Sometimes scale matters – zooming in using multiscale Wendland RBFs

Ian H. Sloan

i.sloan@unsw.edu.au

University of New South Wales, Sydney

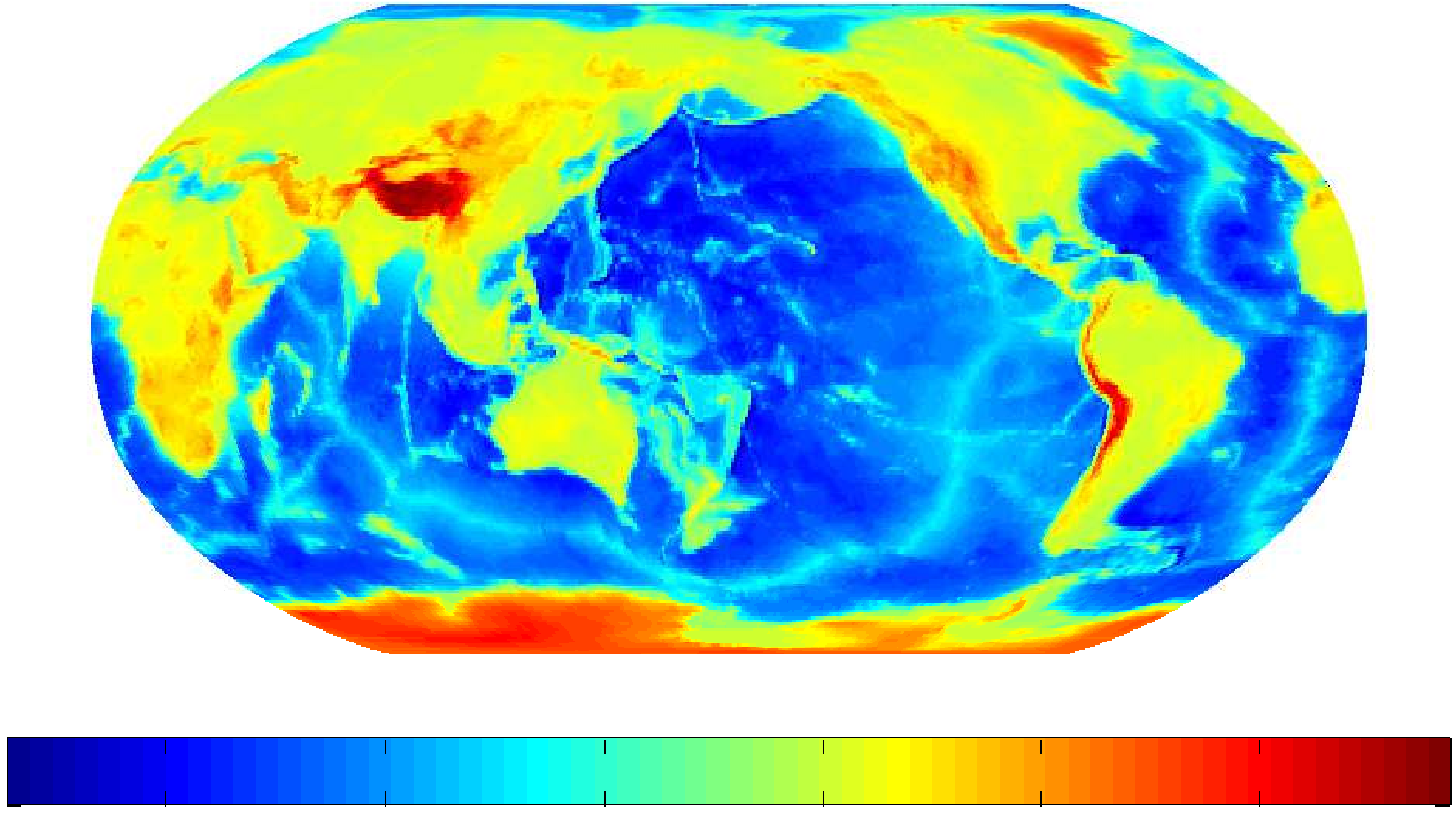
Guangzhou, May 2017

Joint work with Q Thong Le Gia and Holger Wendland

The need for many scales

Physical phenomena often occur on many different scales.

An example: the topography of Earth



Radial basis functions

Radial basis functions (RBFs) in $\mathbb{R}^3$:

$$\boldsymbol{\Psi}(\mathbf{x},\mathbf{y}) = \psi(|\mathbf{x}-\mathbf{y}|)$$

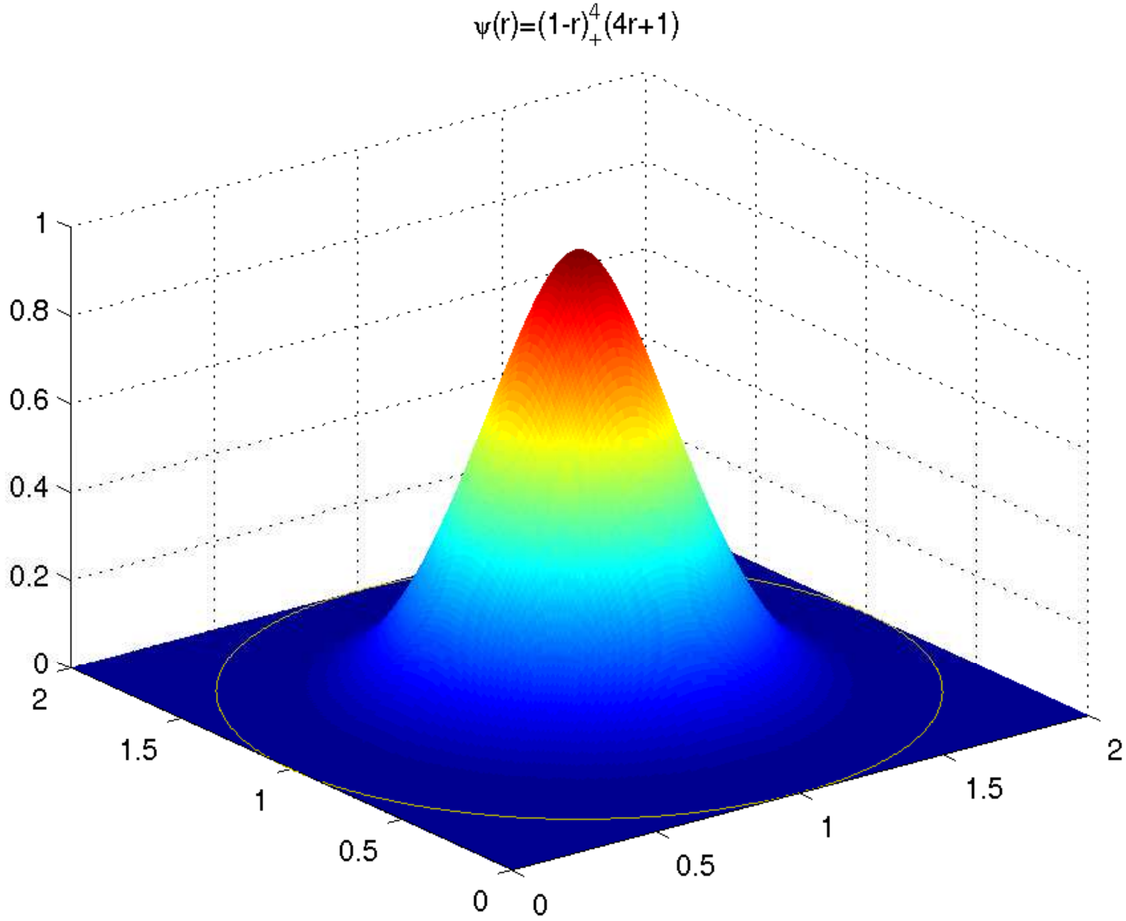
RBFs with compact support:

$$\psi(t) = 0 \text{ for } t \geq 1.$$

Example (Holger Wendland):

$$\psi(t) = (4t+1)(1-t)_+^4.$$

Wendland RBF restricted to the plane



Scaling the RBF

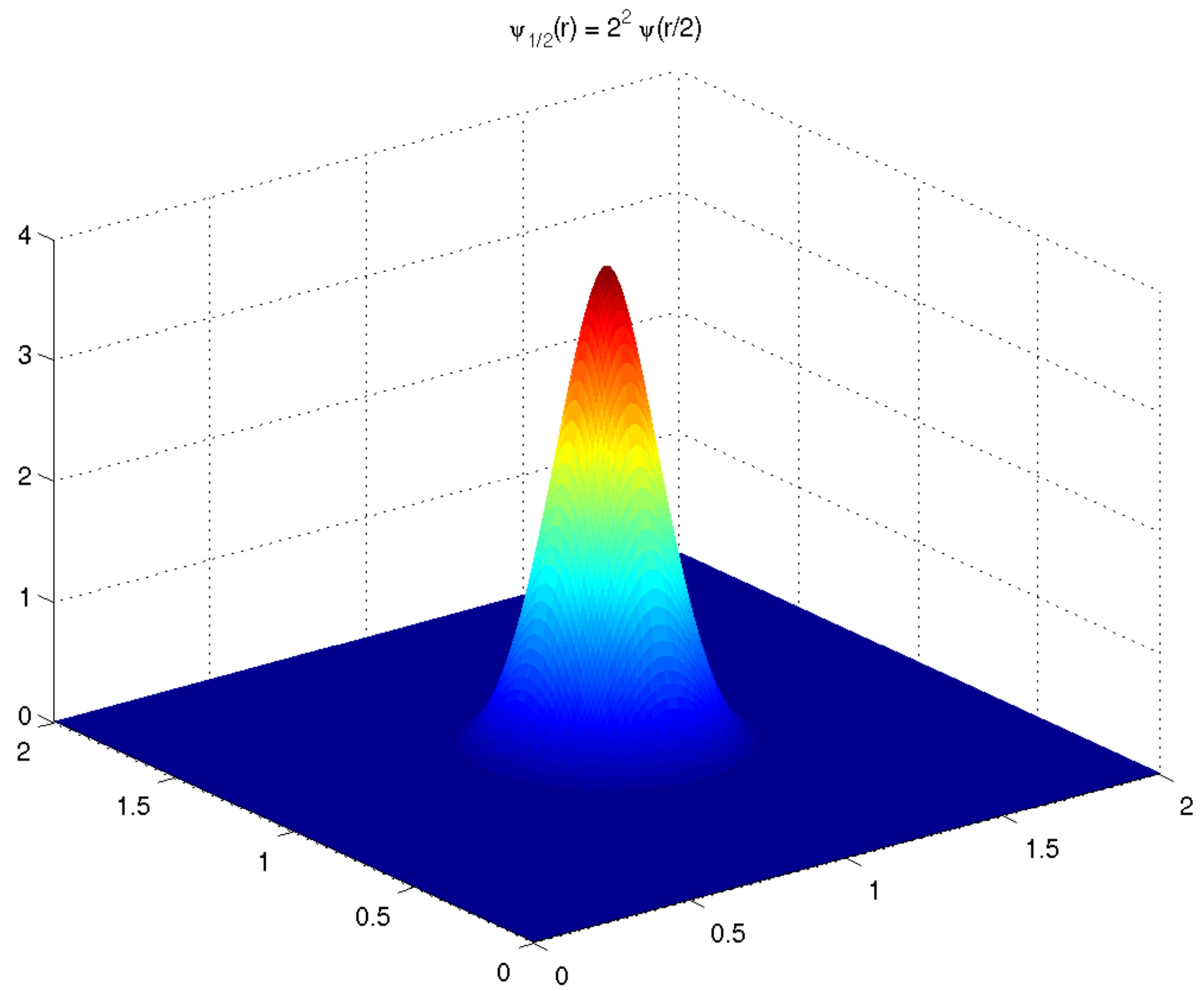
Suppose we replace $\psi(t)$ by

$$\psi_{\frac{1}{2}}(t) := c\psi\left(\frac{t}{1/2}\right) = c\psi(2t) = c(4 \times 2t + 1)(1 - 2t)_+^4.$$

The 'scaled' RBF $\psi_{\frac{1}{2}}(t)$ has support of radius $\frac{1}{2}$.

We say that $\psi_{\frac{1}{2}}$ has the scale $\delta = \frac{1}{2}$.

