Nodal Sets of Solutions to Some Fouth Order Elliptic Equations

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Contents

Nodal sets for solutions of second order elliptic equations

- Nodal set of a function $u(x)$ is $N(u) = \{x|u(x) = 0\}$.
- Yau's conjecture: If $\triangle u + \lambda u = 0$ on an n dimensional C^∞ compact manifold M without boundary, then

$$c\sqrt{\lambda} \leq \mathcal{H}^{n-1}(N(u)) \leq C\sqrt{\lambda}.$$

- J. Brüning and S. T. Yau: two dimensional analytic case, independently.
- H. Donnelly and C. Fefferman: upper bound for the compact higher dimensional analytic manifold without boundary (1988, Invent. Math.).
- F. H. Lin: upper bound for the compact analytic manifold without boundary (1991, Comm. Pure Appl. Math.).
- R. Hardt and L. Simon: upper bound for the n dimensional compact C^∞ manifold without boundary is $C\lambda^{C\sqrt{\lambda}}$ (1989, J. Diff. Geom.).

- H. Donnelly and C. Fefferman: upper bound for 2 dimensional C^∞ compact manifold without boundary is $C\lambda^{3/4}$ (1990, J. Amer. Math. Soc.).
- R. T. Dong: upper bound for 2 dimensional C^∞ compact manifold without boundary is $C\lambda^{3/4}$. By using a very different method (1992, J. Diff. Geom.).
- A. Logunov: upper bound for the 2 dimensional C^∞ compact manifold without boundary is $C\lambda^{3/4-\epsilon}$ (2016, arXiv: 1605. 02595).
- A. Logunov: upper bound for the n dimensional C^∞ compact manifold without boundary is $C\lambda^\beta$ (2018 Annals of Math., 2016, arXiv: 1605. 02587v1).

- T. H. Colding and W. P. Minicozzi: lower bound for the C^∞ compact manifold without boundary is $C\lambda^{(3-n)/4}$ (2011, Comm. Math. Phys.).
- C. D. Sogge and S. Zelditch: lower bound for the C^∞ compact manifold without boundary is $C\lambda^{(7-3n)/8}$ (2011, Math. Res. Lett.).
- A. Logunov: lower bound for Yau's conjecture. (2018 Annals of Math.).
- $\cdots$.

Nodal sets for solutions of high order elliptic equations

- I. Kukavica: upper bound for eigenfunctions of high order linear analytic uniformly elliptic operator (1995, J. d'Analyse. Math.).
- L. Tian and X. P. Yang: upper bound for bi-harmonic and polyharmonic functions (2014, J. Diff. Equ.; 2018, Chinese Ann. Math.).
- L. Tian and X. P. Yang: upper bound for eigenfunctions of bi-harmonic operator on some connected bounded domain $\Omega \subseteq \mathbb{R}^n$ whose boundary is non-analytic (2017, arXiv: 1709. 00153).
- $\cdots$

Difficulties

- If u is a solution of the second order uniformly elliptic equation $\mathcal{L}u = (a_{ij}u_i)_j = 0$, then

$$dim\left(\{u = 0, |\nabla u| = 0\}\right) \le n - 2; \quad singular.$$

- If u is a solution of the fourth order uniformly elliptic equation $\mathcal{L}^2 u = 0$, then

$$dim\left(\{u = 0, \nabla u = 0\}\right) \le n - 1, \quad not \;\; singular.$$
$$dim\left(\left\{u = 0, \nabla u = 0, \nabla^2 u = 0, \nabla^3 u = 0\right\}\right) \le n - 2, \quad singular.$$

- If u satisfies the equation $\mathcal{L}u = 0$, then

$$\|u\|_{L^2(B_r(x_0))} \leq 2^{CN(x_0,r)+C}\|u\|_{L^2(B_{r/2}(x_0))},$$

  where $N(x_0, r)$ is the frequency function of u centered at x_0 with radius r;
- If u is a solution of the fourth order uniformly elliptic equation $\mathcal{L}^2 u = 0$, then it is hard to find a suitable quantity to describe the above doubing condition.

