Non-Convex Methods for Low-Rank Matrix Reconstruction

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Low-rank matrix recovery

Assume $X \in \mathbb{R}^{m\times n}$ and $\mathrm{rank}(X) = r < \min(m, n)$. We want to reconstruct X from its linear measurement $y \in \mathbb{R}^p$

$$y = \mathcal{A}X,$$

where $\mathcal{A} \ : \ \mathbb{R}^{m\times n} \mapsto \mathbb{R}^p$ is a linear operator.

It is challenging to solve this problem because usually $p < mn$.

Example 1: Recommendation System

Netflix problem: Predict the rating of a viewer to a movie based on available ratings.

- X — rating matrix: x_{ij} is the rating of viewer i to movie j.
- Assume X is of low-rank — the rating is given by a few factors

$$x_{ij} = \sum_{k=1}^{r} \text{rating on factor } k = \sum_{k=1}^{r} p_{ik} q_{jk},$$

  where p_{ik} is the opinion of viewer i on factor k and q_{jk} is the quality of movie j on factor k.
- Only a small portion of entries of X is observed

$$\mathcal{A}X = \{x_{ij} \ : \ (i,j) \in \Omega\}, \quad \Omega \subset \{1, \ldots, m\} \times \{1, \ldots, n\}$$

- This problem is also called matrix completion.

Example 2: Phase Retrieval

Only intensities can be recorded by physical instruments. Can we recover the phase information?

- Let $x \in \mathbb{C}^n$ be an unknown vector.
- Intensities of its linear measurements are observed.

$$y_i = |\langle a_i, x\rangle|, \quad i = 1, \ldots, p.$$

- Instead of recovering x, we reconstruct the rank-1 matrix $X = xx^* \in \mathbb{C}^{n\times n}$.
- The observations are linear with respect to X

$$y_i^2 = \langle a_i a_i^*, X\rangle, \quad i = 1, \ldots, p$$

so that

$$\mathcal{A}X = \{\langle a_i a_i^*, X\rangle\}_{i=1}^{p}$$

More examples

- High-dimensional data.
  - Let $X = [\boldsymbol{x}_1, \ \boldsymbol{x}_2, \ \ldots, \ \boldsymbol{x}_n]$, where $\boldsymbol{x}_i \in \mathbb{R}^m$ for all i.
  - Assume all $\boldsymbol{x}_i$ lie on a low dimensional subspace in $\mathbb{R}^n$, which implies X is of low rank.
  - Linear inverse problems on high-dimensional data can be formulated by the problem of low-rank matrix recovery.

More examples

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  - Let $X = [\boldsymbol{x}_1, \ \boldsymbol{x}_2, \ \ldots, \ \boldsymbol{x}_n]$, where $\boldsymbol{x}_i \in \mathbb{R}^m$ for all i.
  - Assume all $\boldsymbol{x}_i$ lie on a low dimensional subspace in $\mathbb{R}^n$, which implies X is of low rank.
  - Linear inverse problems on high-dimensional data can be formulated by the problem of low-rank matrix recovery.
- Homogeneous quadratic inverse problems.
  - Besides phase retrieval, some quadratic inverse problem may be converted to the recovery of a low-rank matrix.
  - Let $\boldsymbol{x} = [x_1, \ x_2, \ \ldots, \ x_m]^T$, and it is measured linear combinations of $x_i x_j^*$ for $1 \leq i, j \leq m$.
  - Let $X = \boldsymbol{x}\boldsymbol{x}^*$. Then the measurements are linear w.r.t. the rank-1 matrix X.
  - Examples: Phase Retrieval, Blind Deconvolution, Euclidean Embedding, Sensor Self-Calibration, ...

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Convex Optimization

To find a low-rank solution of $\mathcal{A}Z = y$

- Solve
$$\min_{Z} \ \mathrm{rank}(Z), \quad \text{s.t.} \quad \mathcal{A}Z = y.$$
- Non-convex, NP-hard.

Convex Optimization

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- Solve

$$\min_{Z} \ \mathrm{rank}(Z), \quad \text{s.t.} \quad \mathcal{A}Z = y.$$

- Non-convex, NP-hard.
- Convex relaxation:

$$\|Z\|_* = \sum_i \sigma_i(Z)$$

  where $\|\boldsymbol{Z}\|_*$ is the nuclear norm of Z, the sum of all singular values of Z.

- Solve

$$\min_{Z} \ \|Z\|_*, \quad \text{s.t.} \quad \mathcal{A}Z = y.$$

Computation of Nuclear Norm Minimization

$$\min_{Z} \ \|Z\|_*, \quad \text{s.t.} \quad \mathcal{A}Z = y.$$

- $\|\cdot\|_*$ is non-smooth: the step size will be extremely small when a forward gradient descent method is used.