The scaled RBF



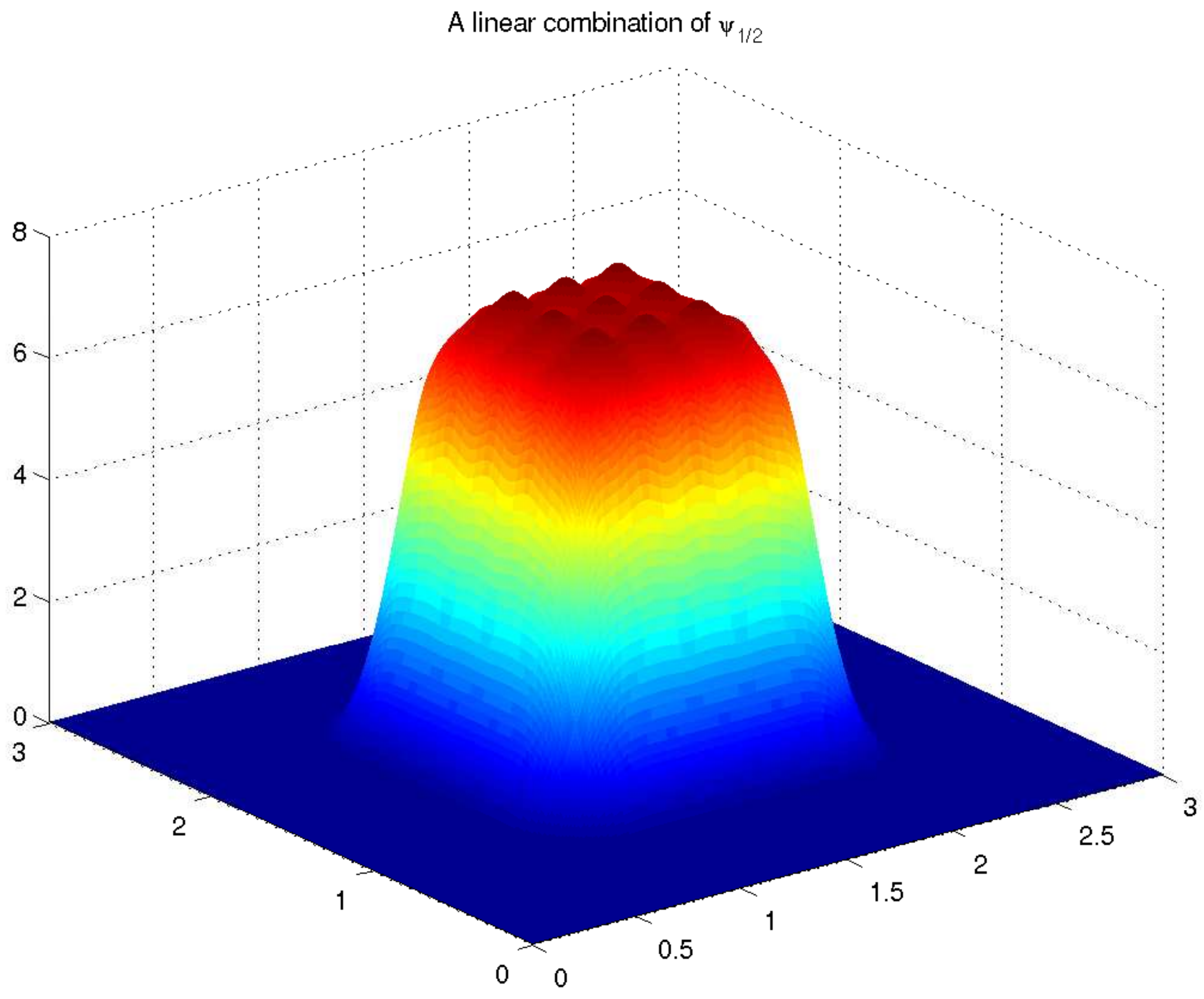
RBF approximation

An RBF approximation (with a single scale) has the form

$$f(\mathbf{x}) \approx \sum_{j=1}^{M} \alpha_j \Psi(\mathbf{x}, \mathbf{x}_j), \qquad \mathbf{x} \in \mathbb{R}^3.$$

Example

Here's an example: It's a sum of nine RBF's on a $\mathbf{3} \times \mathbf{3}$ grid.



Now spherical basis functions

Starting with an RBF $\boldsymbol{\Psi}(\mathbf{x}, \mathbf{y})$ in $\mathbb{R}^{\mathbf{3}}$, we obtain a **spherical radial basis function** (SBF) by restricting $\boldsymbol{\Psi}$ to the sphere $\mathbb{S}^{\mathbf{2}}$:

$$\boldsymbol{\Phi}(\mathbf{x}, \mathbf{y}) := \boldsymbol{\Psi}(\mathbf{x}, \mathbf{y}) \text{ if } |\mathbf{x}| = |\mathbf{y}| = \mathbf{1}.$$

$\boldsymbol{\Phi}(\mathbf{x}, \mathbf{y})$ depends only on the angle between $\mathbf{x}$ and $\mathbf{y}$, thus we can write

$$\boldsymbol{\Phi}(\mathbf{x}, \mathbf{y}) = \phi(\mathbf{x} \cdot \mathbf{y}).$$

Now spherical basis functions

Starting with an RBF $\boldsymbol{\Psi}(\mathbf{x}, \mathbf{y})$ in $\mathbb{R}^3$, we obtain a **spherical radial basis function** (SBF) by restricting $\boldsymbol{\Psi}$ to the sphere $\mathbb{S}^2$:

$$\boldsymbol{\Phi}(\mathbf{x}, \mathbf{y}) := \boldsymbol{\Psi}(\mathbf{x}, \mathbf{y}) \text{ if } |\mathbf{x}| = |\mathbf{y}| = 1.$$

$\boldsymbol{\Phi}(\mathbf{x}, \mathbf{y})$ depends only on the angle between $\mathbf{x}$ and $\mathbf{y}$, thus we can write

$$\boldsymbol{\Phi}(\mathbf{x}, \mathbf{y}) = \phi(\mathbf{x} \cdot \mathbf{y}).$$

An SBF approximation (with a single scale) has the form

$$f(\mathbf{x}) \approx \sum_{j=1}^{M} \alpha_j \phi(\mathbf{x} \cdot \mathbf{x}_j), \qquad \mathbf{x} \in \mathbb{S}^2.$$

Positive definite RBFs

We often prefer RBFs that are (strictly) **positive definite**:

$$\sum_{j=1}^{M} \sum_{k=1}^{M} \alpha_j \alpha_k \Psi(\mathbf{x}_j, \mathbf{x}_k) \geq 0 \qquad \forall\, M, \mathbf{x}_j, \alpha_j,$$

with equality iff $\alpha_1 = \alpha_2 = \cdots = \alpha_M = 0$.

Positive definite RBFs

We often prefer RBFs that are (strictly) **positive definite**:

$$\sum_{j=1}^{M}\sum_{k=1}^{M}\alpha_j\alpha_k\Psi(\mathbf{x}_j,\mathbf{x}_k)\geq 0 \qquad \forall\, M,\mathbf{x}_j,\alpha_j,$$

with equality iff $\alpha_1 = \alpha_2 = \cdots = \alpha_M = 0$.

The Wendland RBF is (strictly) positive definite.

Proof: (Wendland) By showing that its Fourier transform is positive.

How to choose the coefficients α_j?

Recall: an SBF approximation (with a single scale) has the form

$$f(\mathbf{x}) \approx \sum_{j=1}^{M} \alpha_j \phi(\mathbf{x} \cdot \mathbf{x}_j), \qquad \mathbf{x} \in \mathbb{S}^2.$$

How to find good coefficients α_j?

How to choose the coefficients α_j?

Recall: an SBF approximation (with a single scale) has the form

$$f(\mathbf{x}) \approx \sum_{j=1}^{M} \alpha_j \phi(\mathbf{x} \cdot \mathbf{x}_j), \qquad \mathbf{x} \in \mathbb{S}^2.$$

How to find good coefficients α_j?

One way is to interpolate at the points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_M$: then we have to solve

$$f(\mathbf{x}_i) = \sum_{j=1}^{M} \alpha_j \phi(\mathbf{x}_i \cdot \mathbf{x}_j), \qquad i = 1, \ldots, M,$$

a linear system with the $M \times M$ symmetric matrix $(\phi(\mathbf{x}_j \cdot \mathbf{x}_i))$.

How to choose the scale?

If we want to do interpolation with a single RBF and a single scale, and with a given set of centres, the choice of scale is difficult:

How to choose the scale?

If we want to do interpolation with a single RBF and a single scale, and with a given set of centres, the choice of scale is difficult:

If δ is too large, then the approximation can be very good, but the **condition** of the matrix $\phi(\mathbf{x}_j \cdot \mathbf{x}_i)$ will be bad.

How to choose the scale?

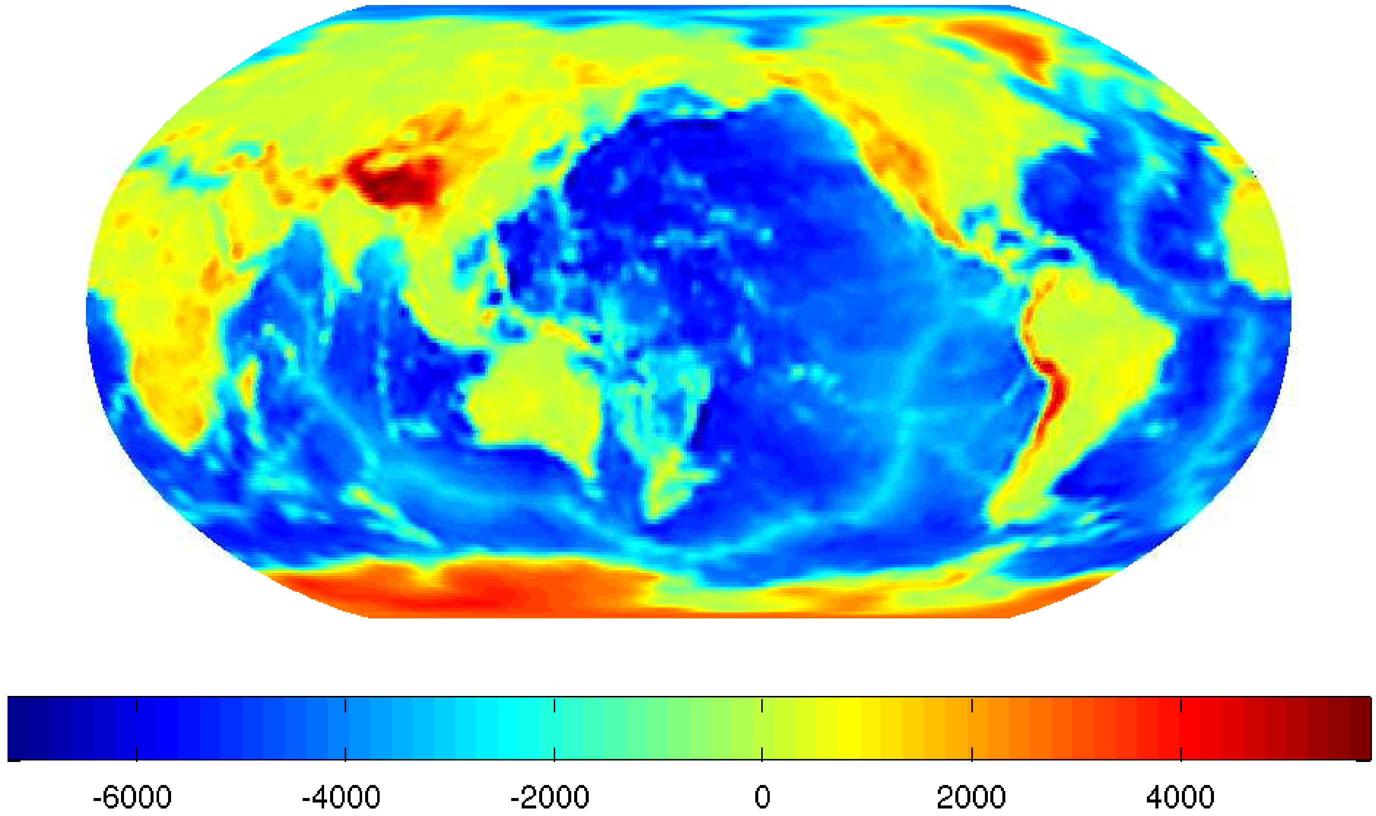
If we want to do interpolation with a single RBF and a single scale, and with a given set of centres, the choice of scale is difficult:

If δ is too large, then the approximation can be very good, but the **condition** of the matrix $\phi(\mathbf{x}_j \cdot \mathbf{x}_i)$ will be bad.

If δ is too small, then the matrix will be well conditioned but the **approximation will be poor**.

Interpolation with $\delta = 1$ and 8000 points

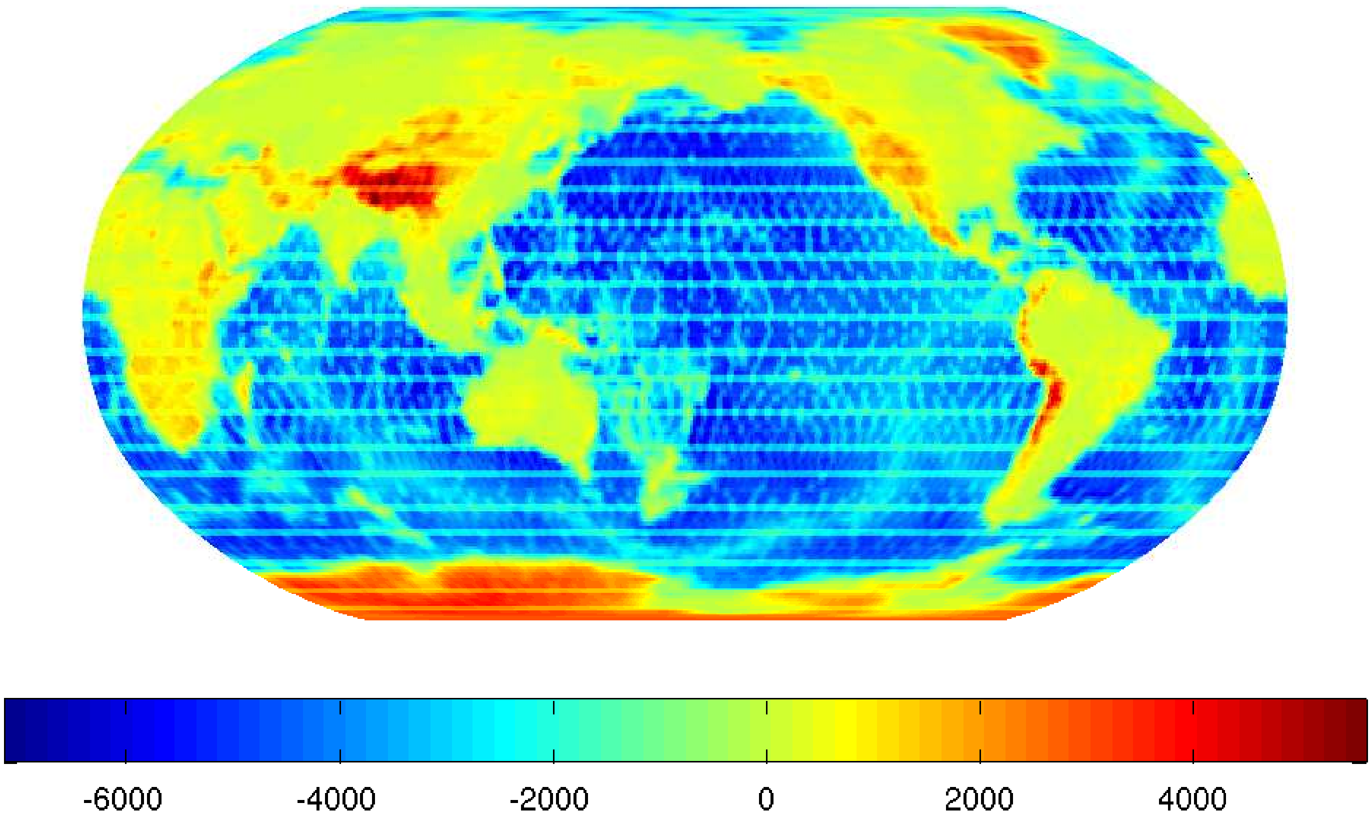
Approximating 'topo', with $\boldsymbol{\delta = 1}$.