Let u satisfy the equation

$$\mathcal{L}^2 u = 0, \tag{1}$$

where

$$\mathcal{L}w = (a_{ij} w_i)_j$$

is a linear elliptic operator of second order in principal part. Also assume that

$$\begin{cases} \Lambda^{-1}|\xi|^2 \le a_{ij}\xi_i\xi_j \le \Lambda|\xi|^2, & \forall\ \xi \in \mathbb{R}^n, \\ \|a_{ij}\|_{C^4} \le K. \end{cases} \tag{2}$$

Theorem (T - X. P. Yang)

Let u be a solution of (1) *in* $B_1(0)$ *and satisfy* (2)*. Assume that*

$$\|u\|_{L^2(B_r(x_0))} \le 2^N \|u\|_{L^2(B_r(x_0))}, \tag{3}$$

for any $B_r(x_0) \subseteq B_1(0)$*. If* $N > 0$ *large enough, then*

$$\mathcal{H}^{n-1}\{x \in B_{1/2}(0) : \; u(x) = 0\} \le (CN)^{CN}, \tag{4}$$

where C *is a positive constant depending on* n*,* Λ *and* K*.*

Nodal sets comparison lemmas

Lemma (R. Hardt, L. Simon, 1989)

There exists a positive constant $\eta_0 < 1/2$ *such that for any* $\eta \in (0, \eta_0)$ *and* $w_1,\ w_2 \in C^{1,1/2}(B_2(0))$*, if* $\|w_1\|_{C^{1,1/2}},\ \|w_2\|_{C^{1,1/2}} \leq 1$ *and* $\|w_1 - w_2\|_{C^1} \leq \eta^5/8$*, then*

$$\begin{aligned} & \mathcal{H}^{n-1}\left(B_{2-\eta}(0) \cap \{w_1 = 0,\ |\nabla w_1| \geq \eta\}\right) \\ \leq\ & (1 + c\sqrt{\eta})\mathcal{H}^{n-1}\left(B_2(0) \cap \{w_2 = 0,\ |\nabla w_2| \geq \eta/2\}\right). \end{aligned}$$

Here c is a positive constant depending only on n.

Lemma

There exists a positive constant $\eta_0 < 1/2$ *such that for any* $\eta \in (0, \eta_0)$ *and* $w_1 \in C^{2,1/2}(B_2(0))$, *if*

$$\begin{cases} \|w_1\|_{C^{2,1/2}},\ \|w_2\|_{C^{2,1/2}} \le 1, \\ \|w_1 - w_2\|_{C^2} \le \eta^5/8, \end{cases}$$

where w_2 *is a polynomial of degree* d, *then*

$$\mathcal{H}^{n-1}\left(B_{2-\eta}(0) \cap \left\{w_1 = 0,\ |\nabla w_1| \le \kappa\eta^5,\ |\nabla^2 w_1| \ge \eta\right\}\right) \le Cd,$$

where η_0, C *and* κ *are positive constants depending only on* n.

Sketch of the proof:

- Step 1. Let $S = \left\{w_1 = 0,\ |\nabla w_1| \leq \kappa\eta^5,\ |\nabla^2 w_1| \geq \eta\right\}$. Let ν_j be the direct such that

$$|\nabla_{\nu_j\nu_j} w_j(y)| = \max_{\nu\in\partial B_1} |\nabla_{\nu\nu} w_j(y)|,\quad j = 1, 2,$$

  such that $\nu_1 \cdot \nu_2 \geq 0$. For $x^* \in S$, there are at most two nodal surfaces of w_1 between the two hyperplanes $\left\{x_n = x_n^* \pm \eta^4\right\}$ in the ball $B_{\eta^3}(x^*)$. We use $\Psi^k_{x^*}(y)$ to denote them.
- Step 2. We also find that in $B_{\eta^3}(x^*)$, there exists a nodal surface $\nabla_{\nu_1} w_2$.

- Step 3. Let $\mathcal{C}$ be the collection of orthonormal bases of $\mathbb{R}^n$ such that for any unit director ν of $\mathbb{R}^n$, there exists an orthonormal basis $\{\tau_1, \cdots, \tau_n\}$, such that $|\nu - \tau_j| \le c$ for some $j = 1, \cdots, n$, where c is a positive constant depending only on n to be chosen.
- Step 4. Through some calculation, there exists $\Phi_x \in C^1(T_x \cap B_{\eta^3}(x))$, such that

$$B_{\eta^3}(x) \cap graph_{T_x}\Phi_x \subseteq \left\{\nabla_{\tau_{j_0}} w_2 = 0\right\} \cap B_{\eta^3}(x),$$

where τ_{j_0} is a direct of one of the basis contained in $\mathcal{C}$. Moreover,

$$|\nabla\Phi_x(y) - \nabla\Psi_x^k(y)| \le C, \quad y \in B_{\eta^3}(x), \quad k = 1, 2.$$

