha_0 - \alpha_{N+1}$ or $\alpha_0 + \alpha_{N+1}$ is an integer. Moreover, we have

(i). *If $\alpha_0 - \alpha_{N+1}$ is an integer, then $|\alpha_0 - \alpha_{N+1}| \leq \sum_{i=1}^{N} \alpha_i$.*

(ii). *If $\alpha_0 + \alpha_{N+1}$ is an integer, then $|\alpha_0 + \alpha_{N+1}| \leq \sum_{i=1}^{N} \alpha_i$.*

Improved result on the local mass

Based on the result of Eremenko-Gabrielov-Tarasov, we can improve the previous result on computing the local mass of $SU(3)$ Toda system

Theorem (Lin-Y.)

Let $\sigma_i,\ i=1,2$ be the local masses of a sequence of blow up solutions of (8) and (9) hold. Then

$$(\sigma_1,\sigma_2)=(\sigma_1^*+2N_1,\sigma_2^*+2N_2),\ \ (\sigma_1^*,\sigma_2^*)\in\Gamma(\mu_1,\mu_2),\ \ N_1,N_2\in\mathbb{Z}.$$

Furthermore there exists at least one component has the concentration property and the corresponding local mass satisfies

$$\sigma_i=\sigma_i^*+2N_i,\quad N_i\in\mathbb{N}.$$

The idea of the proof

We have already seen

$$(\sigma_1, \sigma_2) = (\sigma_1^* + 2N_1, \sigma_2^* + 2N_2),$$

with $(\sigma_1^*, \sigma_2^*) \in \Gamma(\mu_1, \mu_2)$, $N_1, N_2 \in \mathbb{Z}$. For the local mass contributed in the bubbling disk centered at 0, we have

$$(\hat{\sigma}_1(\boldsymbol{\tau}), \hat{\sigma}_2(\boldsymbol{\tau})) \in \Gamma(\mu_1, \mu_2).$$

We separate our discussion into three parts according to the value of $(\hat{\sigma}_1(\boldsymbol{\tau}), \hat{\sigma}_2(\boldsymbol{\tau}))$:

Case 1. At least one of $\hat{\sigma}_i(\boldsymbol{\tau})$ is $2\mu_1 + 2\mu_2$,

Case 2. $(\hat{\sigma}_1(\boldsymbol{\tau}), \hat{\sigma}_2(\boldsymbol{\tau})) = (2\mu_1, 0)$ or $(\hat{\sigma}_1(\boldsymbol{\tau}), \hat{\sigma}_2(\boldsymbol{\tau})) = (0, 2\mu_2)$,

Case 3. $(\hat{\sigma}_1(\boldsymbol{\tau}), \hat{\sigma}_2(\boldsymbol{\tau})) = (0, 0)$.

The idea of the proof

Case 1. We have $\max_i(\sigma_1, \sigma_2) \geq 2\mu_1 + 2\mu_2$.

Case 2. The key observation: after each step of transformations, we could prove that at least one component of $\hat{\sigma}_i(\mathbf{r})$ satisfies that $2\mu_i - \sum_{j=1}^{2} a_{ij}\hat{\sigma}_j(\mathbf{r}) < 0$, and

$$\hat{\sigma}_i(\mathbf{r}) \in \{2\bar{N}_{i,1},\ 2\mu_1 + 2\bar{N}_{i,2},\ 2\mu_1 + 2\mu_2 + 2\bar{N}_{i,3}\}$$

with $\bar{N}_{i,l} \in \mathbb{N} \cup \{0\},\ i = 1, 2,\ l = 1, 2, 3.$

Case 3. Similar to the Case 2.

A priori estimate of (2)

When $\alpha_p^i \in \mathbb{N},\ i = 1, 2$ for $p \in \mathcal{S}$, we have the following a priori estimate

Theorem

Suppose $\alpha_p^i \in \mathbb{N}$ for any $p \in \mathcal{S}$ and $\rho_i \notin 4\pi\mathbb{N},\ i = 1, 2$, then there exists a constant $C(\rho_1, \rho_2)$ such that for any solution in $\mathring{H}^1(M) \times \mathring{H}^1(M)$ of (2),

$$|\tilde{u}_i| \leq C, \quad i = 1, 2.$$

Remark: We can also derive the same result of $\mathbf{B}_2$ and $\mathbf{G}_2$ Toda system.

SU(3) Toda system (without singularity)

From the compactness result, we can define the Topological degree of (2) if $\rho_i \notin 4\pi\mathbb{N},\ i = 1, 2$.