The approximation is quite good, but the condition number is $\mathbf{4 \times 10^6}$

Interpolation with $\delta = \frac{1}{16}$ and 8000 points

Approximating ‘topo’, with $\delta = \frac{1}{16}$.

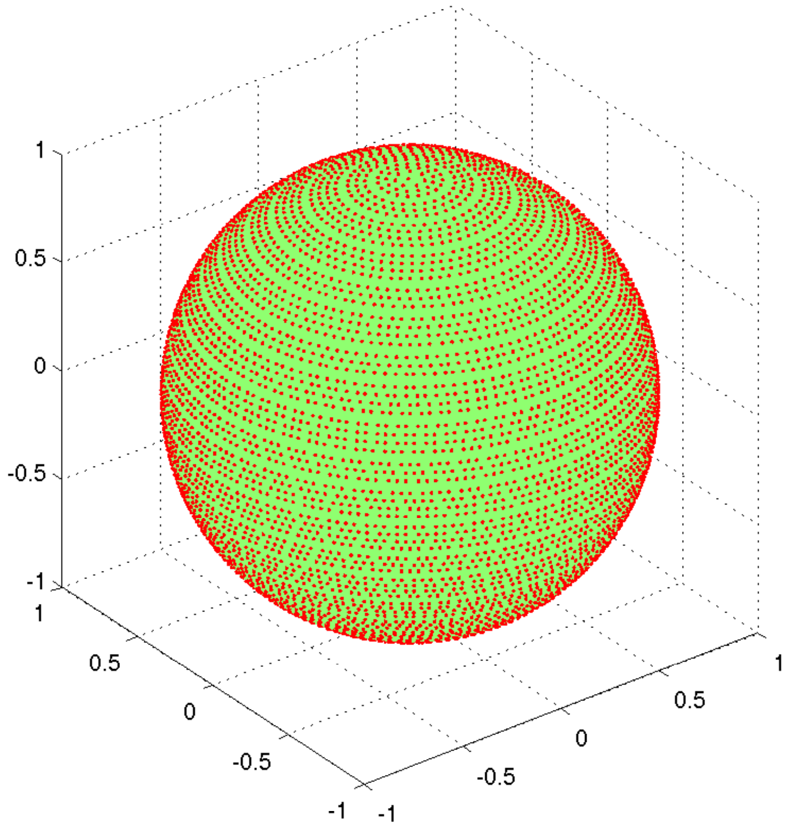


The approximation is terrible, but the condition number is only 8.

Which points?

The examples use Rakhmanov-Saff-Zhou ‘equal-area’ points, rounded to nearest degree (because that is where the values of ‘topo’ are available.)

The figure shows 8000 equal area points:



Convergence of interpolation for a single scale

The (global) mesh norm of the point set $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ is:

$$h_X := \max_{\mathbf{x} \in \mathbb{S}^2} \min_{1 \le j \le M} \cos^{-1}(\mathbf{x} \cdot \mathbf{x}_j).$$

Convergence of interpolation for a single scale

The (global) mesh norm of the point set $X = \{\mathrm{x}_1, \ldots, \mathrm{x}_N\}$ is:

$$h_X := \max_{\mathrm{x} \in \mathbb{S}^2} \min_{1 \le j \le M} \cos^{-1}(\mathrm{x} \cdot \mathrm{x}_j).$$

The following is well known:

Theorem. Let X be a point set on $\mathbb{S}^2$, with mesh norm h_X. Then

$$\|f - I_X f\|_{L_2} \le c h_X^\sigma \|f\|_{H^\sigma(\mathbb{S}^2)}, \quad f \in H^\sigma.$$

where $\sigma > 0$ depends on Φ,

For the Wendland function $\psi(t) = (4t + 1)(1 - t)_+^4$ we have $\sigma = 5/2$. For smoother Wendland functions σ is larger.

What is H^σ?

Think of $H^\sigma(\mathbb{S}^2)$ as the space of L^2 functions on the sphere with σ square-integrable derivatives.

For $f \in L^2(\mathbb{S}^2)$ we can write

$$f(\mathbf{x}) = \sum_{\ell=0}^{\infty} \sum_{k=1}^{2\ell+1} \widehat{f}_{\ell,k} Y_{\ell,k}(\mathbf{x}),$$

with $\{Y_{\ell,k}\}$ an orthonormal set of spherical harmonics, and

$$\widehat{f}_{\ell,k} = \int_{\mathbb{S}^2} f(\mathbf{x}) Y_{\ell,k(\mathbf{x})} \mathrm{d}\omega(\mathbf{x}).$$

What is H^σ?

Think of $H^\sigma(\mathbb{S}^2)$ as the space of L^2 functions on the sphere with σ square-integrable derivatives.

For $f \in L^2(\mathbb{S}^2)$ we can write

$$f(\mathbf{x}) = \sum_{\ell=0}^{\infty} \sum_{k=1}^{2\ell+1} \widehat{f}_{\ell,k} Y_{\ell,k}(\mathbf{x}),$$

with $\{Y_{\ell,k}\}$ an orthonormal set of spherical harmonics, and

$$\widehat{f}_{\ell,k} = \int_{\mathbb{S}^2} f(\mathbf{x}) Y_{\ell,k(\mathbf{x})} \mathrm{d}\omega(\mathbf{x}).$$

Then

$$\|f\|_{H^\sigma(\mathbb{S}^2)} := \left[\sum_{\ell=0}^{\infty} \sum_{k=1}^{2\ell+1} (1+\ell)^{2\sigma} |\widehat{f}_{\ell,k}|^2 \right]^{1/2}.$$

Wendland functions and Sobolev spaces

Each Wendland function is associated with a particular Sobolev space:

Given $\psi(|\mathbf{x}-\mathbf{y}|)=\phi(\mathbf{x}\cdot\mathbf{y})$ for $\mathbf{x},\mathbf{y}\in\mathbb{S}^2$, define $\widehat{\phi}(\ell)$ by

$$\phi(t)=\frac{1}{4\pi}\sum_{\ell=0}^{\infty}(2\ell+1)\,\widehat{\phi}(\ell)P_\ell(t),\quad t\in[-1,1]. \tag{1}$$

Then the Wendland functions have the property that for some $\sigma>0$

$$\widehat{\phi}(\ell)\asymp(1+\ell)^{-2\sigma}.$$

Example: $\sigma=5/2$ for $\phi(t)=(4t+1)(1-t)_+^4$.

The Sobolev space for a Wendland Φ

With a positive definite SBF kernel Φ we can associate a special Hilbert space (the "native space"), with norm

$$\|f\|_\Phi := \left[\sum_{\ell=0}^{\infty}\sum_{k=1}^{2\ell+1}\frac{|\widehat{f}_{\ell,k}|^2}{\widehat{\phi}(\ell)}\right]^{1/2}.$$

MATHEMATICS & STATISTICS
THE UNIVERSITY OF NEW SOUTH WALES

The Sobolev space for a Wendland Φ

With a positive definite SBF kernel Φ we can associate a special Hilbert space (the “native space”), with norm

$$\|f\|_{\Phi} := \left[\sum_{\ell=0}^{\infty} \sum_{k=1}^{2\ell+1} \frac{|\widehat{f}_{\ell,k}|^2}{\widehat{\phi}(\ell)} \right]^{1/2}.$$

Thus for the Wendland functions, if $\widehat{\phi}(\ell) \asymp (1+\ell)^{-2\sigma}$ then $\|f\|_{\Phi}$ is equivalent to $\|f\|_{H^{\sigma}(\mathbb{S}^2)}$.

So each Wendland function is associated with a unique Sobolev spaces $H^{\sigma}(\mathbb{S}^2)$.

Mixing different scales?

Mixing of different scales is not often tried. The problem is that one loses symmetry and positive definiteness of the interpolation matrix, and hence may lose the unique solvability property.

(Bozzini, Lenarduzzi, Rossini and Schaback 2004 showed that unique solvability is retained if the matrix perturbation is small enough.)

The multiscale approximation

- uses scaled versions of a single compactly supported **radial basis function** (RBF) $\Psi(\mathbf{x},\mathbf{y}) = \psi(|\mathbf{x}-\mathbf{y}|),\ \mathbf{x},\mathbf{y} \in \mathbb{R}^3$

  The scaled version, with scale δ, is $\Psi_\delta(\mathbf{x},\mathbf{y}) = \delta^{-2}\psi\left(\left|\frac{\mathbf{x}-\mathbf{y}}{\delta}\right|\right)$.

The multiscale approximation

- uses scaled versions of a single compactly supported **radial basis function** (RBF) $\Psi(\mathbf{x},\mathbf{y}) = \psi(|\mathbf{x}-\mathbf{y}|)$, $\mathbf{x},\mathbf{y} \in \mathbb{R}^3$
  The scaled version, with scale δ, is $\Psi_\delta(\mathbf{x},\mathbf{y}) = \delta^{-2}\psi\left(\left|\frac{\mathbf{x}-\mathbf{y}}{\delta}\right|\right)$.

- It uses a sequence of scales $\delta_1, \delta_2, \ldots$ with $\delta_j \to 0$.

MATHEMATICS & STATISTICS THE UNIVERSITY OF NEW SOUTH WALES

The multiscale approximation

- uses scaled versions of a single compactly supported **radial basis function** (RBF) $\Psi(\mathbf{x},\mathbf{y}) = \psi(|\mathbf{x}-\mathbf{y}|),\ \mathbf{x},\mathbf{y} \in \mathbb{R}^3$
  The scaled version, with scale δ, is $\Psi_\delta(\mathbf{x},\mathbf{y}) = \delta^{-2}\psi\left(\left|\frac{\mathbf{x}-\mathbf{y}}{\delta}\right|\right)$.
- It uses a sequence of scales $\delta_1, \delta_2, \ldots$ with $\delta_j \to 0$.
- And a sequence of ever denser point sets $X_1, X_2, \ldots$ (which need not be nested).

The multiscale approximation

- uses scaled versions of a single compactly supported **radial basis function** (RBF) $\Psi(\mathbf{x},\mathbf{y}) = \psi(|\mathbf{x}-\mathbf{y}|)$, $\mathbf{x},\mathbf{y} \in \mathbb{R}^3$
  The scaled version, with scale δ, is $\Psi_\delta(\mathbf{x},\mathbf{y}) = \delta^{-2}\psi\left(\left|\frac{\mathbf{x}-\mathbf{y}}{\delta}\right|\right)$.
- It uses a sequence of scales $\delta_1, \delta_2, \ldots$ with $\delta_j \to 0$.
- And a sequence of ever denser point sets $X_1, X_2, \ldots$ (which need not be nested).
- At stage j the error from the previous stage is approximated using radial basis functions of smaller scale δ_j, and with closer spaced centres $X_j = \{\mathbf{x}_1, \ldots, \mathbf{x}_{N_j}\}$.

Multiscale RBF is not a new idea, but . . .

Schaback 1995 and Floater & Iske 1996 considered a special case with nested point sets (for the radial basis function centres).

Hales & Levesley 2002 used polyharmonic spines and special point sets.

Narcowich, Schaback & Ward 1999 analysed a different method, based on repeated convolutions of a single RBF.

Multiscale RBF is not a new idea, but . . .

Schaback 1995 and Floater & Iske 1996 considered a special case with nested point sets (for the radial basis function centres).

Hales & Levesley 2002 used polyharmonic spines and special point sets.

Narcowich, Schaback & Ward 1999 analysed a different method, based on repeated convolutions of a single RBF.

But theory was lacking, for multiscale based on scaled versions of a single compactly supported RBF on either Euclidean spaces or spheres, and scattered data points.