Computation of Nuclear Norm Minimization

$$\min_{Z} \|Z\|_*, \quad \text{s.t.} \quad \mathcal{A}Z = y.$$

- $\|\cdot\|_*$ is non-smooth: the step size will be extremely small when a forward gradient descent method is used.
- It will be faster to use a backward method, where we need the proximity operator $(I + \lambda\partial\|\cdot\|_*)^{-1}$.

Theorem (C., Candes, Shen, 2010)

The proximal operator of $\|\cdot\|_*$ is the singular value thresholding (SVT). More precisely, let $Y = U\Sigma V^T \in \mathbb{R}^{m\times n}$ be a given matrix and its SVD. Then,

$$\mathcal{S}_\lambda(Y) = \arg\min_{Z} \frac{1}{2}\|Y - Z\|_F^2 + \lambda\|Z\|_*,$$

where

$$\mathcal{S}_\lambda(Y) = U\max(\Sigma - \lambda I, 0)_+ V^T.$$

Computation of Nuclear Norm Minimization

SVT is a fundamental element in many popular nuclear norm minimization algorithms.

- SVT algorithm [C., Candes, Shen, 2010]

$$\begin{cases} Y_{k+1} = Y_k - \delta\mathcal{A}^*(\mathcal{A}X_k - y) \\ X_{k+1} = \mathcal{S}_\delta(Y_{k+1}). \end{cases}$$

- Iterative soft-threhsolding [Ma, Goldfarb, Chen, 2011]

$$X_{k+1} = \mathcal{S}_{\lambda\delta}(X_k - \delta\mathcal{A}^*(\mathcal{A}X_k - y))$$

- ADMM [Chen, He, Yuan, 2012] [Lin, Chen, Ma, 2011]

$$\cdots$$

- Many others.

Computation of Nuclear Norm Minimization

The bottleneck of these algorithms is the computation of SVT $\mathcal{S}_\lambda(Y)$.

- All singular values exceeding λ and their associated singular vectors are computed.
- For large scale computation, a small rank of $\mathcal{S}_\lambda(Y)$ is needed at each iteration.

Computation of Nuclear Norm Minimization

The bottleneck of these algorithms is the computation of SVT $\mathcal{S}_\lambda(Y)$.

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Disadvantage: The computation is expensive, and it consumes large memory.

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Non-Convex Optimization

Assume the rank r is known or estimated. There are two possible formulations.

- Factorization based methods:

$$\min_{L\in\mathbb{R}^{n\times r},R\in\mathbb{R}^{m\times r}} \|\mathcal{A}(LR^T)-y\|_2^2.$$

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$$\min_{L\in\mathbb{R}^{n\times r}, R\in\mathbb{R}^{m\times r}} \|\mathcal{A}(LR^T) - y\|_2^2.$$

  - Alternating minimization
  - Gradient descent
  - Alternating gradient descent
  - ...

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    - Alternating minimization
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    - ...

- Rank constrained methods:

$$\min_{Z\in\mathbb{R}^{m\times n}} \|\mathcal{A}Z - y\|_2^2, \quad \text{s.t.} \quad \mathrm{rank}(Z) = r.$$

Iterative Hard-Thresholding.

We solve the rank constrained minimization

$$\min_{Z \in \mathbb{R}^{m \times n}} \|\mathcal{A}Z - y\|_2^2, \quad \text{s.t.} \quad \mathrm{rank}(Z) = r.$$

by projected gradient descent

$$X_{l+1} = \mathcal{H}_r(X_l - \alpha_l \mathcal{A}^*(\mathcal{A}X_l - y)),$$

where $\mathcal{H}_r(\cdot)$ is the r-truncated SVD.

- When α_l is fixed, it is known as Singular Value Projection (SVP).
- When α_l is the steepest descent stepsize, it is called normalized IHT (NIHT).

IHT (cont.)

- In each iteration, SVD of $m \times n$ matrices is still needed in each step.
- How to avoid large size SVD?

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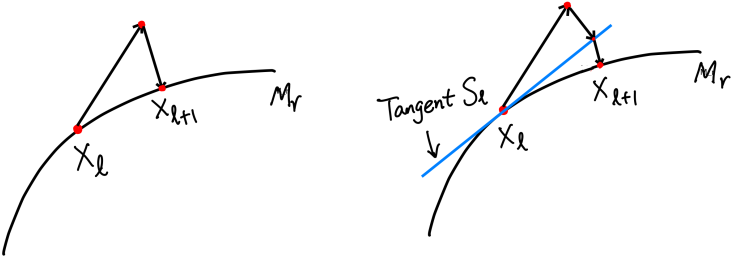
- In each iteration, SVD of $m \times n$ matrices is still needed in each step.
- How to avoid large size SVD?