The degree of the following operator $I + T_{\rho_1,\rho_2}$ is well defined in $\mathring{H}^1(M) \times \mathring{H}^1(M)$, provided $\rho_i \notin 4\pi\mathbb{N},\ i = 1, 2$. Here

$$T_{\rho_1,\rho_2}\begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix} = \Delta^{-1}\begin{pmatrix} 2\left(\frac{\tilde{h}_1 e^{\tilde{u}_1}}{\int_M \tilde{h}_1 e^{\tilde{u}_1} dv_g} - 1\right) - \left(\frac{\tilde{h}_2 e^{\tilde{u}_2}}{\int_M \tilde{h}_2 e^{\tilde{u}_2} dv_g} - 1\right) \\ \\ 2\left(\frac{\tilde{h}_2 e^{\tilde{u}_2}}{\int_M \tilde{h}_2 e^{\tilde{u}_2} dv_g} - 1\right) - \left(\frac{\tilde{h}_1 e^{\tilde{u}_1}}{\int_M \tilde{h}_1 e^{\tilde{u}_1} dv_g} - 1\right) \end{pmatrix}.$$

Generating function

When $\mathcal{S} = \emptyset$, $\Gamma_i = 4\pi\mathbb{N}$, $i = 1, 2$. Then the degree d_{ρ_1,ρ_2} is a homotopic invariant for $\rho_1 \in (4i\pi, 4(i+1)\pi)$ and $\rho_2 \in (4j\pi, 4(j+1)\pi)$, $i, j \in \mathbb{N} \cup \{0\}$.

If $\rho_1 \in (0, 4\pi)$, $\rho_2 \in (4m\pi, 4(m+1)\pi)$, we write d_{ρ_1,ρ_2} as $d_{1,m}$ and have $d_{1,m} = \begin{pmatrix} m - \chi(M) \\ m \end{pmatrix}$.

If $\rho_1 \in (4\pi, 8\pi)$, $\rho_2 \in (4m\pi, 4(m+1)\pi)$, we write d_{ρ_1,ρ_2} as $d_{2,m}$ and define the function

$$g(x) = \sum_{i=0}^{\infty} d_{2,i} x^i. \tag{16}$$

Degree formula

Theorem (Lee-Lin-Wei-Y.)

$$g(x) = \sum_{i=0}^{\infty} d_{2,i} x^i = (1+x+x^2+\cdots)^{1-\chi(M)}(1-\chi(M)(1+x)) \quad (17)$$

The main idea of the proof. We need to compute the difference between $d_{2,i}$ and $d_{1,i}$.

$$d_{2,i} - d_{1,i} = \text{the degree contributed by the bubbling solution.} \quad (18)$$

Shadow system

Consider a sequence of bubbling solution u_{1k}, u_{2k} of the $SU(3)$ Toda system with ρ_1 crosses 4π, $\rho_2 \notin 4\pi\mathbb{N}$. We have u_{1k} blow up and u_{2k} is uniformly bounded from above. We rewrite the system as

$$\begin{cases} \Delta v_{1k} + \rho_1 \left(\frac{h_1 e^{2v_{1k}-v_{2k}}}{\int_M h_1 e^{2v_{1k}-v_{2k}}} - 1 \right) = 0, \\ \Delta v_{2k} + \rho_2 \left(\frac{h_2 e^{2v_{2k}-v_{1k}}}{\int_M h_2 e^{2v_{2k}-v_{1k}}} - 1 \right) = 0, \end{cases} \tag{19}$$

where $v_{1k} = \frac{1}{3}(2u_{1k} + u_{2k})$ and $v_{2k} = \frac{1}{3}(u_{1k} + 2u_{2k})$.

As ρ_{1k} crosses 4π, v_{1k} blow up at p and $v_{2k} \to \frac{1}{2}w$, where (p, w) satisfies the following shadow system

$$\begin{cases} \Delta w + 2\rho_2 \left(\frac{h_2 e^{w-4\pi G(x,p)}}{\int_M h_2 e^{w-4\pi G(x,p)}} - 1 \right) = 0, \\ \nabla \left(\log(h_1 e^{-\frac{1}{2}w})(x) + 4\pi R(x,x) \right) |_{x=p} = 0. \end{cases} \tag{20}$$

Shadow system

Consider a sequence of bubbling solution u_{1k}, u_{2k} of the $SU(3)$ Toda system with ρ_1 crosses 4π, $\rho_2 \notin 4\pi\mathbb{N}$. We have u_{1k} blow up and u_{2k} is uniformly bounded from above. We rewrite the system as

$$\begin{cases} \Delta v_{1k} + \rho_1 \left(\frac{h_1 e^{2v_{1k}-v_{2k}}}{\int_M h_1 e^{2v_{1k}-v_{2k}}} - 1 \right) = 0, \\ \Delta v_{2k} + \rho_2 \left(\frac{h_2 e^{2v_{2k}-v_{1k}}}{\int_M h_2 e^{2v_{2k}-v_{1k}}} - 1 \right) = 0, \end{cases} \tag{19}$$

where $v_{1k} = \frac{1}{3}(2u_{1k} + u_{2k})$ and $v_{2k} = \frac{1}{3}(u_{1k} + 2u_{2k})$.