Multiscale scheme LeGia, IHS, Wendland 2010

The following multilevel scheme achieves good approximation AND good (even constant!) condition number:

Suppose we have a sequence of point sets $X_1, X_2, \ldots \subset \mathbb{S}^2$, with mesh norm h_{X_j} of X_j approaching zero.

The mesh norm of X is

$$h_X := \max_{\mathrm{x} \in \mathbb{S}^2} \min_{1 \le j \le M} \cos^{-1}(\mathrm{x} \cdot \mathrm{x}_j).$$

Correspondingly, we take scales $\delta_j =$ const $\times h_{X_j}$.

Then the algorithm is . . .

The algorithm

- Step 1. The first approximation is $\boldsymbol{f_1 = s_1 := I_{X_1,\delta_1} f}$

  $\boldsymbol{I_{X_1,\delta_1} f}$ is the interpolant of $\boldsymbol{f}$ on the point set X_1 with scale $\boldsymbol{\delta_1}$.

  The error is $\boldsymbol{e_1 := f - f_1}$.

The algorithm

- Step 1. The first approximation is $f_1 = s_1 := I_{X_1,\delta_1} f$

  $I_{X_1,\delta_1} f$ is the interpolant of f on the point set X_1 with scale δ_1.

  The error is $e_1 := f - f_1$.

- Step 2 computes $s_2 = I_{X_2,\delta_2} e_1$ (interpolant of e_1 on X_2 at scale δ_2).

  The new approximation is $f_2 := f_1 + s_2 = f_1 + I_{X_2,\delta_2} e_1$.

  The new error is $e_2 := e_1 - I_{X_2,\delta_2} e_1$.

The algorithm

- Step 1. The first approximation is $f_1 = s_1 := I_{X_1,\delta_1} f$

  $I_{X_1,\delta_1} f$ is the interpolant of f on the point set X_1 with scale δ_1.

  The error is $e_1 := f - f_1$.

- Step 2 computes $s_2 = I_{X_2,\delta_2} e_1$ (interpolant of e_1 on X_2 at scale δ_2).

  The new approximation is $f_2 := f_1 + s_2 = f_1 + I_{X_2,\delta_2} e_1$.

  The new error is $e_2 := e_1 - I_{X_2,\delta_2} e_1$.

- Step 3 ...

The algorithm

- Step 1. The first approximation is $f_1 = s_1 := I_{X_1,\delta_1} f$

  $I_{X_1,\delta_1} f$ is the interpolant of f on the point set X_1 with scale δ_1.

  The error is $e_1 := f - f_1$.

- Step 2 computes $s_2 = I_{X_2,\delta_2} e_1$ (interpolant of e_1 on X_2 at scale δ_2).

  The new approximation is $f_2 := f_1 + s_2 = f_1 + I_{X_2,\delta_2} e_1$.

  The new error is $e_2 := e_1 - I_{X_2,\delta_2} e_1$.

- Step 3 . . .

Finally, after n steps

$$f \approx f_n = s_1 + s_2 + \ldots + s_n.$$

The multiscale convergence theorem

Theorem (LeGia, IHS, Wendland, 2010).

Let $X_1, X_2, \ldots$ be a sequence of point sets belonging to $\mathbb{S}^2$, with mesh norms $h_1, h_2, \ldots$ satisfying $h_{j+1} = \mu h_j$, for $\mu \in (0, 1)$.

Let $\delta_1, \delta_2, \ldots$ be scales ≤ 1 satisfying $\delta_j = \frac{\beta}{\mu} h_j$ for some fixed $\beta > 0$.

Let Φ be a Wendland function associated with $H^\sigma(\mathbb{S}^2)$, for some $\sigma > 1$.

The multiscale convergence theorem

Theorem (LeGia, IHS, Wendland, 2010).

Let $X_1, X_2, \ldots$ be a sequence of point sets belonging to $\mathbb{S}^2$, with mesh norms $h_1, h_2, \ldots$ satisfying $h_{j+1} = \mu h_j$, for $\mu \in (0,1)$.

Let $\delta_1, \delta_2, \ldots$ be scales ≤ 1 satisfying $\delta_j = \frac{\beta}{\mu} h_j$ for some fixed $\beta > 0$.

Let Φ be a Wendland function associated with $H^\sigma(\mathbb{S}^2)$, for some $\sigma > 1$.

Then with $\alpha = C\mu^\sigma$ and $C = \ldots$ we have

$$\|f - f_n\|_{L_2} \leq c\alpha^n \|f\|_{H^\sigma} \text{ for all } f \in H^\sigma$$

.

The multiscale convergence theorem

Theorem (LeGia, IHS, Wendland, 2010).

Let $X_1, X_2, \ldots$ be a sequence of point sets belonging to $\mathbb{S}^2$, with mesh norms $h_1, h_2, \ldots$ satisfying $h_{j+1} = \mu h_j,$ for $\mu \in (0,1)$.

Let $\delta_1, \delta_2, \ldots$ be scales ≤ 1 satisfying $\delta_j = \frac{\beta}{\mu} h_j$ for some fixed $\beta > 0$.

Let Φ be a Wendland function associated with $H^\sigma(\mathbb{S}^2)$, for some $\sigma > 1$.

Then with $\alpha = C\mu^\sigma$ and $C = \ldots$ we have

$$\|f - f_n\|_{L_2} \leq c\alpha^n \|f\|_{H^\sigma} \text{ for all } f \in H^\sigma$$

.

And hence for μ sufficiently small f_n converges to f linearly .

Condition numbers

Theorem

Under the conditions in the theorem above, and provided the sequence of point sets is quasi-uniform, the condition number of the linear system at stage j is independent of j.

Quasiuniform means: Minimum separation distance scales like h_X.

Condition numbers

Theorem

Under the conditions in the theorem above, and provided the sequence of point sets is quasi-uniform, the condition number of the linear system at stage j is independent of j.

Quasiuniform means: Minimum separation distance scales like $h_{\mathbf{X}}$.

Largest eigenvalue: the matrix becomes larger, but the number and size of non-zero entries in a row remain roughly constant.

Smallest eigenvalue: the smallest eigenvalue is essentially unchanged because the minimum separation distance between points scales proportionally to δ.

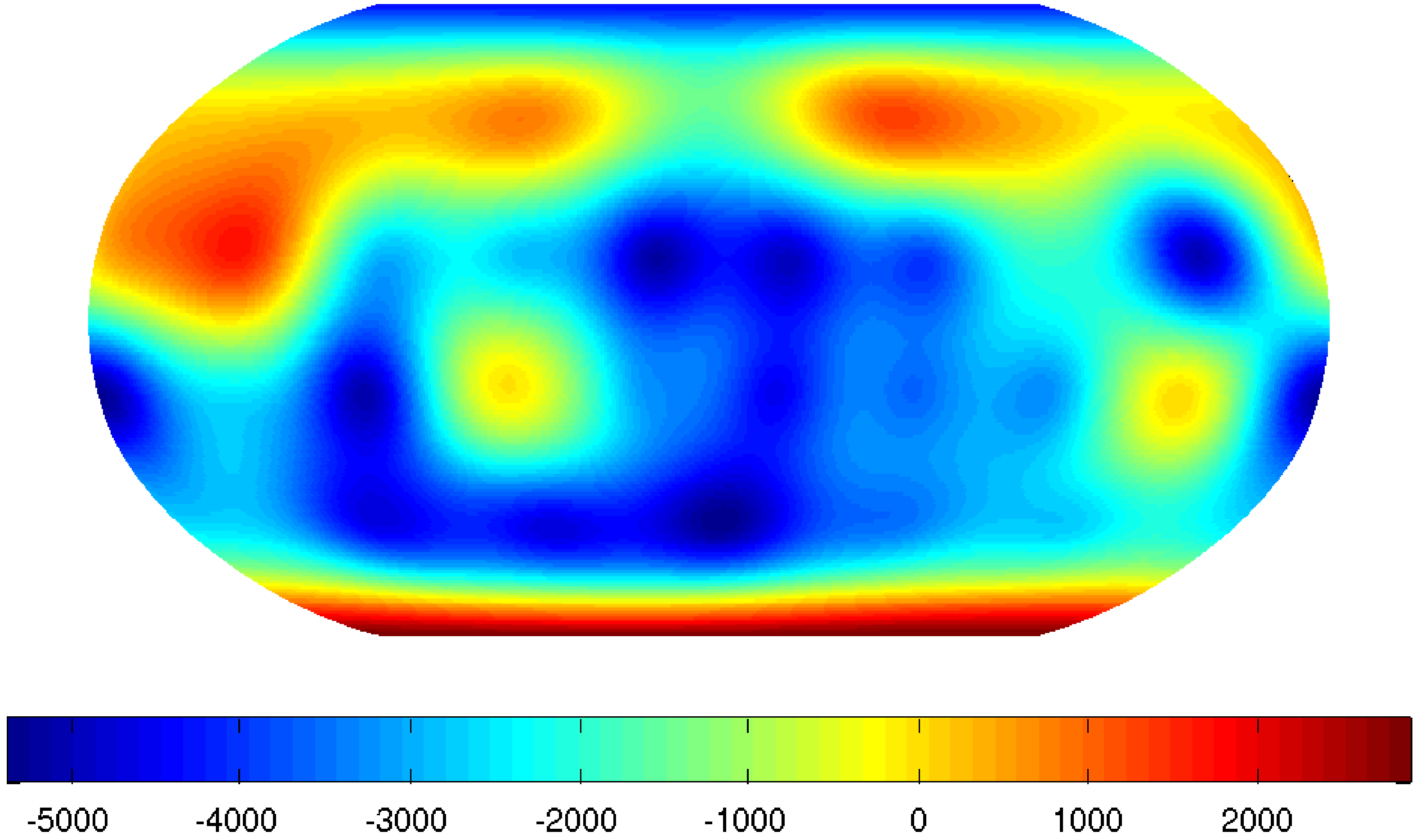
A multiscale experiment with topo

The following experiment uses these values:

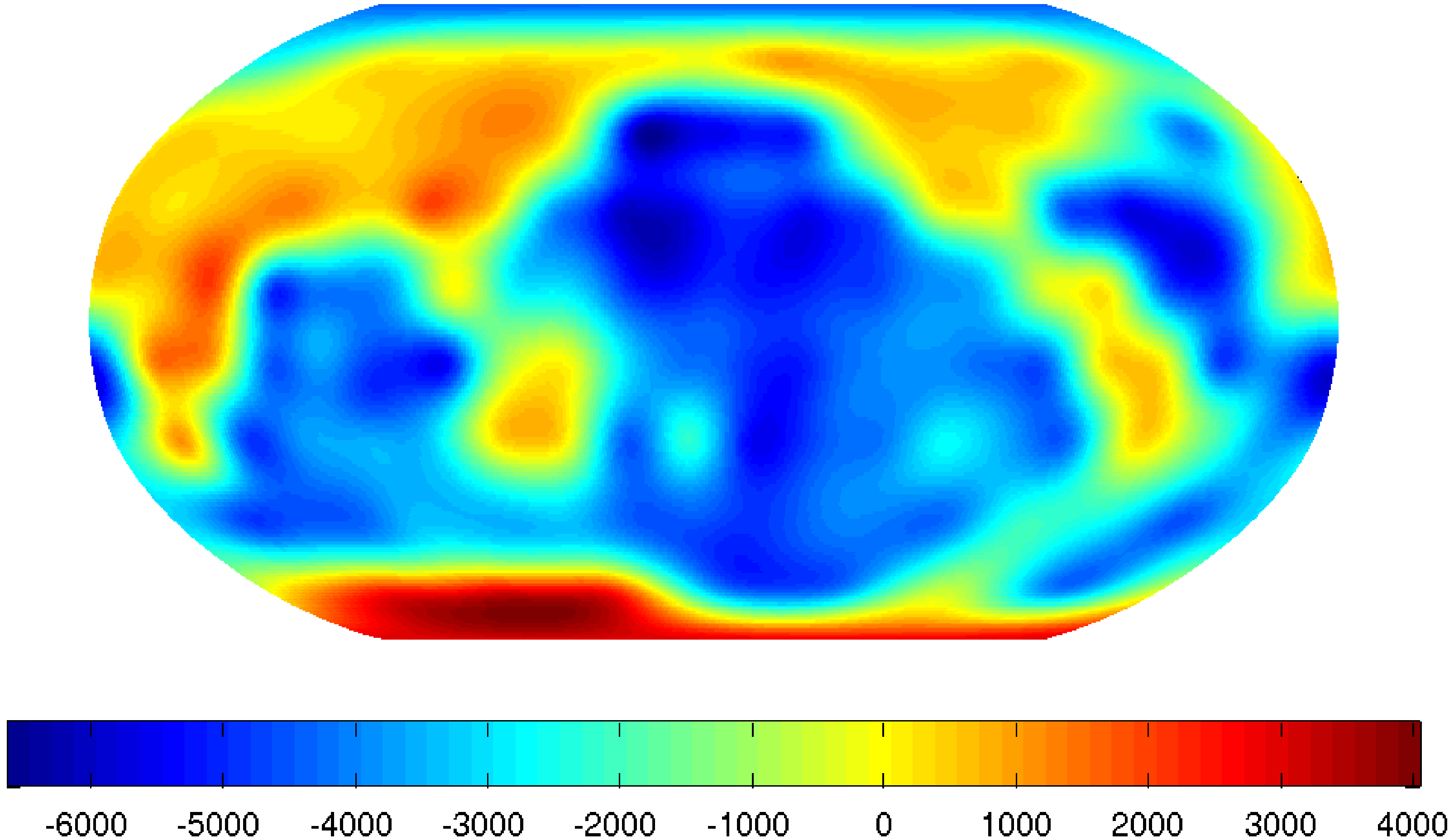
j	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{4}$	$\mathbf{5}$
δ_j	$\mathbf{1}$	$\frac{\mathbf{1}}{\mathbf{2}}$	$\frac{\mathbf{1}}{\mathbf{4}}$	$\frac{\mathbf{1}}{\mathbf{8}}$	$\frac{\mathbf{1}}{\mathbf{16}}$
n	$\mathbf{32}$	$\mathbf{125}$	$\mathbf{500}$	$\mathbf{2000}$	$\mathbf{8000}$

and the Wendland basis function $\psi(t) = (4t + 1)(1 - t)_+^4$.

