Lecture Fifty-One :

Hardy's multiquadratics and Interpolation on a multivariate Grid

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Outline

1. Hardy's Multiquadratics
2. Cardinal Interpolation
3. Flat Interpolation

Hardy's Multiquadratics

$$x^1, x^2, \ldots, x^n \in \mathbb{R}^d$$

at

$$y_1, y_2, \ldots, y_n \in \mathbb{R}$$

by a function of the form

$$f(x) = a_1\sqrt{c^2 + \|x - x^1\|^2} + \cdots + a_n\sqrt{c^2 + \|x - x^n\|^2}$$

so that

$$f(x^i) = y_i, \quad i \in \mathbb{N}_n.$$

Hardy's Multiquadratics

Fact

If

$$A_{ij} = \sqrt{c^2 + \|x^i - x^j\|^2}, \quad i,j \in \mathbb{N}_n$$

then

$$(-1)^{n-1} \det A > 0$$

for any $x^1, x^2, \ldots, x^n \in \mathbb{R}^d$.

Gamma function trick

$$\Gamma\left(\frac{1}{2}\right) = \int_{[0,\infty]} e^{-t} t^{-\frac{1}{2}}\, dt.$$

Change to $t = \sigma x$

$$\frac{1}{\sigma^{\frac{1}{2}}} = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_{[0,\infty]} e^{-\sigma t} t^{-\frac{1}{2}}\, dt.$$

Integrate

$$\int_{[0,\sigma]} \frac{d\alpha}{\alpha^{\frac{1}{2}}} = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_{[0,\infty]} \left(\int_{[0,\sigma]} e^{-\alpha t}\, d\alpha \right) t^{-\frac{1}{2}}\, dt$$

$$2\sigma^{\frac{1}{2}} = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_{[0,\infty]} \frac{1 - e^{-\sigma t}}{t} t^{-\frac{1}{2}}\, dt.$$

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Now, do a replacement

$$\sqrt{c^2 + \|x^i - x^j\|^2} = \frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_{[0,\infty]} \frac{1-e^{-\left(c^2+\|x^i-x^j\|^2\right)t}}{t} t^{-\frac{1}{2}}\, dt$$

and inportant fact

$$\sum_{i\in\mathbb{N}_n}\sum_{j\in\mathbb{N}_n} b_i b_j \left(c^2 + \|x^i - x^j\|^2\right)^{\frac{1}{2}}$$

$$= -\frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_{[0,\infty]} \sum_{i\in\mathbb{N}_n}\sum_{j\in\mathbb{N}_n} b_i b_j e^{-t\|x^i - x^j\|^2} \frac{e^{-c^2 t}}{t^{\frac{1}{2}}}\, dt$$

$$\leq 0$$

when $\sum_{j\in\mathbb{N}_n} b_j = 0$.

Therefore, the quadratic form for the matrix

$$A = \left(c^2 + \|x^i - x^j\|^2\right)^{\frac{1}{2}}, \qquad c \in \mathbb{R},$$

is conditionally negative definite.

Also, we have that

$$\lambda_n = \max \frac{(Ax, x)}{(x, x)} \geq \frac{(Ae, e)}{(e, e)}$$
$$= \frac{1}{n} \sum_{i \in \mathbb{N}_n} \sum_{j \in \mathbb{N}_n} A_{ij} > \frac{1}{n}$$

and

$$\lambda_{n-1} = \min_{y \in \mathbb{N}^n} \max_{(x,y)=0} \frac{(Ax, x)}{(x, x)} \leq \max_{(x,e)=0} \frac{(Ax, x)}{(x, x)} < 0.$$

Cardinal Interpolation

Cardinal Functions

$$\Phi(x) = \phi(\|x\|)$$

$$\|x\| = \text{ euclidean norm of } x \in \mathbb{R}^n$$

$$\chi(x) = \sum_{j \in \mathbb{Z}^n} c_j \Phi(x - j), \quad x \in \mathbb{R}^n$$