- Step 5. Let S be the nodal set of w_1. Let $\left\{\phi_j\right\}_{j=1}^{D}$ be a partition of unity for $S_1 \cap B_{2-\eta}(0)$, such that

$$supp\phi_j \subseteq B_{\eta^3}(x_j), \;\; \phi_j \geq c^{-1} \;\; on \;\; B_{\eta^3/2}(x_j), \;\; |\nabla\phi_j| \leq c/\eta^3,$$

where the balls $\left\{B_{\eta^3/2}(x_j)\right\}$ cover $S \cap B_{2-\eta}(0)$, and $\left\{B_{\eta^3/4}(0)\right\}$ are pairwise-disjoint. Also define

$$\begin{cases} F_j^k(y) = \phi_j(y + \Psi_{x_j}^k(y)\nu_1(x_j))\sqrt{1 + |\nabla\Psi_{x_j}^k(y)|^2}, \\ G_j(y) = \phi_j(y + \Phi_{x_j}(y)\nu_1(x_j))\sqrt{1 + |\nabla\Phi_{x_j}(y)|^2}, \end{cases}$$

for $j = 1, \cdots, D$ and $k = 1, 2$.

- Step 6. Then

$$\mathcal{H}^{n-1}\left(S \cap B_{2-\eta}(0)\right) \leq \sum_{j=1}^{D} \sum_{k=1}^{2} \int_{T_{x_j} \cap B_{\eta^3}(x_j)} F_j^k(y)dy$$

$$\leq \quad 2\sum_{j=1}^{D} \int_{T_{x_j} \cap B_{\eta^3}(x_j)} G_j(y)dy + \sum_{j=1}^{D} \sum_{k=1}^{2} \int_{T_{x_j} \cap B_{\eta^3}(x_j)} (F_j^k - G_j)(y)dy$$

$$\leq \quad C \sum_{\{\tau_j\} \in \mathcal{C}} \sum_{j=1}^{n} \mathcal{H}^{n-1}\left(B_2 \cap \left\{\nabla_{\tau_j} w_2 = 0\right\}\right)$$

$$+ \quad \sum_{j=1}^{D} \sum_{k=1}^{2} \int_{T_{x_j} \cap B_{\eta^3}(x_j)} (F_j^k - G_j)(y)dy \leq Cd.$$

Lemma

There eixsts a positive constant $\eta_0 < 1/2$ depending only on n such that for any $\eta \in (0, \eta_0)$, any $w_1 \in C^{3,1/2}$ and any polynomial w_2 of degree d, with

$$\begin{cases} \|w_1\|_{C^{3,1/2}},\ \|w_2\|_{C^{3,1/2}} \le 1, \\ \|w_1 - w_2\|_{C^3} \le \eta^5/8. \end{cases}$$

Then

$$\mathcal{H}^{n-1}\left(B_{2-\eta}(0) \cap \left\{w_1 = 0,\ |\nabla w_1| \le \kappa\eta^9,\ |\nabla^2 w_1| \le \kappa\eta^5, |\nabla^3 w_1| \ge \eta\right\}\right) \le Cd.$$

Here η_0, C and κ are positive constants depending only on n.

Two other preparations

- Approximate u by a bi-harmonic function.

Lemma

Let $a_{ij}(0) = \delta_{ij}$, $|a_{ij}(x) - a_{ij}(0)| \leq \epsilon^{2N}$ for $\rho < 1$, $y \in B_1(0)$ such that $B_\rho(y) \subseteq B_1(0)$, and $\mathcal{L}^2 u = 0$ on $B_1(0)$ satisfies the assumption (3). Then there exist $\epsilon_0 > 0$ depending only on n and bi-harmonic polynomial ϕ_B of degree $2N + 3$ such that

$$|u_B - \phi_B|_{C^3(B_2(0))} \leq (c\epsilon)^N,$$

where $u_B(x) = (\|u\|_{L^2(B)}\|)^{-1} u(y + \rho x)$, $B = B_\rho(y) \subseteq B_1(0)$, and c is a positive constant depending only on n.

- Quantitative estimate for the volume of the neighbor- hood to the singular set of u.