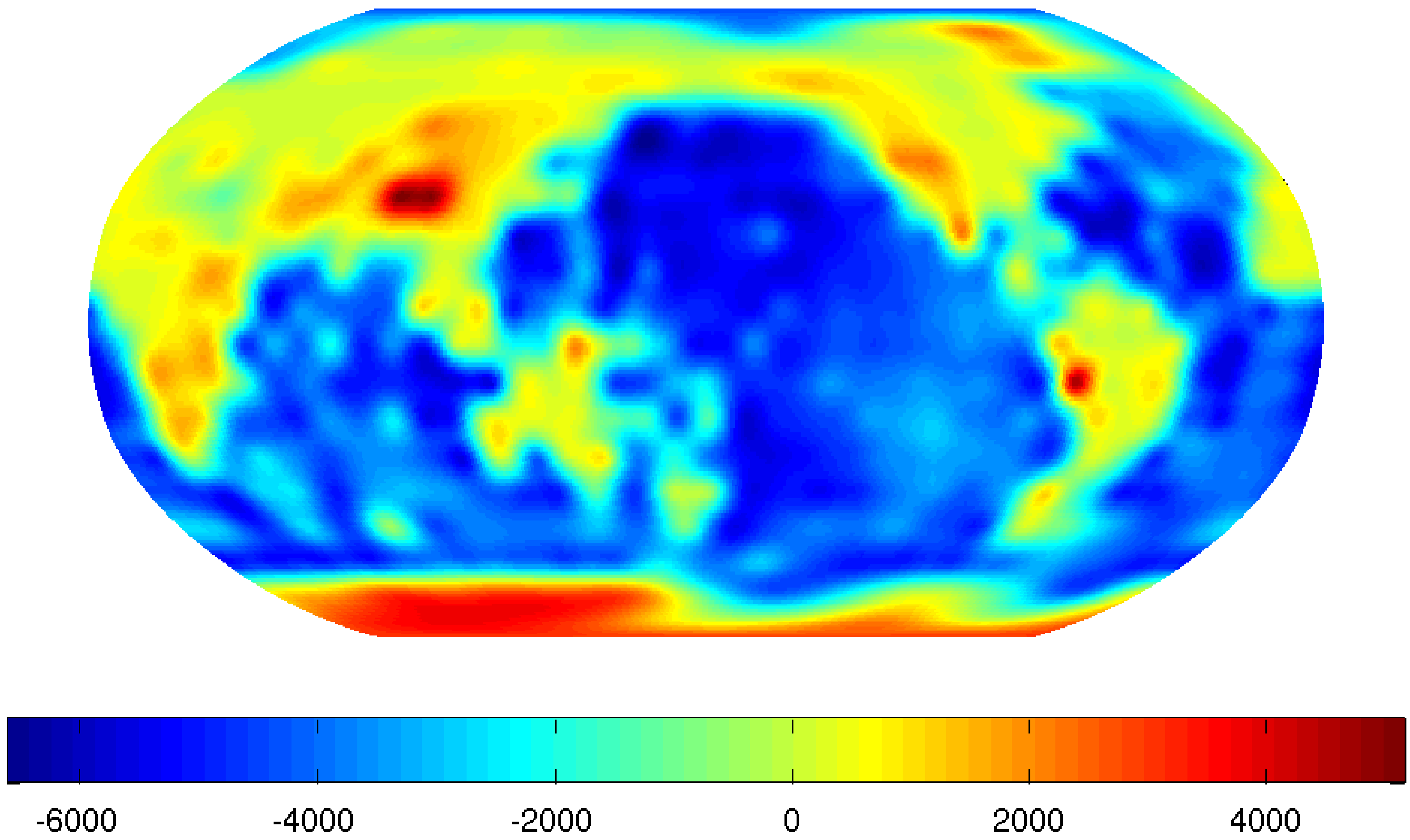
First approximation $f_1 = s_1$



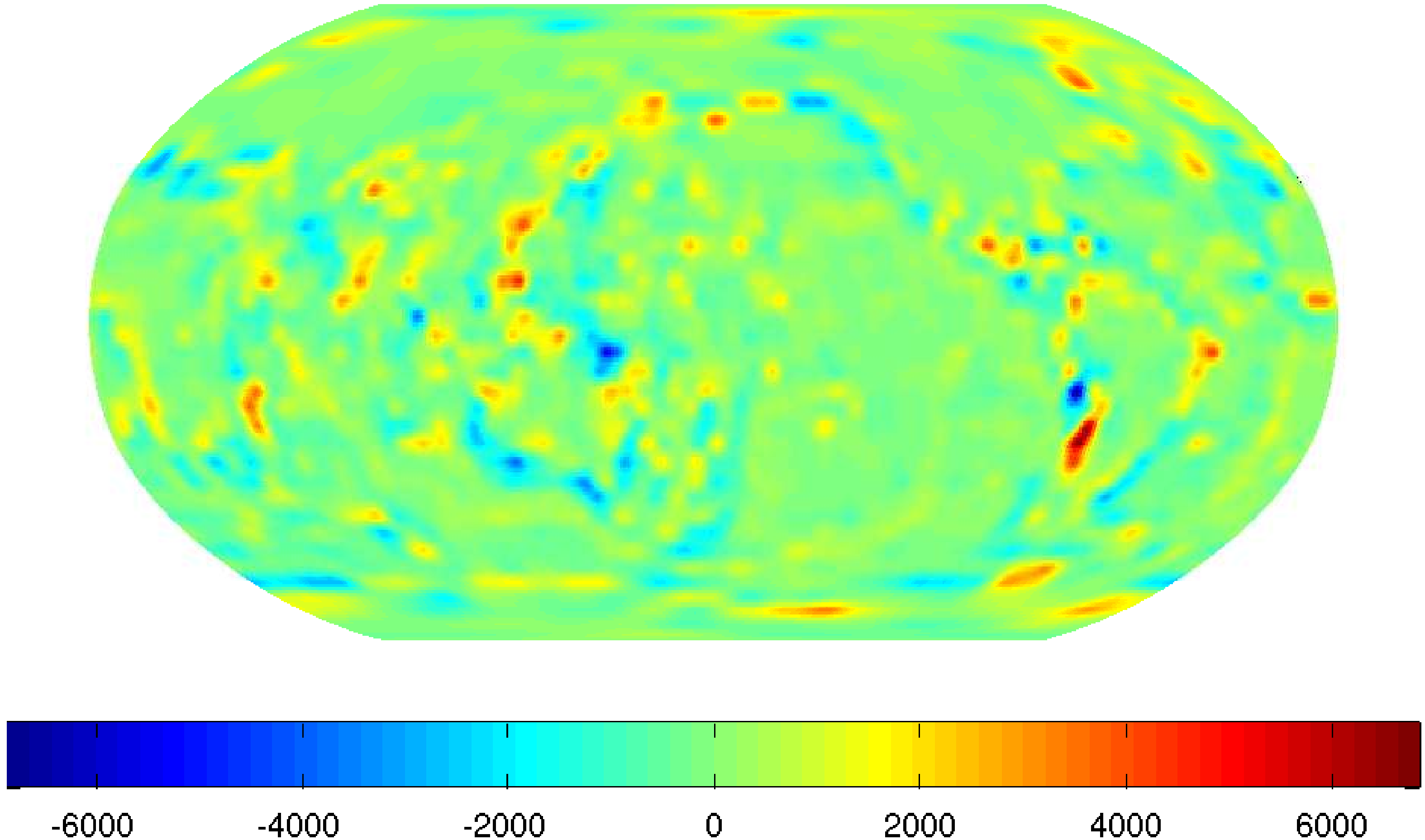
Second approximation f_2



Third approximation f_3

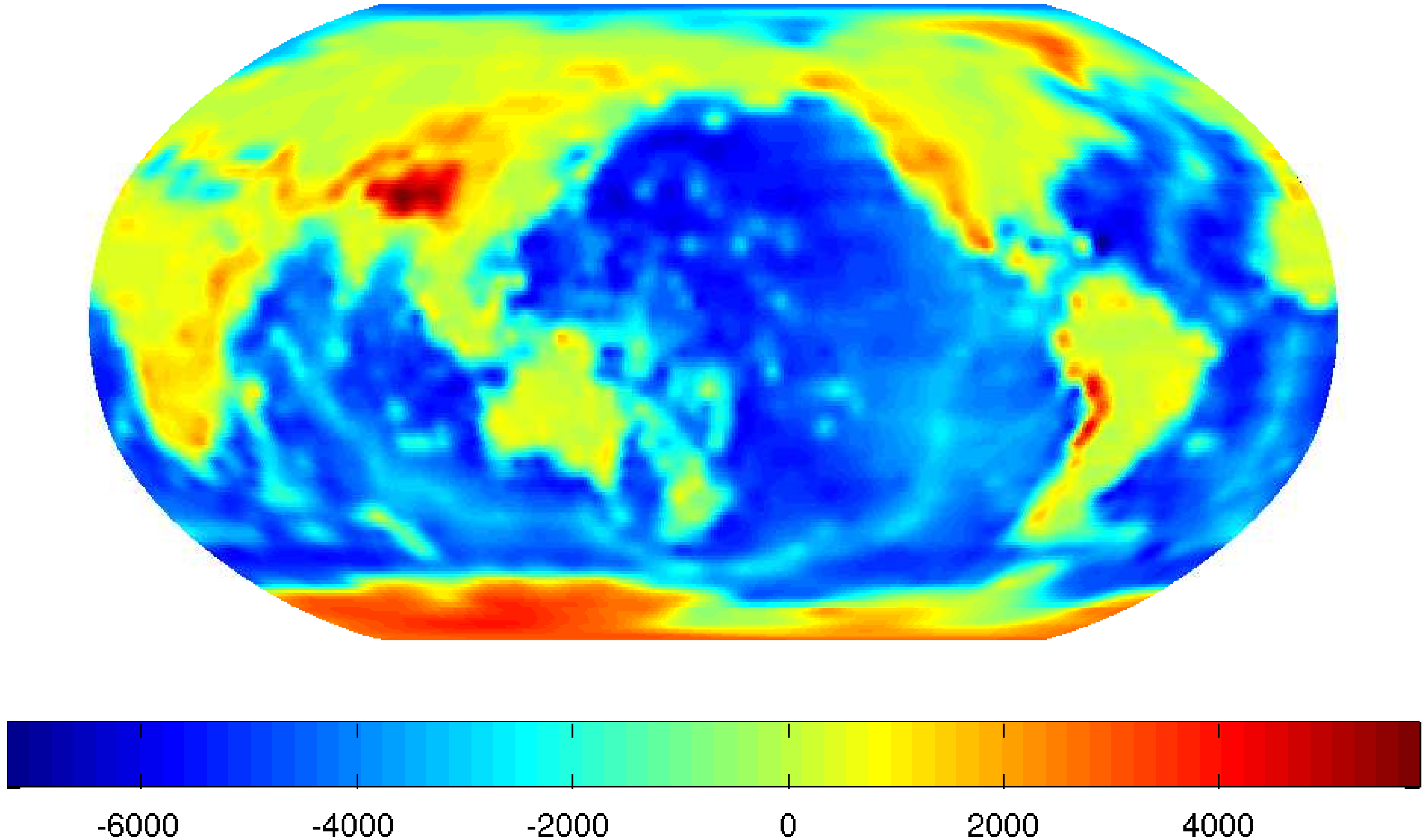


Detail s_4

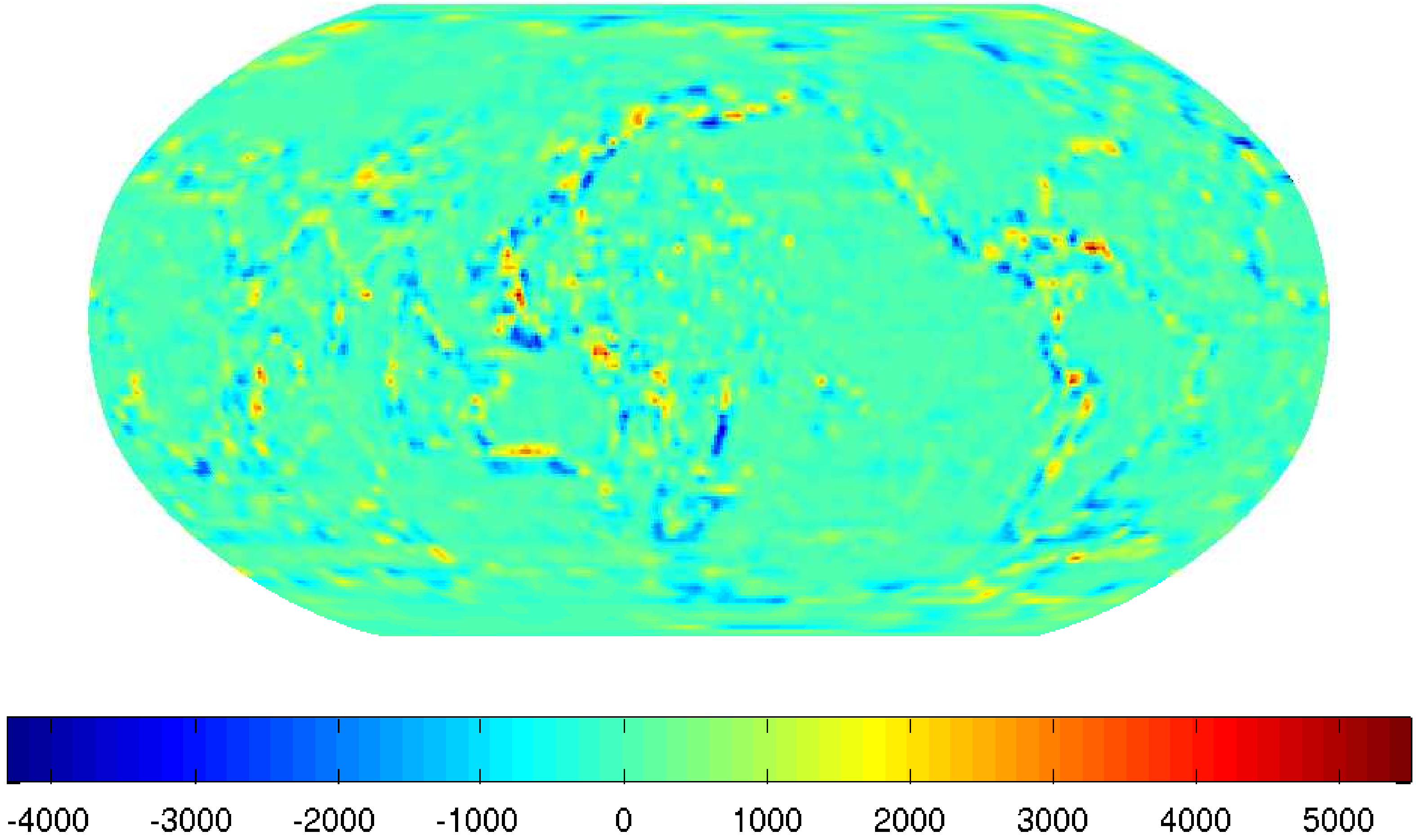


Condition number $= 4.1$

Fourth approximation f_4

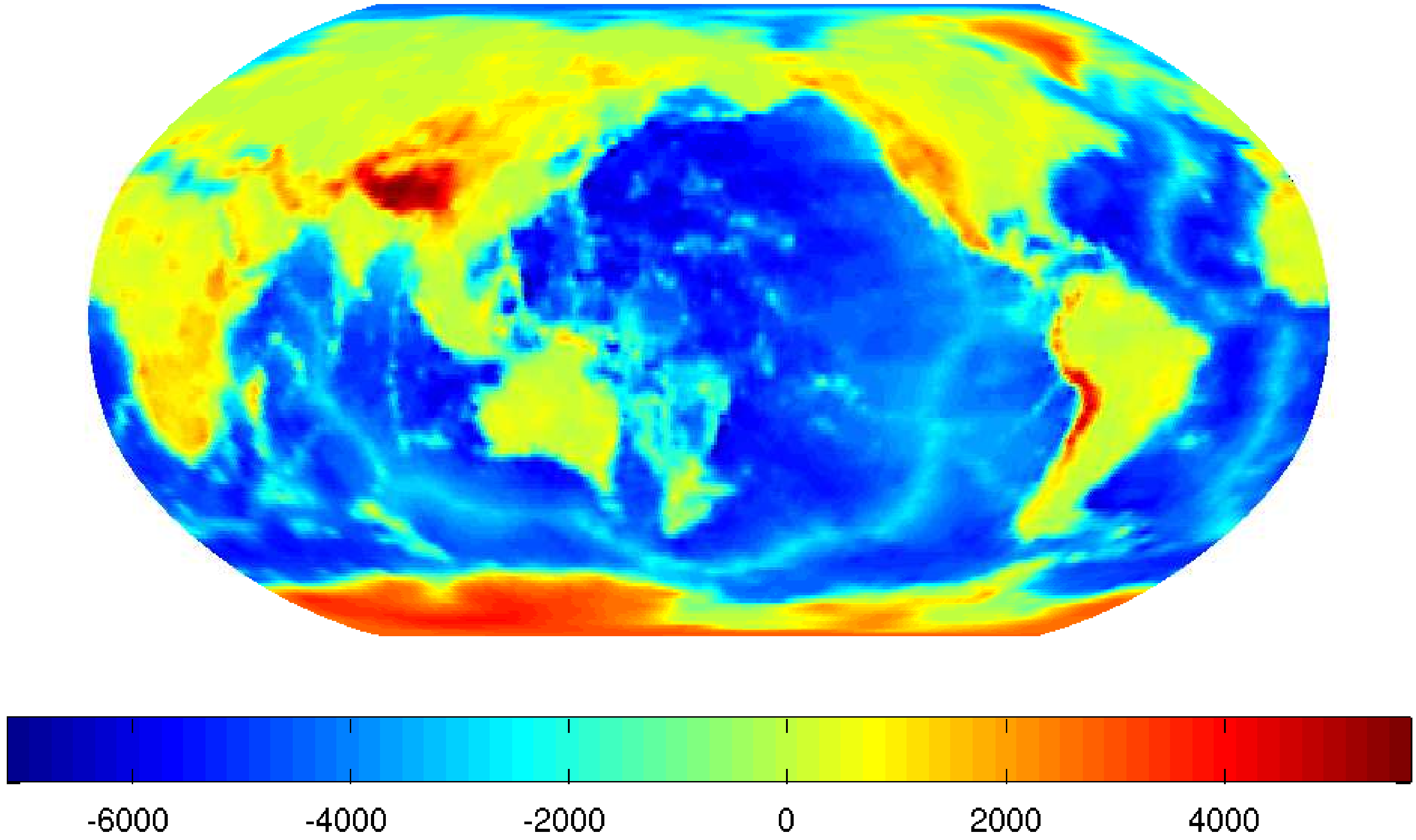


Detail s_5



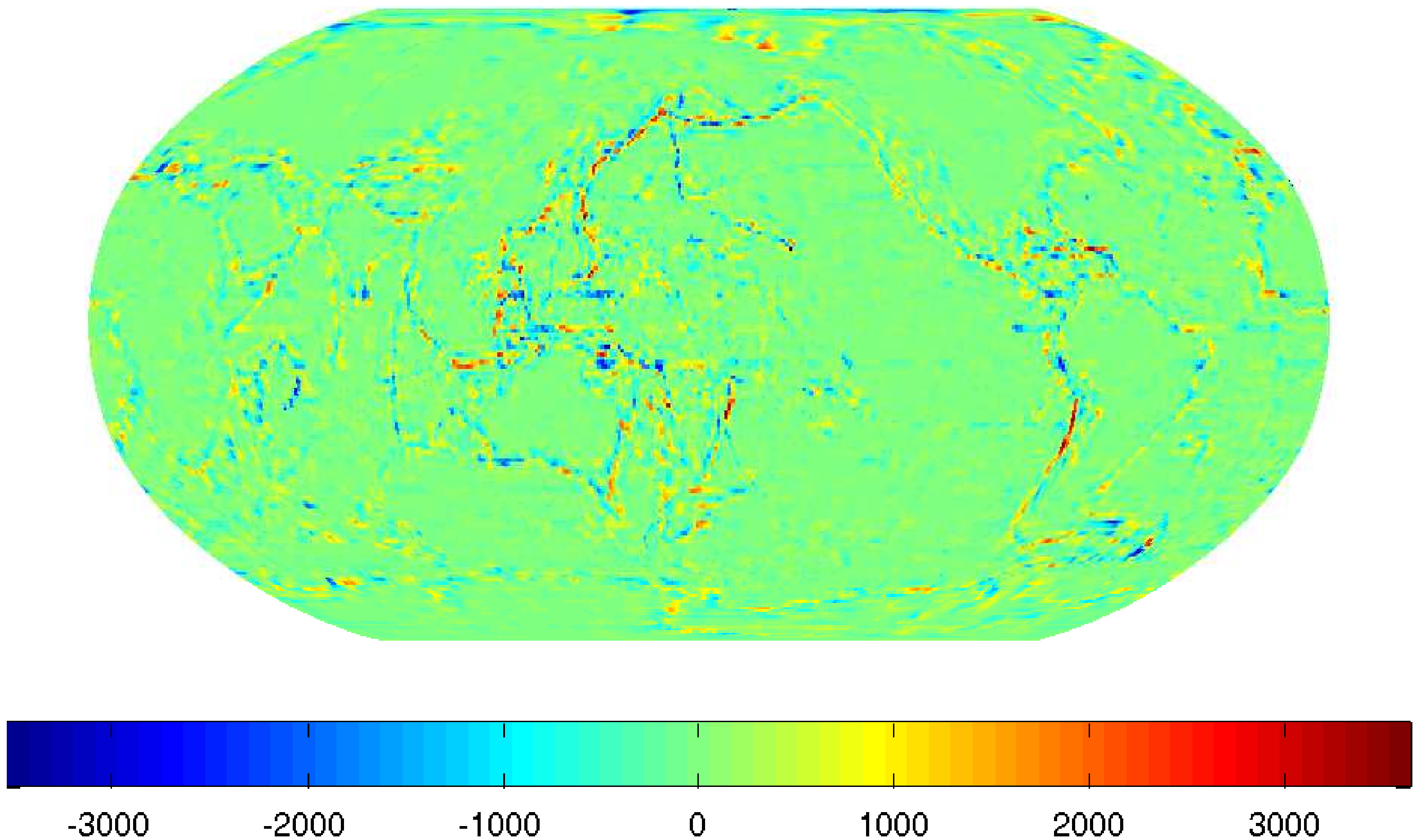
Condition number $= \mathbf{7.8}$

Fifth approximation f_5



The total number of points is $\mathbf{8000 + 2000 + 500 + 125 + 32 = 10,657}$.

The error e_5



The condition numbers are tiny (all $\leq \mathbf{10}$), so we could clearly keep on going indefinitely, but ...