Our algorithm [Wei, C., Chan, Leung, 2016]

$$X_{l+1} = \mathcal{H}_r \mathcal{P}_{\mathcal{S}_l}(X_l - \alpha_l \mathcal{A}^*(\mathcal{A}X_l - y)).$$

No large scale SVD

- The subspace

$$\mathcal{S}_l = \{U_l P^T + Q V_l^T \ : \ P \in \mathbb{R}^{n\times r}, Q \in \mathbb{R}^{m\times r}\},$$

  is the tangent space of smooth manifold $\mathcal{M}_r$ at $X_l = U_l \Sigma_l V_l^T$, where $\mathcal{M}_r$ is the set of all rank-r matrices embedded in $\mathbb{R}^{m\times n}$.
- The projection $\mathcal{P}_{\mathcal{S}_l}$ is just matrix products.

No large scale SVD

- The subspace

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- The projection $\mathcal{P}_{\mathcal{S}_l}$ is just matrix products.
- SVD of size only $2r \times 2r$ is needed.

$$W_l \in \mathcal{S}_l \Longrightarrow W_l = \overbrace{[U_l \ \ Q]}^{2r} \left.\begin{bmatrix} P^T \\ V_l^T \end{bmatrix}\right\} 2r,$$

  where W_l is the matrix before the application of $\mathcal{H}_r$.
  - First compute QR decompostion of $[U_l \ \ Q]$ and $\begin{bmatrix} P^T \\ V_l^T \end{bmatrix}$ respectively.
  - Then compute SVD of the product of R factors, which is of size $2r \times 2r$.

Riemannian optimization

The algorithm

$$X_{l+1} = \mathcal{H}_r(X_l - \alpha_l \mathcal{P}_{\mathcal{S}_l}\mathcal{A}^*(\mathcal{A}X_l - y)),$$

can be interpreted as a Gradient Descent Algorithm on the Riemannian manifold $\mathcal{M}_r$. [Vandereycken, 2013; Mishra, Apuroop, Sepulchre, 2013; Mishra, Meyer, Bonnabel, Sepulchre, 2013]

Riemannian Gradient Descent (RGad)

$$G_l = \mathcal{P}_{\mathcal{S}_l}(\mathcal{A}^*(\mathcal{A}X_l - y)) \qquad \text{(Gradient on the tangent space)}$$

$$\alpha_l = \frac{\|G_l\|_F^2}{\|\mathcal{A}G_l\|_2^2} \qquad \text{(Steepest Descent Step Size)}$$

$$W_l = X_l - \alpha_l G_l \qquad \text{(Update along the gradient)}$$

$$X_{l+1} = H_r(W_l) \qquad \text{(Retraction)}$$

Riemannian Conjugate Gradient

The algorithm can be further improved by conjugate gradient on Riemannian manifold for solving $\min_{Z\in\mathcal{M}_r} \|\mathcal{A}Z - y\|_2^2$

Riemannian Conjugate Gradient (RCG)

$$
\begin{aligned}
G_l &= \mathcal{P}_{\mathcal{S}_l}(\mathcal{A}^*(\mathcal{A}X_l - y)) && \text{(Gradient on the tangent space)}\\
\beta_l &= -\frac{\langle \mathcal{A}G_l, \mathcal{A}P_{l-1}\rangle}{\|\mathcal{A}P_{l-1}\|_2^2} && \text{(novel formula for } \beta \text{ [Wei, C., Chan, Leung, 2016])}\\
P_l &= \mathcal{P}_{\mathcal{S}_l}(G_l + \beta_l P_{l-1}) && (P_l \text{ is conjugate to } P_{l-1})\\
\alpha_l &= \frac{\langle G_l, P_l\rangle}{\|\mathcal{A}P_l\|_2^2} && \\
W_l &= X_l - \alpha_l P_l && \text{(Update along the search direction)}\\
X_{l+1} &= H_r(W_l) && \text{(Retraction)}
\end{aligned}
$$

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Theoretical Gurantee

Will RGrad and RCG find the true low-rank matrix X from $y = \mathcal{A}X$?

- Yes.

  $$\text{Good Initialization} + \text{Local Convergence} \implies \text{Theoretical Guanrantee}$$

- The analysis depends on applications.

Initialization

- We choose
$$X_0 = H_r(\mathcal{A}^* y)$$
- X_0 is one step of IHT with initial guess 0.