Lemma

Let ϕ be a d degree bi-harmonic polynomial with $d \geq 3$. Assume that $\sup_{B_1} |\phi(x) - \phi(0)| = 1$. Also assume that $|\nabla \phi(0)|$, $|\nabla^2 \phi(0)|$, $|\nabla^3 \phi(0)|$ are less than or equal to $(\theta\epsilon)^{d-1}$, $(\theta\epsilon)^{d-2}$, $(\theta\epsilon)^{d-3}$, respectively. Then there exist positive constants θ and ϵ_0 depending only on n, such that for any $\epsilon \in (0, \epsilon_0)$,

$$\mathcal{H}^n\left(\left\{x \in B_1(0) \,|dist\left(x, \left\{ |\nabla^3 \phi| \leq (\theta\epsilon)^{d-3}\right\}\right) < \epsilon\right\}\right) \leq C d^{2n+2} \epsilon^2 \log \epsilon^{-1},$$

where C is also a positive constant depending only on n.

The sketch of the measure estimates of nodal sets

The idea of the proof of the measure estimates of nodal sets.

- Step 1. Assume that $|a_{ij}(x) - a_{ij}(0)| \leq \epsilon^{2N}$ in $B_1(0)$, and $a_{ij}(0) = \delta_{ij}$, where ϵ is a positive constant to be chosen. Let $\eta = \epsilon^{N/9}/2$.
- Step 2. From the above lemmas, we have

$$\mathcal{H}^{n-1}\left\{u_B^{-1}(0) \cap B_1 \cap \left\{|\nabla^3 u_B| \geq \eta/2\right\}\right\} \leq c(2N+3),$$

where $B = B_\rho(y)$ and $y \in u^{-1}(0) \cap B_1(0)$.

- Step 3. Let $\mathcal{Q}_0 = \{B_1(0)\}$. Assume that $\mathcal{Q}_j$, $j = 0, 1, \cdots, l-1$ are already defined such that each ball in the collection $\mathcal{Q}_j$ is centered in $B_1 \cap u^{-1}(0)$ with radius ϵ^{j-1}, such that

$$B_1(0) \cap |\nabla^3 u|^{-1}(0) \cap u^{-1}(0) \subseteq \cup_{\mathcal{Q}_i} B, \quad i = 0, \cdots, l-1.$$

  For each $B \in \mathcal{Q}_{l-1}$, cover $u^{-1}(0) \cap B$ with a collection $\mathcal{Q}_l^B$ of balls with centers in $u^{-1}(0) \cap B_1(0)$ and radius ϵ^{l-1} such that the balls of the same centers and $1/2$ radius are pairwise disjoint. Let $\mathcal{Q}_l = \cup_{B \in \mathcal{Q}_{l-1}} \mathcal{Q}_l^B$.

- Step 4. Then by the dilation and iteration,

$$\begin{aligned}\mathcal{H}^{n-1}\left\{u^{-1}(0)\cap B_1\right\} &\leq c(2N+3)\sum_{l=1}^{\infty}(c(2N+3)^{2n+2}\epsilon\log\epsilon^{-1})^{l-1}\\ &\leq c(2N+3),\end{aligned}$$

if we choose $\epsilon > 0$ small enough such that

$$c(2N+3)^{2n+2}\epsilon\log\epsilon^{-1} < 1/2.$$

- Step 5. Choose $\epsilon = c(2N+3)^{-(4n+4)}$, $\rho_0 = c\epsilon^{2N}$, we have

$$\mathcal{H}^{n-1}\left\{x\in B_{\rho_0}(0)|\, u=0\right\} \leq c(2N+3)\rho_0^{n-1},$$

and then

$$\mathcal{H}^{n-1}\{x\in B_{1/2}(0)|\, u=0\} \leq c(2N+3)\rho_0^{n-1}\frac{C}{\rho_0^n} \leq (CN)^{CN},$$

provided that N is large enough.

Eigenvalue problem 1

$$(\mathcal{L}+\lambda)^2 u = 0 \quad in \quad B_1(0), \tag{5}$$

where $\mathcal{L}$ is the second order uniformly elliptic operator of divergence form as before.

Theorem (T - X. P. Yang)

Let u satisfy (5) in $B_1(0)$. Then for $\lambda > 0$ large enough,

$$\mathcal{H}^{n-1}\left(x \in B_{1/2}(0) : \ u(x) = 0\right) \leq (C\lambda)^{C\sqrt{\lambda}}, \tag{6}$$

where C is a positive constant depending on n, K and Λ.