The condition numbers are tiny (all $\leq \mathbf{10}$), so we could clearly keep on going indefinitely, but ...

We are running out of data!

The condition numbers are tiny (all $\leq \mathbf{10}$), so we could clearly keep on going indefinitely, but ...

We are running out of data!

In any case, do we REALLY want more precision everywhere? Why not just look only at places where the error is large?

The condition numbers are tiny (all $\leq \mathbf{10}$), so we could clearly keep on going indefinitely, but ...

We are running out of data!

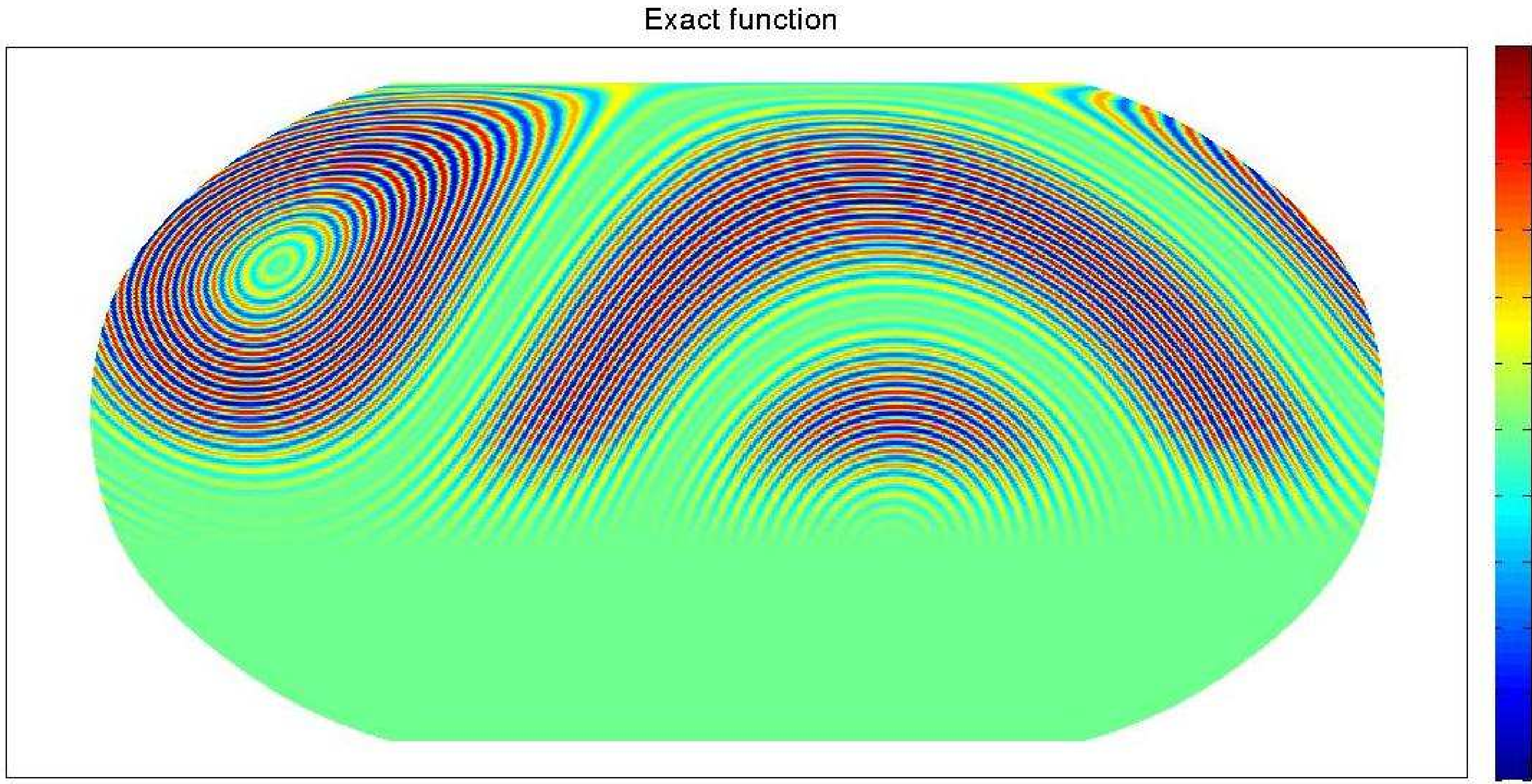
In any case, do we REALLY want more precision everywhere? Why not just look only at places where the error is large?