Initialization

- We choose
$$X_0 = H_r(\mathcal{A}^* y)$$
- X_0 is one step of IHT with initial guess 0.
- A probabilistic explanation: Assume A_i, $i = 1, \ldots, p$ have i.i.d. entries with expectation 0 and variance $1/p$. Then

$$\mathbf{E}([\mathcal{A}^* y]_{jk}) = \mathbf{E}\left(\sum_i \langle A_i, X\rangle [A_i]_{jk}\right) = \mathbf{E}\left(\sum_{i,a,b} [A_i]_{jk}[A_i]_{ab} X_{ab}\right)$$
$$= \mathbf{E}\left(\sum_{i=1}^{p} [A_i]_{jk}^2\right) \cdot X_{jk} = X_{jk}$$

Case I: Guarantee for $\mathcal{A}$ satisfying RIP

Restricted Isometric Property (s-RIP)

There exists a constant $R_s \in (0,1)$ such that

$$(1-R_s)\|Z\|_F^2 \le \|\mathcal{A}Z\|_F^2 \le (1+R_s)\|Z\|_F^2, \quad \forall Z \in \mathcal{M}_s.$$

Theorem (Wei, C., Chan, Leung, SIMAX, 2016)

Assume $\mathcal{A}$ satisfies RIP with

$$R_{3r} \le \frac{1}{\mathrm{Cond}^2(X)} \frac{1}{25\sqrt{r}}.$$

Then the RGrad algorithm with initial guess $X_0 = H_r(\mathcal{A}^* y)$ converges linearly to X, provided the rank of X is r and $y = \mathcal{A}X$.

When $\mathcal{A}$ satisfies RIP

Theorem (Wei, C., Chan, Leung, SIMAX, 2016)

Assume $\mathcal{A}$ satisfies RIP with

$$R_{3r} \le \frac{1}{\mathrm{Cond}^2(X)} \frac{1}{40\sqrt{r}}.$$

Then the Riemannian conjugate gradient algorithm with:

- *Initial guess $X_0 = H_r(\mathcal{A}^* y)$*
- *Restarting when either $\frac{\langle G_l, P_{l-1}\rangle}{\|G_l\|_F \|P_{l-1}\|_F} \le 0.1$ or $\|G_l\|_F \le \|P_{l-1}\|_F$ violated*

converges linearly to X, provided the rank of X is r and $y = \mathcal{A}X$.

Numerical Experiments

$\mathcal{A}$ is the random Gaussian.

Table: Average computational time (seconds) and average number of iterations of RGrad, RCG, RCG restarted, and ASD over ten random rank r matrices per (m, n, p, r) tuple for $m = n \in \{80, 160\}$, $r \in \{5, 10\}$ and $p/(m+n-r)r \in \{2, 3\}$; Gaussian sensing.

r	5						10					
$1/\rho$	2			3			2			3		
	rel.err	iter	time	rel.err	iter	time	rel.err	iter	time	rel.err	iter	time
	$m = n = 80$											
RGrad	3.3e-05	137	8.52	2.2e-05	58	5.61	3.2e-05	130	24.9	2.1e-05	57	15.5
RCG	2.2e-05	34	2.38	1.4e-05	22	2.71	2.1e-05	34	8.47	1.4e-05	22	7.72
RCG res.	2.2e-05	35	2.81	1.5e-05	22	2.79	2.2e-05	36	8.95	1.3e-05	23	8.12
ASD	2.5e-05	143	10.3	1.7e-05	73	9.37	2.4e-05	210	53.8	1.7e-05	224	82.2
	$m = n = 160$											
RGrad	3.3e-05	142	103	2e-05	61	66.2	3.2e-05	135	194	2.1e-05	58	123
RCG	2.3e-05	35	33.0	1.5e-05	22	31.2	2.2e-05	35	65.7	1.4e-05	23	62.9
RCG res.	2.4e-05	36	33.9	1.5e-05	23	32.1	2.2e-05	36	67.7	1.4e-05	24	66.1
ASD	2.5e-05	147	140	1.8e-05	81	117	2.4e-05	213	407	1.6e-05	149	426

Table: Phase transition table for Gaussian sensing with $m = n = 80$. For each (m, n, p) with $p = \delta \cdot mn$, the algorithm can recover all of the ten random test matrices when $r \leq r_{\min}$, but fails to recover each of the randomly drawn matrices when $r \geq r_{\max}$.