The doubling condition

Lemma

Let u be a solution of (5). Then for any $r < r_0/\sqrt{\lambda}$ with $B_r(x_0) \subseteq B_1(0)$ and $\lambda > 0$ large enough,

$$\|u\|_{L^2(B_r(x_0))} \leq 2^{C\sqrt{\lambda}} \|u\|_{L^2(B_{r/2}(x_0))},$$

where C and r_0 are positive constant depending on n, K and Λ.

Sketch of the proof.

- Step 1.

$$\begin{cases} \mathcal{L}u + \lambda u = v, \\ \mathcal{L}v + \lambda v = 0. \end{cases}$$

Since v is an eigenfunction of $\mathcal{L}$,

$$\|v\|_{L^2(B_r)(x_0)} \leq 2^{C\sqrt{\lambda}} \|v\|_{L^2(B_{r/2}(x_0))},$$

provided that $r \leq r_0$ and $B_r(x_0) \subseteq B_1(0)$.

- Step 2.
Through some interior estimates, for any $r \leq r_0/\sqrt{\lambda}$,

$$\|v\|_{L^2(B_{r/2}(x_0))} \leq \frac{C}{r^2} \|u\|_{L^2(B_r(x_0))}.$$

- Step 3.
  For a given $B_r(x_0)$, let u_1 and u_2 satisfy the following two equations respectively.

$$\begin{cases} \mathcal{L}u_1 + \lambda u_1 = 0 \ \ in \ \ B_r(x_0), & u_1 = u \ \ on \ \ \partial B_r(x_0), \\ \mathcal{L}u_2 + \lambda u_2 = v \ \ in \ \ B_r(x_0), & u_2 = 0 \ \ on \ \ \partial B_r(x_0). \end{cases}$$

  Since u_2 satisfies the homogeneous Dirichlet boundary condition,

$$\begin{aligned} \|u_2\|_{L^2(B_r(x_0))} &\le Cr^2 \|v\|_{L^2(B_r(x_0))} \le Cr^2 2^{C\sqrt{\lambda}} \|v\|_{L^2(B_{r/4}(x_0))} \\ &\le Cr^2 2^{C\sqrt{\lambda}} \frac{C}{r^2} \|u\|_{L^2(B_{r/2}(x_0))} \\ &\le 2^{C\sqrt{\lambda}} \|u\|_{L^2(B_{r/2}(x_0))}. \end{aligned}$$

- Step 4.
  Since u_1 is an eigenfunction of $\mathcal{L}$,

$$\begin{aligned}
\|u_1\|_{L^2(B_r(x_0))} &\leq 2^{C\sqrt{\lambda}}\|u_1\|_{L^2(B_{r/2}(x_0))} \\
&\leq C2^{C\sqrt{\lambda}}\left(\|u\|_{L^2(B_{r/2}(x_0))} + \|u_2\|_{L^2(B_r(x_0))}\right) \\
&\leq C2^{C\sqrt{\lambda}}\left(\|u\|_{L^2(B_{r/2}(x_0))} + 2^{C\sqrt{\lambda}}\|u\|_{L^2(B_{r/2}(x_0))}\right) \\
&\leq 2^{C\sqrt{\lambda}}\|u\|_{L^2(B_{r/2}(x_0))}.
\end{aligned}$$

- Step 5.
  Then from the above calculation,

$$\|u\|_{L^2(B_r(x_0))} \leq C\left(\|u_1\|_{L^2(B_r(x_0))} + \|u_2\|_{L^2(B_r(x_0))}\right) \leq 2^{C\sqrt{\lambda}}\|u\|_{L^2(B_{r/2}(x_0))}.$$

Some extensions

- Measure estimates of nodal sets of solutions to the equation

$$(\triangle + \lambda)^2 u = 0,$$

on a C^∞ bounded compact mainifold without boundary.
- For the following equations, the same method can also be used.