We should be able to “zoom in” anywhere we want to.

Zooming in

Le Gia, IHS and Wendland "Zooming from global to local: a multiscale RBF approach" (Advances in Computational Math., to appear).

A new example



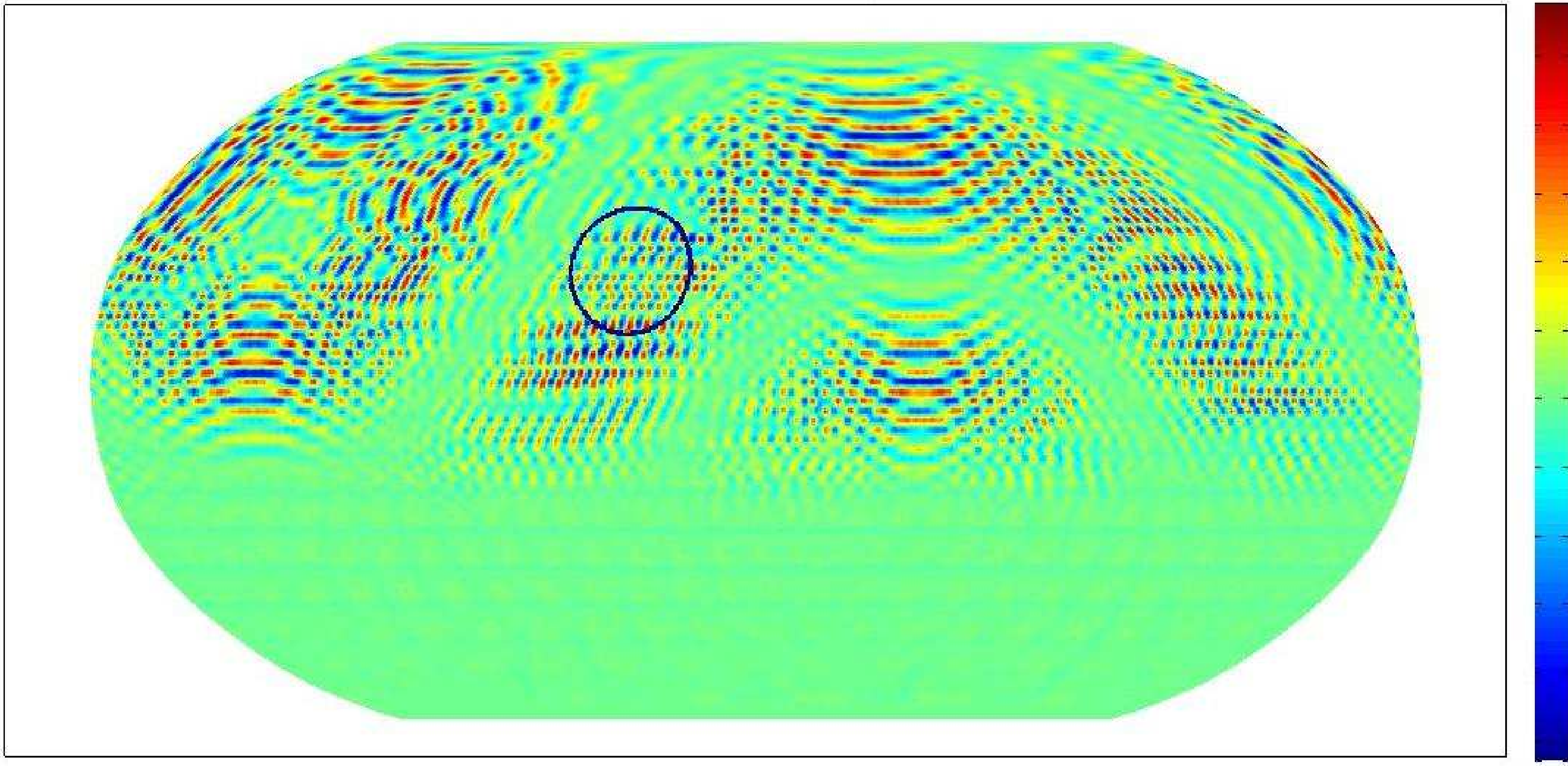
The function is

$$f(\mathrm{x}) = 2 + \left[\sin t \cos(100t) + (1 - \pi s/144)_+^2 \cos(2000\theta)\right] S(\theta).$$

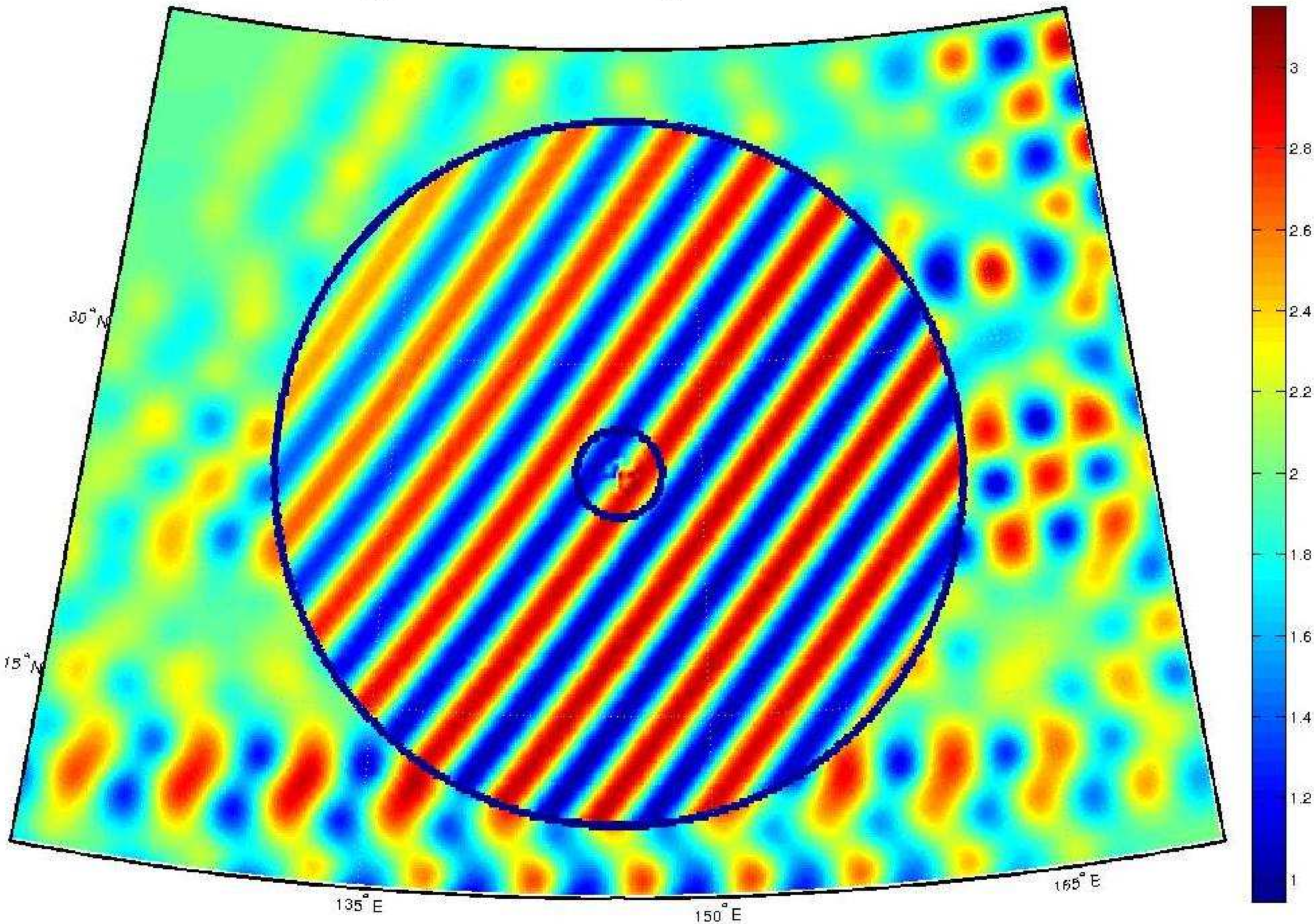
- $S(\theta)$ is a cubic spline which equals 1 for $\theta \in [0, \pi/2]$ and equals 0 for $\theta \in [2\pi/3, \pi]$.
- $t = \cos^{-1}(\mathrm{p} \cdot \mathrm{x})$, with $\mathrm{p} = (1/\sqrt{3},\ 1/\sqrt{3},\ 1/\sqrt{3})^T$
- $s = \cos^{-1}(\mathrm{q} \cdot \mathrm{x})$, with $\mathrm{q} = (-0.7476,\ 0.5069,\ 0.4289)^T$

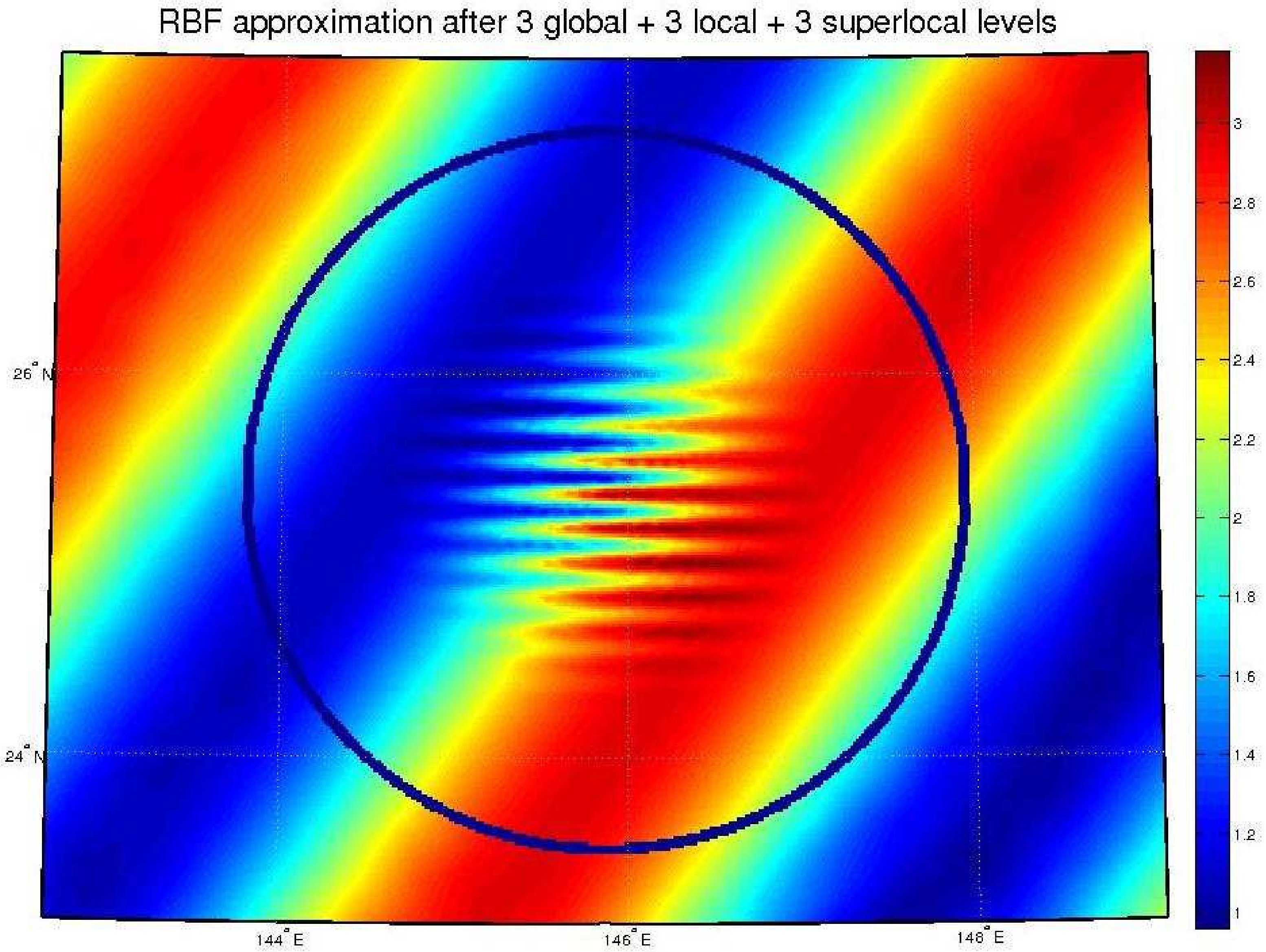
It has both slowly varying and fine scale features.