	RGrad				RCG				RCG restarted			
δ	$r_{\min}$	$r_{\max}$	$\rho_{\min}$	$\rho_{\max}$	$r_{\min}$	$r_{\max}$	$\rho_{\min}$	$\rho_{\max}$	$r_{\min}$	$r_{\max}$	$\rho_{\min}$	$\rho_{\max}$
0.1	3	4	0.74	0.97	3	4	0.74	0.97	3	4	0.74	0.97
0.15	4	6	0.65	0.96	4	6	0.65	0.96	4	6	0.65	0.96
0.2	6	8	0.72	0.95	6	8	0.72	0.95	6	8	0.72	0.95
0.25	8	10	0.76	0.94	8	10	0.76	0.94	8	10	0.76	0.94
0.3	11	12	0.85	0.93	11	13	0.85	1	11	13	0.85	1
0.35	12	15	0.79	0.97	12	15	0.79	0.97	11	15	0.73	0.97
0.4	14	17	0.8	0.95	14	17	0.8	0.95	14	17	0.8	0.95
0.45	17	19	0.84	0.93	17	19	0.84	0.93	17	19	0.84	0.93
0.5	20	22	0.88	0.95	20	22	0.88	0.95	20	22	0.88	0.95
0.55	22	24	0.86	0.93	22	24	0.86	0.93	22	24	0.86	0.93
0.6	25	27	0.88	0.94	26	28	0.91	0.96	26	28	0.91	0.96
0.65	28	30	0.89	0.94	28	32	0.89	0.98	28	32	0.89	0.98
0.7	31	33	0.89	0.94	31	35	0.89	0.98	31	35	0.89	0.98
0.75	34	36	0.89	0.93	35	38	0.91	0.97	35	38	0.91	0.97
0.8	38	40	0.91	0.94	40	42	0.94	0.97	40	42	0.94	0.97
0.85	42	44	0.91	0.94	44	47	0.94	0.98	44	47	0.94	0.98
0.9	47	50	0.92	0.95	50	53	0.95	0.98	50	53	0.95	0.98
0.95	52	54	0.92	0.94	57	61	0.97	0.99	57	61	0.97	0.99

Case II: Guarantee for matrix completion

- The operator $\mathcal{A} = \mathcal{P}_\Omega$ doesn't satisfies RIP.
- Matrix completion may fail for any algorithms.
  Example: if the $(1,1)$-entry is not sampled, then any algorithm cannot distinct the following matrices
  $$e_1 e_1^T, \quad 2e_1 e_1^T, \quad \dots$$
- The singular vectors cannot be too sparse.

Assumption 1 [Candes, Recht, 2009; Candes, Tao, 2010]

Let X be an $n \times n$, rank-r matrix with compact SVD $X = U\Sigma V^T$. Assume there exist two positive constants μ_0 and μ_1 such that

$$\frac{n}{r} \max_{1\leq i\leq n} \max\left\{\|\mathcal{P}_U(e_i)\|_2^2, \|\mathcal{P}_V(e_i)\|_2^2\right\} \leq \mu_0, \quad \|X\|_\infty \leq \mu_1 \sqrt{\frac{r}{n^2}}\|X\|_2.$$

Matrix completion

- Initialization: $X_0 = H_r(\mathcal{A}^* y)$
- The first $O(\log N)$ steps uses resampling and trimming.

Theorem (Wei, C., Chan, Leung, *preprint*, 2016)

Let X be fixed and satisfying Assumption 1. Suppose Ω is sampled uniformly at random with $|\Omega| = m$. Then both RGrad and restarted RCG converges linearly to X with probability at least $1 - n^{-2}$ provided

$$m \geq Cnr^2 \log^2 n$$

for some constant $C > 0$.

Matrix completion

Key inequalities in the proof

- RIP in the tangent space of $\mathcal{M}_r$ at X

$$\left\| \mathcal{P}_{\mathcal{T}} \left(\mathcal{I} - \frac{mn}{p} \mathcal{P}_{\Omega} \right) \mathcal{P}_{\mathcal{T}} \right\| \le \epsilon$$

[Candes, Rechet, 2009; Candes, Tao, 2010]

- "asymmetric" isometric property

$$\left\| \mathcal{P}_{\widehat{\mathcal{T}}_\ell} \left(\mathcal{I} - \frac{mn}{p} \mathcal{P}_{\widehat{\Omega}_{\ell+1}} \right) \left(\mathcal{P}_U - \mathcal{P}_{\widehat{U}_\ell} \right) \right\| \le \epsilon$$

[Wei, C., Chan, Leung, 2016]

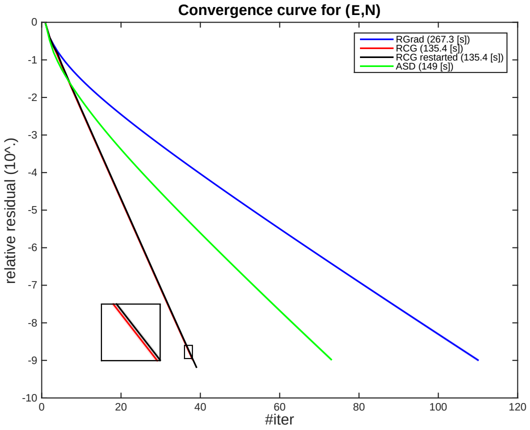
Numerical Experiments: Matrix Completion

Table: Average computational time (seconds) and average number of iterations of RGrad, RCG, RCG restarted, and ASD over ten random rank r matrices per (m, n, p, r) tuple for $m = n \in \{8000, 16000\}$, $r \in \{50, 100\}$ and $p/(m + n - r)r \in \{2, 3\}$; Entry sensing.