1. $(\mathcal{L} + \lambda_1)(\mathcal{L} + \lambda_2)u = 0;$

2. $(\mathcal{L}_1 + \lambda_1)(\mathcal{L}_2 + \lambda_2)u = 0;$

3. $\mathcal{L}w = (a_{ij}w_i)_j + b_i w_i + cw.$

Eigenvalue problem 2

$$\mathcal{L}^2 u = \lambda^2 u \ \ in \ \ \Omega, \tag{7}$$

where $\Omega \in C^2$. Also assume that one of the boundary condition is

$$a(\mathcal{L}u + \lambda u) + (\mathcal{L}u + \lambda u)_\nu = 0 \ \ on \ \ \partial\Omega. \tag{8}$$

Theorem (T - X. P. Yang)

Let u be a solution of (7) in a bounded connected domain $\Omega \in C^2$ with (8) as one of its boundary codition. Then for $\lambda > 0$ large enough,

$$\mathcal{H}^{n-1}\left(x \in \Omega' : \ u(x) = 0\right) \leq (C\lambda)^{C\sqrt{\lambda}}, \tag{9}$$

where $\Omega' = \left\{x \in \Omega : \ dist(x, \partial\Omega) > c/\sqrt{\lambda}\right\}$, C and c are positive constants depending only on n, K, Λ, a and Ω.

The doubling condition

Lemma

Let u be a solution of (7) and (8) is one of its boundary condition. Then for any $r < r_0/\sqrt{\lambda}$ with $B_r(x_0) \subseteq \Omega$ and $\lambda > 0$ large enough,

$$||u||_{L^2(B_r(x_0))} \leq 2^{C(\sqrt{\lambda}+M(x_0,r))}||u||_{L^2(B_{r/2}(x_0))},$$

where C and r_0 are positive constant depending on n, K and Λ. Here $M(x_0, r)$ is the frequency function of $v = \mathcal{L}u + \lambda u$ defined as follows.

$$M(x_0, r) = r\frac{D(x_0, r)}{H(x_0, r)} = r\frac{\int_{B_r(x_0)}(a_{ij}v_iv_j + \lambda v^2)dx}{\int_{\partial B_r(x_0)} \mu v^2 d\omega},$$

where $\mu(x) = a_{ij}x_ix_j/|x|^2$.

The upper bound for $M(x_0, r)$

Lemma

Let v be a solution of the equation $\mathcal{L}v - \lambda v = 0$ in Ω with the boundary condition $av + v_\nu = 0$. Also assume that $\partial\Omega \in C^2$. Then for $0 < r < r_0$ and $B_r(x_0) \subseteq \Omega$,

$$M(x_0, r) \le C,$$

where C and r_0 are positive constants depending on n, K, Λ, a and Ω.

Sketch of the proof.

- Step 1. Monotonicity formula.

$$\frac{M'(x_0, r)}{M(x_0, r)} \geq -C,$$

for $0 < r < r_0$.

- Step 2. Doubling condition for v.
For $0 < r_1 \leq \frac{r_2}{\Gamma} \leq r_2 \leq r_0$,

$$\|v\|_{L^2(B_{r_2}(x_0))} \leq \left(\frac{r_2}{r_1}\right)^{CM(x_0, r_2)+C} \|v\|_{L^2(B_{r_1}(x_0))},$$

$$\|v\|_{L^2(B_{r_2}(x_0))} \geq \left(\frac{r_2}{r_1}\right)^{C'M(x_0, r_1)-C'} \|v\|_{L^2(B_{r_1}(x_0))}.$$

- Step 3.
  By the global estimate of v, we have

$$\|v\|_{W^{1,2}(\Omega)} \leq C\|v\|_{L^2(\Omega)}.$$

- Step 4.
  From the global propagation of smallness,

$$\|v\|_{L^2(B_{r_0/\Gamma}(x_0))} \geq C.$$

  Then the upper bound of $M(x_0, r)$ comes from the doubling condition for v.

Propagation of smallness

Lemma (G. Alessandrini et al., 2009)

Let v satisfies the equation $\mathcal{L}v = 0$. Then for any $B_{r_0}(x_0) \subseteq \Omega$,

$$\|v\|_{L^2(\Omega)} \leq \|v\|_{W^{1,2}(\Omega)} \frac{C}{\left(\log\left(\frac{\|v\|_{W^{1,2}(\Omega)}}{\|v\|_{L^2(B_{r_0/\Gamma}(x_0))}}\right)\right)^{\sigma}},$$

where σ and C are positive constant depending on n, K, Λ, Ω, Γ and r_0.

Further Works

- Use Logunov's method to improve our results.
- Find a suitable quantity to describe the index of doubling condition for solution of the first equation (1).
- Find a method to give a global estimate of nodal sets for solutions of eigenvalue problems of the fourth order elliptic operator as in Section 3.
- Consider the corresponding Steklov problems and other higher order equations.

Thank you!