As ρ_{1k} crosses 4π, v_{1k} blow up at p and $v_{2k} \to \frac{1}{2}w$, where (p, w) satisfies the following shadow system

$$\begin{cases} \Delta w + 2\rho_2 \left(\frac{h_2 e^{w-4\pi G(x,p)}}{\int_M h_2 e^{w-4\pi G(x,p)}} - 1 \right) = 0, \\ \nabla \left(\log(h_1 e^{-\frac{1}{2}w})(x) + 4\pi R(x,x) \right) |_{x=p} = 0. \end{cases} \tag{20}$$

The relation between shadow system and blowup solution

- Description of the bubbling solution: we can prove v_{1k} is a perturbation of the standard bubble around the singular point, and a perturbation of the Green function outside the singular point; while v_{2k} is a perturbation of $\frac{1}{2}w$, i.e,

$$v_{1k} = \underbrace{\text{standard bubble}}_{\text{inside}} + \underbrace{\text{Green function}}_{\text{outside}} + e.s.t,$$
$$v_{2k} = \frac{1}{2}w + e.s.t..$$

- Let $d_T(p, w)$ denote the degree contributed by the bubbling solution, $d_S(p, w)$ denote the degree of the system (20). Then

$$d_T(p, w) = -d_S(p, w).$$

The relation between shadow system and blowup solution

- Description of the bubbling solution: we can prove v_{1k} is a perturbation of the standard bubble around the singular point, and a perturbation of the Green function outside the singular point; while v_{2k} is a perturbation of $\frac{1}{2}w$, i.e,

$$\begin{aligned} v_{1k} &= \underbrace{\text{standard bubble}}_{\text{inside}} + \underbrace{\text{Green function}}_{\text{outside}} + e.s.t, \\ v_{2k} &= \frac{1}{2}w + e.s.t.. \end{aligned}$$

- Let $d_T(p, w)$ denote the degree contributed by the bubbling solution, $d_S(p, w)$ denote the degree of the system (20). Then

$$d_T(p, w) = -d_S(p, w).$$

Topological degree of shadow system

Inserting a parameter **t** into the second equation of (20), we have

$$\begin{cases} \Delta w + 2\rho_2 \left(\frac{h_2 e^{w-4\pi G(x,p)}}{\int_M h_2 e^{w-4\pi G(x,p)}} - \frac{1}{|M|} \right) = 0, \\ \\ \nabla \log \left(h_1 e^{-\mathbf{t}\frac{w}{2}}(x) + 4\pi R(x,x) \right) |_p = 0. \end{cases}$$

Proposition

Let $\rho_2 \in (4m\pi, 4(m+1)\pi)$, $m \in \mathbb{N}$, then we get

$$d_S(p,w) = \chi(M)(b_m + b_{m-1}),$$

where $b_m = \binom{m-\chi(M)}{m}$.

Topological degree of shadow system

Inserting a parameter **t** into the second equation of (20), we have

$$\begin{cases} \Delta w + 2\rho_2 \left(\frac{h_2 e^{w-4\pi G(x,p)}}{\int_M h_2 e^{w-4\pi G(x,p)}} - \frac{1}{|M|} \right) = 0, \\ \\ \nabla \log \left(h_1 e^{-\mathbf{t}\frac{w}{2}}(x) + 4\pi R(x,x) \right) |_p = 0. \end{cases}$$

Proposition

Let $\rho_2 \in (4m\pi, 4(m+1)\pi)$, $m \in \mathbb{N}$, *then we get*

$$d_S(p,w) = \chi(M)(b_m + b_{m-1}),$$

where $b_m = \binom{m - \chi(M)}{m}$.

Future Problem (with singularity)

Same things happen for the Toda system with singularity, and we will also get a shadow system

$$\begin{cases} \Delta w + 2\rho_2 \left(\dfrac{\tilde{h}_2 e^{w - 4\pi G(x,p)}}{\int_M \tilde{h}_2 e^{w - 4\pi G(x,p)}} - 1 \right) = 0, \\ \nabla \left(\log(\tilde{h}_1 e^{-\frac{1}{2}w})(x) + 4\pi R(x,x) \right) |_{x=p} = 0, \end{cases} \tag{21}$$

where

$$\tilde{h}_i = h_i e^{-\sum_{p \in S} 4\pi \alpha_p^i G(x,p)}, \ \ i = 1, 2.$$

Difficult point: How to compute the degree.

Thank you!