r	50						100					
$1/\rho$	2			3			2			3		
	rel.err	iter	time	rel.err	iter	time	rel.err	iter	time	rel.err	iter	time
	$m = n = 8000$											
RGrad	3.2e-05	116	58.9	2.1e-05	52	37.9	3.1e-05	107	184	1.9e-05	54	129
RCG	2.2e-05	36	27.59	1.5e-05	23	24.7	2.1e-05	33	82.0	1.1e-05	22	75.2
RCG res.	2.3e-05	36	27.6	1.6e-05	23	24.6	1.9e-05	34	83.0	1.2e-05	22	75.3
ASD	3.2e-05	89	71.4	2.1e-05	40	38.0	3e-05	74	119	1.9e-05	35	70.0
	$m = n = 16000$											
RGrad	3.2e-05	116	151	2e-05	48	89.3	3.1e-05	97	453	2.1e-05	55	353
RCG	2.3e-05	36	66.9	1.3e-05	24	61.7	2.2e-05	34	209	1.5e-05	22	187
RCG res.	2.1e-05	37	67.4	1.2e-05	24	62.3	2.2e-05	34	211	1.6e-05	22	187
ASD	3.3e-05	92	262	2.1e-05	41	132	3.1e-05	76	351	1.9e-05	36	204

Table: Phase transition table for entry sensing with $m = n = 800$. For each (m, n, p) with $p = \delta \cdot mn$, the algorithm can recover all of the ten random test matrices when $r \leq r_{\min}$, but failes to recover each of the randomly drawn matrices when $r \geq r_{\max}$.

	RGrad				RCG				RCG restarted			
δ	$r_{\min}$	$r_{\max}$	$\rho_{\min}$	$\rho_{\max}$	$r_{\min}$	$r_{\max}$	$\rho_{\min}$	$\rho_{\max}$	$r_{\min}$	$r_{\max}$	$\rho_{\min}$	$\rho_{\max}$
0.1	36	38	0.88	0.93	35	37	0.86	0.9	36	37	0.88	0.9
0.15	55	59	0.89	0.95	55	57	0.89	0.92	55	57	0.89	0.92
0.2	76	78	0.9	0.93	74	77	0.88	0.92	74	77	0.88	0.92
0.25	97	99	0.91	0.93	96	98	0.9	0.92	96	98	0.9	0.92
0.3	119	121	0.92	0.93	117	119	0.9	0.92	117	119	0.9	0.92
0.35	142	143	0.92	0.93	140	142	0.91	0.92	140	142	0.91	0.92
0.4	166	167	0.93	0.93	163	166	0.91	0.93	163	166	0.91	0.93
0.45	190	192	0.93	0.94	188	191	0.92	0.93	188	191	0.92	0.93
0.5	217	219	0.94	0.95	214	217	0.93	0.94	214	217	0.93	0.94
0.55	244	248	0.94	0.95	242	246	0.93	0.95	242	245	0.93	0.94
0.6	274	276	0.95	0.95	272	274	0.94	0.95	272	274	0.94	0.95
0.65	306	308	0.95	0.96	302	306	0.94	0.95	304	306	0.95	0.95
0.7	340	343	0.96	0.96	338	340	0.95	0.96	338	340	0.95	0.96
0.75	378	380	0.96	0.97	374	378	0.96	0.96	374	378	0.96	0.96
0.8	418	422	0.96	0.97	416	420	0.96	0.97	416	420	0.96	0.97
0.85	466	470	0.97	0.98	464	468	0.97	0.97	464	468	0.97	0.97
0.9	524	527	0.98	0.98	522	526	0.98	0.98	522	526	0.98	0.98
0.95	600	604	0.99	0.99	600	604	0.99	0.99	600	604	0.99	0.99


Convergence curve for (E,N)
RGrad (267.3 [s])
RCG (135.4 [s])
RCG restarted (135.4 [s])
ASD (149 [s])
relative residual (10^.)
#iter
0
-1
-2
-3
-4
-5
-6
-7
-8
-9
-10
0
20
40
60
80
100
120

Case III: Guarantee for Phase Retrieval

- Phase Retrieval: Solve $\boldsymbol{x} \in \mathbb{C}^n$ from $|\boldsymbol{A}\boldsymbol{x}| = \boldsymbol{y}$ with known $\boldsymbol{A} \in \mathbb{C}^{m\times n}$ and $\boldsymbol{y} \in \mathbb{R}_+^m$.
- The problem can be reformulated as

$$\mathcal{A}\boldsymbol{X} = \boldsymbol{b},$$

where $X = \boldsymbol{x}\boldsymbol{x}^*$, $[\mathcal{A}\boldsymbol{X}]_i = \boldsymbol{a}_i^* X \boldsymbol{a}_i$, and $\boldsymbol{b} = \boldsymbol{y}^2$.

- $\mathcal{A}$ doesn't satisfies RIP.