$$\chi(\ell) = \delta_{0\ell}, \quad \ell \in \mathbb{Z}^n$$

$$(If)(x) = \sum_{j\in\mathbb{Z}^n} f(j)\chi(x-j)$$

completely monotonic Functions

$$(-1)^j g^{(j)}(t) \geq 0, \qquad t \geq 0, \quad j \in \mathbb{Z}_+.$$

Notation

$$f(t) \sim At^{-n}, \quad t \to a$$

$$A, \alpha \in \mathbb{R}$$

$$a \in \mathbb{R} \cup \{+\infty\}$$

$$\left|f(t) - At^{-\alpha}\right| = o\left(t^{-\alpha}\right), \quad t \to a$$

Hypotheses

1. ϕ is continuous on $(0,\infty)$ and C^∞ on $[0,\infty)$.
2. $$\xi(t) = \frac{d^k}{dt^k}\left(\phi(\sqrt(t))\right), t > 0$$
   for some nonnegative integer is completely monotonic.
3. If $k = 0$ we also require $\xi(0^+)$ is bounded
4. There are constants α and α' with
   (i) $\alpha, \alpha' \leq \frac{1}{2}n + k - 1$
   (ii) $0 < \alpha < \frac{1}{2}n + 1$
   (iii) For $k > 0$, $0 \leq \alpha' < k$
   (iv) $\xi(t) \sim At^{-\alpha}, t \to +\infty$
   $\xi(t) \sim A^0 t^{-\alpha'}, t \to 0^+$

Conclusion

$$\chi(x) = \sum_{j \in \mathbb{Z}^n} c_j \cdot \Phi(x - j) \tag{2.1}$$

a. $|c_j| = o\left(\|j\|^{-n-\mu}\right)$ and (2.1) converges where

$$\mu = 2k + n - 2\alpha$$

b. $\chi(x)$ above in (2.1) satisfies

$$\chi(j) = \delta_{oj}$$

and

$$|\chi(x)| == o\left(\|x\|^{-n-\mu}\right)$$

c.

$$(If)(x) = \sum_{j \in \mathbb{Z}^n} c_j \cdot \Phi(x - j).$$

I preserves $P_n^{\lfloor\mu\rfloor}$ the linear space of polynomial of total degree $\lfloor\mu\rfloor$ on $\mathbb{R}^n$

d.

$$(I_n f)(x) = \sum_{j \in \mathbb{Z}^n} f(jn)\chi\left(\frac{x}{h} - j\right)$$

scaled interpolation

$$\|I_n f - f\|_\infty = o\left(n^{\lfloor\mu\rfloor} |\log n|\right), \qquad n \to 0^+.$$

Example 1

q nonnegative integer

$$k = q + 1$$

$$\alpha = \alpha' = \frac{1}{2}$$

$$\mu = 2q + n + 1$$

$$\phi(r) = r^{2q+1}$$

Example 2

$$k = 1, \quad \alpha = \frac{1}{2}, \quad \alpha' = 0, \quad \mu = n + 1$$

$$\phi(r) = \sqrt{r^2 + c^2}$$

Example 3

q nonnegative integer

$$k = q + 1$$
$$\alpha = \alpha' = 1$$
$$\mu = 2q + n$$
$$\phi(r) = r^{2q} \log n$$

Are we finished?
NO!

More grids

$$\{\alpha_i + j : 0 \leq i \leq m, j \in \mathbb{Z}^n\}$$

$$\alpha_1, \ldots, \alpha_N \in (0,1)^n$$

$$\alpha_0 = 0$$

$$L_k(x) = \sum_{r \in \mathbb{Z}_{m+1}} \sum_{\ell \in \mathbb{Z}^n} c_{r\ell} \Phi(x - r, -\ell), \qquad x \in \mathbb{Z}^n.$$

But, we can do the following :

Flat Interpolation

$$f(x) = \sum_{i \in \mathbb{N}_n} a_i \phi(x - x_i)$$

$$f(x_i) = y_i$$

Lagrange vectors

$$u(x) = (u_i(x) : i \in \mathbb{N}_n)$$

$$u_i(j) = \delta_{ij}$$

$$u^{\varepsilon}(x) = (u_i(\varepsilon x) : i \in \mathbb{N}_n)$$

What is the limit

$$\lim_{\varepsilon \to 0} u^{\varepsilon}(x) = ?$$

We shall identify

$$\lim_{\varepsilon \to 0} u^{\varepsilon}(x)$$

when

$$\lim_{\theta \to 0} H_{2\gamma}(\theta)\widehat{\phi}(\theta) = c$$

$$c > 0$$

$$2\gamma > d$$

H_{2r} is positive and homogeneous function of 2γ.

Fact

$\lim_{\varepsilon \to 0} u(\varepsilon x)$ is the Lagrange vector for polyharmonic interpolation.

Thank You!