Phase Retrieval

- Initialization: $X_0 = H_1(\mathcal{A}^* y)$
- Use only "good" measurements at each iteration.

Theorem (C., Wei, *working paper*, 2017)

Assume entries of $\boldsymbol{A} \in \mathbb{C}^{m \times n}$ *are i.i.d. complex Gaussian. Then the RGrad algorithm converges linearly to* X *with probability at least* $1 - c_0 e^{-c_1 n}$*, provided* $m \geq Cn$*.*

- Experimental results show RCG is much faster than popular non-convex methods, e.g., Wirtinger flow, truncated Wirtinger flow.

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Spectrally Sparse Signal Reconstruction

- Spectrally Sparse Signal:

$$x(t) = \sum_{k=1}^{r} d_k e^{2\pi\imath f_k t} e^{-\tau_k t}$$

  - $r \in \mathbb{Z}_+$ — sparsity
  - $f_k \in [0,1)$, $k = 1, \ldots, r$ — (normalized) frequencies.
  - $d_k \in \mathbb{C}$, $k = 1, \ldots, r$ — complex magnitudes.
  - $\tau_k \in \mathbb{R}_+$, $k = 1, \ldots, r$ — damping factors. When $\tau_k = 0$ for all k, no damping.
- The signal is uniquely determined by its uniform samples at integer times.

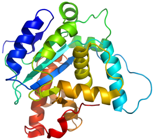
$$\boldsymbol{x} = [x(0),\ x(1),\ x(2), \ldots,\ x(n-1)]^T \in \mathbb{C}^n.$$

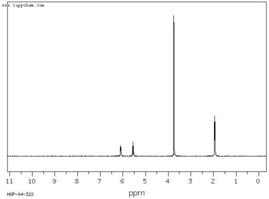
- In many applications, only partial entries of $\boldsymbol{x}$ can be observed. We need to

  Find $\boldsymbol{x}$ from its partial entries $\boldsymbol{x}_\Omega$

  where $\Omega \subset \{0, 1, \ldots, n-1\}$ and $|\Omega| = m < n$.

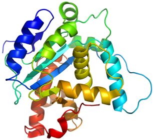
Motivating Example: NMR spectroscopy

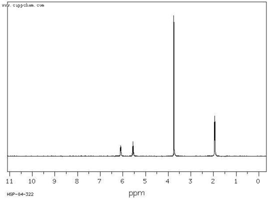




- The signal can be modelled well by a multidimensional spectrally sparse signal with damping.
- Unfortunately, the full sampling of $\boldsymbol{x}$ for a specimen may take a few weeks.
- To save time and cost, non-uniform sampling (NUS) is popular in NMR spectroscopy.

Motivating Example: NMR spectroscopy





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Some other examples include band limited signal sampling, where the frequency of the signal is ultra high and the sampling hardware may not work reliably.

Finite vs. Infinite Dictionary

- Conventional Compressed Sensing (for undamped signals). [CandesRombergTao2006]
  - Use a finite dictionary by discretizing the frequency domain $[0,1)$ with a uniform grid.
  - However, it suffers basis mismatch of the discrete dictionary, and using finer grid doesn't help too much.
- Off-the-grid approaches
  - Uses an infinite dictionary with atoms of frequencies on the continuous domain $[0,1)$, and no gridding is needed.
  - Spectral Super resolution
  - Solved by semi-definite programming (SDP).
  - Atomic Norm Minimization [TangBhaskarShahRecht2013], Enhanced Matrix Completion [ChenChi2014], Super Resolution [CandesFernandez-Granda2012]...
  - They need to solve problems with $O(n^2)$ unknowns. Slow, and intractable for multidimensional signals of moderate size.

From Spectrally Sparsity to Low-Rank Hankel Matrix

- Define the Hankel matrix formed by $\boldsymbol{x}$ as

$$\mathcal{H}\boldsymbol{x} = \left[x_{j+k}\right]_{j,k} \in \mathbb{C}^{n_1 \times n_2},$$

where n_1 and n_2 are prescribed integers satisfying $n_1 + n_2 = n - 1$.

- Then $\operatorname{rank}(\mathcal{H}\boldsymbol{x}) = r$ because of the following Vandermonde decomposition:

$$\mathcal{H}\boldsymbol{x} = \underbrace{\boldsymbol{V}_L}_{n_1 \times r} \underbrace{\boldsymbol{D}}_{r \times r} \underbrace{\boldsymbol{V}_R^T}_{r \times n_2},$$

where $\boldsymbol{V}_L = [e^{2\pi\imath f_k j} e^{-\tau_k j}]_{j,k}$, $\boldsymbol{V}_R = [e^{2\pi\imath f_k j} e^{-\tau_k j}]_{j,k}$ are Vandermonde matrices, and $\boldsymbol{D} = \mathrm{diag}(d_1, d_2, \ldots, d_r)$

- The spectrally sparse signal reconstruction can be converted to Low-rank Hankel Matrix Completion:

Find the rank-r Hankel matrix $\mathcal{H}\boldsymbol{x}$

from its partially known anti-diagonals $\boldsymbol{x}_\Omega$.

Our algorithms

We solve the non-convex optimization

$$\min_{\boldsymbol{z}} \sum_{j\in\Omega} |z_j - x_j|^2 \quad \text{s.t.} \quad \mathrm{rank}(\mathcal{H}\boldsymbol{z}) = r.$$

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$$\min_{\boldsymbol{z}} \sum_{j\in\Omega} |z_j - x_j|^2 \quad \text{s.t.} \quad \mathrm{rank}(\mathcal{H}\boldsymbol{z}) = r.$$

- Iterative Hard Thresholding (IHT)

$$\boldsymbol{x}_{\ell+1} = \mathcal{H}^{\dagger}\mathcal{T}_r\mathcal{H}(\boldsymbol{x}_\ell - p^{-1}\mathcal{P}_\Omega(\boldsymbol{x}_\ell - \boldsymbol{x})),$$

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$$\boldsymbol{x}_{\ell+1} = \mathcal{H}^{\dagger}\mathcal{T}_r\mathcal{P}_{\mathcal{S}_\ell}\mathcal{H}(\boldsymbol{x}_\ell - p^{-1}\mathcal{P}_\Omega(\boldsymbol{x}_\ell - \boldsymbol{x})),$$

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  - Every step can be implemented by FFTs.
  - Can use Takagi decompostion to save half computational cost when the matrix is square.

Theoretical Guarantee of FIHT

We will show that FIHT converges to $\boldsymbol{x}$ linearly provided $m \sim O(r^2 \log^2 n)$

- Assumptions:
  - The elements of Ω is sampled independently and uniformly from $\{0, 1, \ldots, n-1\}$ with replacement.
  - $\mathcal{H}\boldsymbol{x}$ is μ_0-incoherent, which may be viewed as a condition on the separation of frequencies.

Definition

The Hankel matrix $\mathcal{H}\boldsymbol{x}$ with the Vandermonde decomposition $\mathcal{H}\boldsymbol{x} = \boldsymbol{V}_L \boldsymbol{D} \boldsymbol{V}_R^T$ is said of μ_0-incoherent if

$$\sigma_{\min}(\boldsymbol{V}_L^* \boldsymbol{V}_L) \geq \frac{n_1}{\mu_0}, \quad \sigma_{\min}(\boldsymbol{V}_R^* \boldsymbol{V}_R) \geq \frac{n_2}{\mu_0}$$

Guarantee of FIHT

FIHT converges linearly to the correct solution when it is initialized by $L = O(\log n)$ resampling and trimming, provided $m \sim O(r^2 \log^2 n)$.

Theorem (Theoretical Guarantee of FIHT,[C., Wang, Wei, 2016])

Assume $\mathcal{H}\boldsymbol{x}$ is μ_0-incoherent. Let $0 < \varepsilon_0 < \frac{1}{10}$ and $L = \left\lceil 6 \log\left(\frac{\sqrt{n}\log(n)}{16\varepsilon_0}\right)\right\rceil$. Define $\nu = 10\varepsilon_0 < 1$. Then with probability at least $1 - (2L+3)\, n^{-2}$, the iterates generated by FIHT with our initialization satisfies

$$\|\boldsymbol{x}_\ell - \boldsymbol{x}\| \leq \nu^\ell \|\boldsymbol{L}_0 - \mathcal{H}\boldsymbol{x}\|_F,$$

provided

$$m \geq C\mu_0 c_s \kappa^6 r^2 \log(n) \log\left(\frac{\sqrt{n}\log(n)}{16\varepsilon_0}\right)$$

for some universal constant $C > 0$.

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- The new framework of combining $\mathcal{H}_r$ with $\mathcal{P}_{\mathcal{S}_\ell}$ is better than $\mathcal{H}_r$ solely for low-rank matrix recovery problems.

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- The new framework of combining $\mathcal{H}_r$ with $\mathcal{P}_{\mathcal{S}_\ell}$ is better than $\mathcal{H}_r$ solely for low-rank matrix recovery problems.
- The projection $\mathcal{P}_{\mathcal{S}_\ell}$ onto the tangent space helps
  - Computationally: reduce SVD of size $n \times n$ to $O(r) \times O(r)$.
  - Theoretically: help to prove the theoretical guarantee, because the isometric property holds true only in the tangent space.

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[2] K. Wei, **J.-F. Cai**, T.F. Chan and S. Leung, Guarantees of Riemannian Optimization for Low Rank Matrix Completion, *preprint*, 2016.

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Thanks for your attention!

Questions?