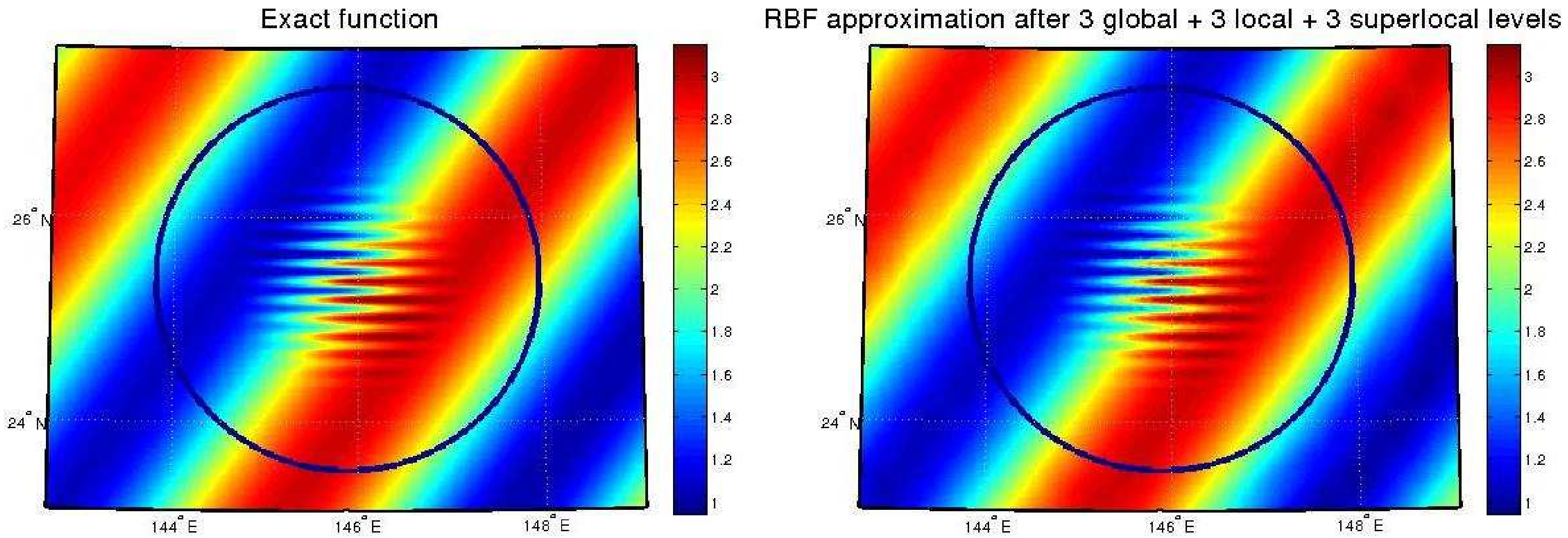
RBF approximation after 3 global levels



RBF approximation after 3 global + 3 local levels




RBF approximation after 3 global + 3 local + 3 superlocal levels
26° N
24° N
144° E
146° E
148° E
3
2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
1


Exact function
RBF approximation after 3 global + 3 local + 3 superlocal levels
26° N
24° N
144° E
146° E
148° E
3
2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
1

Some numbers

Level	N	δ_j	h_j	$\|e_j\|_{L_2(\Omega_2)}$	κ_j
1	500	$1/4$	0.113	4.2e-02	1.7
2	2000	$1/8$	0.057	4.2e-02	1.7
3	8000	$1/16$	0.028	3.4e-02	1.7
4	500	$1/32$	0.019	1.2e-02	3.2
5	2000	$1/64$	0.0089	9.8e-03	3.4
6	8000	$1/128$	0.0041	8.9e-03	3.3
7	500	$1/256$	0.0018	7.9e-03	3.2
8	2000	$1/512$	0.0009	2.9e-03	3.4
9	8000	$1/1024$	0.0005	8.0e-04	3.3

Black: the global levels

Blue: The local levels (zooming to the large spherical cap Ω_1)

Red: The superlocal levels (zooming to the small spherical cap Ω_2)

Here h_j is the **global** mesh norm or **local** mesh norm, as appropriate.

Algorithm 1: Multiscale global/local algorithm

Data: Right hand side f,
total number of levels n,
number of zoomed levels $m < n$,
spherical cap Ω for zooming to,
$X_j \in \mathbb{S}^2, \delta_j$ for $j = 1, \ldots, m$,
$X_j \in \Omega, \delta_j$ for $j = m+1, \ldots, n$.

begin

Set $f_0 = 0$, $e_0 = f$.

for $j = 1, 2, \ldots, n$ **do**

Determine the (global or local) interpolant $s_j \in W_j$ to e_{j-1}

Set $f_j = f_{j-1} + s_j$.

Set $e_j = e_{j-1} - s_j$.

Result: Approximation solution $f_n|_\Omega = s_1 + s_2 + \cdots + s_n$

Theorem

(LeGia, IHS, Wendland, Adv. Comp. Math., to appear).

Let $X_1, X_2, \ldots, X_m$ be point sets belonging to $\mathbb{S}^2$, $X_{m+1}, \ldots, X_n$ be point sets belonging to a spherical cap Ω, and h_j be the (global or local) mesh norm of X_j, where the mesh norms satisfy satisfy $h_{j+1} = \mu h_j$, for some $\mu \in (0, 1)$.

Let $\delta_1, \delta_2, \ldots$ be scales ≤ 1 satisfying $\delta_j = \frac{\beta}{\mu} h_j$ for some fixed $\beta > 0$.

Let Φ be a Wendland function associated with $H^\sigma(\mathbb{S}^2)$, for some $\sigma > 1$.

Then . . .

Theorem (continued)

Then with $\alpha = C\mu^{\sigma}$ and $C = \ldots$ we have

$$\|f - f_k\|_{L_2(\Omega)} \leq c\alpha^n \|f\|_{H^{\sigma}} \text{ for all } f \in H^{\sigma}$$

.

Theorem (continued)

Then with $\alpha = C\mu^\sigma$ and $C = \dots$ we have

$$\|f - f_k\|_{L_2(\Omega)} \leq c\alpha^n \|f\|_{H^\sigma} \text{ for all } f \in H^\sigma$$

.

And hence for μ sufficiently small f_n converges to f linearly as $n \to \infty$.

Main ideas in the proof

First let's pretend that all levels are global, i.e. $\Omega = \mathbb{S}^2$.

1. Define $e_n := f - f_n$.

Because $e_n(\mathbf{x}) = 0$ for $\mathbf{x} \in X_n$, it follows from the **zeros theorem** (Hangelbroek, Narcowich, Ward 2012, Le Gia, Narcowich, Ward, Wendland 2006) that

$$\|e_n\|_{L_2(\mathbb{S}^2)} \leq c h_n^{\sigma} \|e_n\|_{H^{\sigma}(\mathbb{S}^2)}.$$

Main ideas - scaling

2. **Scaling**: All the scaled norms Φ_{δ_j} are equivalent, but different:

$$\boxed{c_1 \|u\|_{\Phi_\delta} \leq \|u\|_{H^\sigma(\mathbb{S}^2)} \leq c_2 \delta^{-\sigma} \|u\|_{\Phi_\delta}}$$

$$\Longrightarrow \|e_n\|_{L_2(\mathbb{S}^2)} \leq c h_n^\sigma \|e_n\|_{H^\sigma(\mathbb{S}^2)} \leq c c_2 h_n^\sigma \delta_{n+1}^{-\sigma} \|e_n\|_{\Phi_{\delta_{n+1}}}$$

and because $\delta_{n+1} = \frac{\beta}{\mu} h_{n+1} = \frac{\beta}{\mu} \cdot \mu h_n = \beta h_n$, we have now

$$\|e_n\|_{L_2(\mathbb{S}^2)} \leq c' \|e_n\|_{\Phi_{\delta_{n+1}}}.$$

Main ideas - recursion

3. The key step is a (not-so-obvious) recursion:

$$\|e_j\|_{\Phi_{\delta_{j+1}}} \leq \alpha \|e_{j-1}\|_{\Phi_{\delta_j}}, \quad j = n, n-1, \ldots 1,$$

with $\boxed{\alpha = C\mu^\sigma}$. So $\alpha < 1$ for μ sufficiently small.

Putting things together, we have $\|e_n\|_{L_2(\Omega)} \leq c'\|e_n\|_{\Phi_{n+1}}$, and now

$$\begin{aligned}\|e_n\|_{\Phi_{n+1}} &\leq \alpha\|e_{n-1}\|_{\Phi_n} \\ &\cdots \\ &\leq \alpha^n \|e_0\|_{\Phi_1} \\ &\leq c\alpha^n \|f\|_{H^\sigma}(\mathbb{S}^2)\ .\end{aligned}$$

Except for the local aspect, **that's it**!

Main ideas – extension

To get results for errors on $bos\Omega$ we need an

extension operator $\boldsymbol{E}$:

$$\boldsymbol{E} : \boldsymbol{H^{\nu}(\Omega) \to H^{\nu}(\mathbb{S}^2)}, \quad \boldsymbol{0 \le \nu \le \sigma},$$

with $\boldsymbol{E}$ independent of $\boldsymbol{\nu}$, and such that

$$\boldsymbol{Eu|_{\Omega} = u|_{\Omega}} \quad \text{for all } \boldsymbol{u \in H^{\nu}(\Omega)},$$

and

$$\boldsymbol{\|Eu\|_{H^{\nu}(\mathbb{S}^2)} \le c_{\nu} \|u\|_{H^{\nu}(\Omega)}}.$$

Proof of existence of $\boldsymbol{E}$: Hubbert and Morton (2004) for integer $\boldsymbol{\nu}$, plus interpolation.

Applying zeros theorem and extension

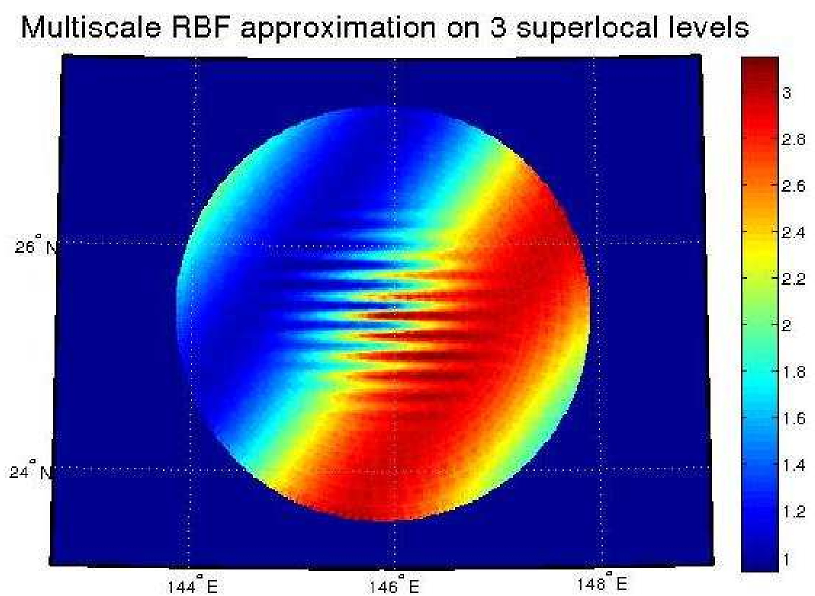
So now we have

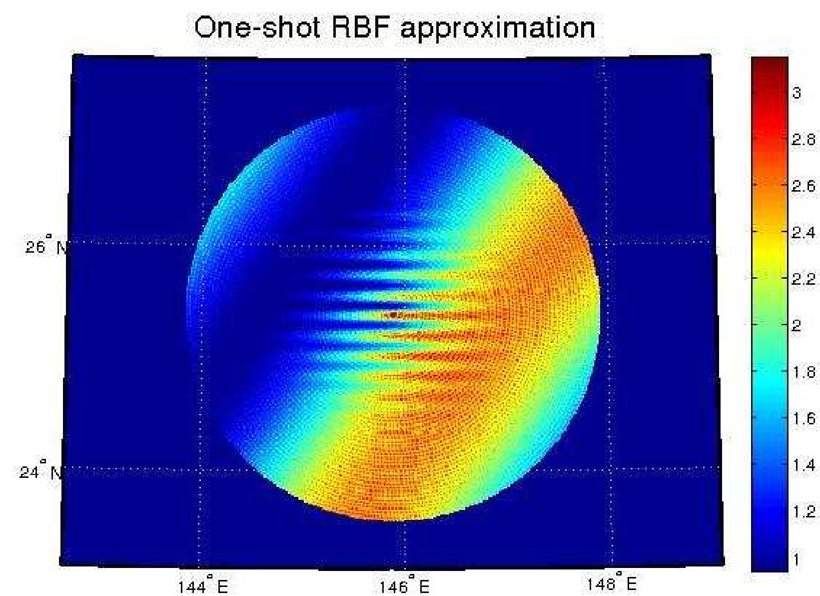
$$\begin{aligned}\|e_n\|_{L_2(\Omega)} &\le c h_n^{\sigma} \|e_n\|_{H^{\sigma}(\Omega)} \\ &= c h_n^{\sigma} \|E e_n\|_{H^{\sigma}(\Omega)} \\ &\le c h_n^{\sigma} \|E e_n\|_{H^{\sigma}(\mathbb{S}^2)} \\ &\le \text{etc}\end{aligned}$$

Two alternative calculations

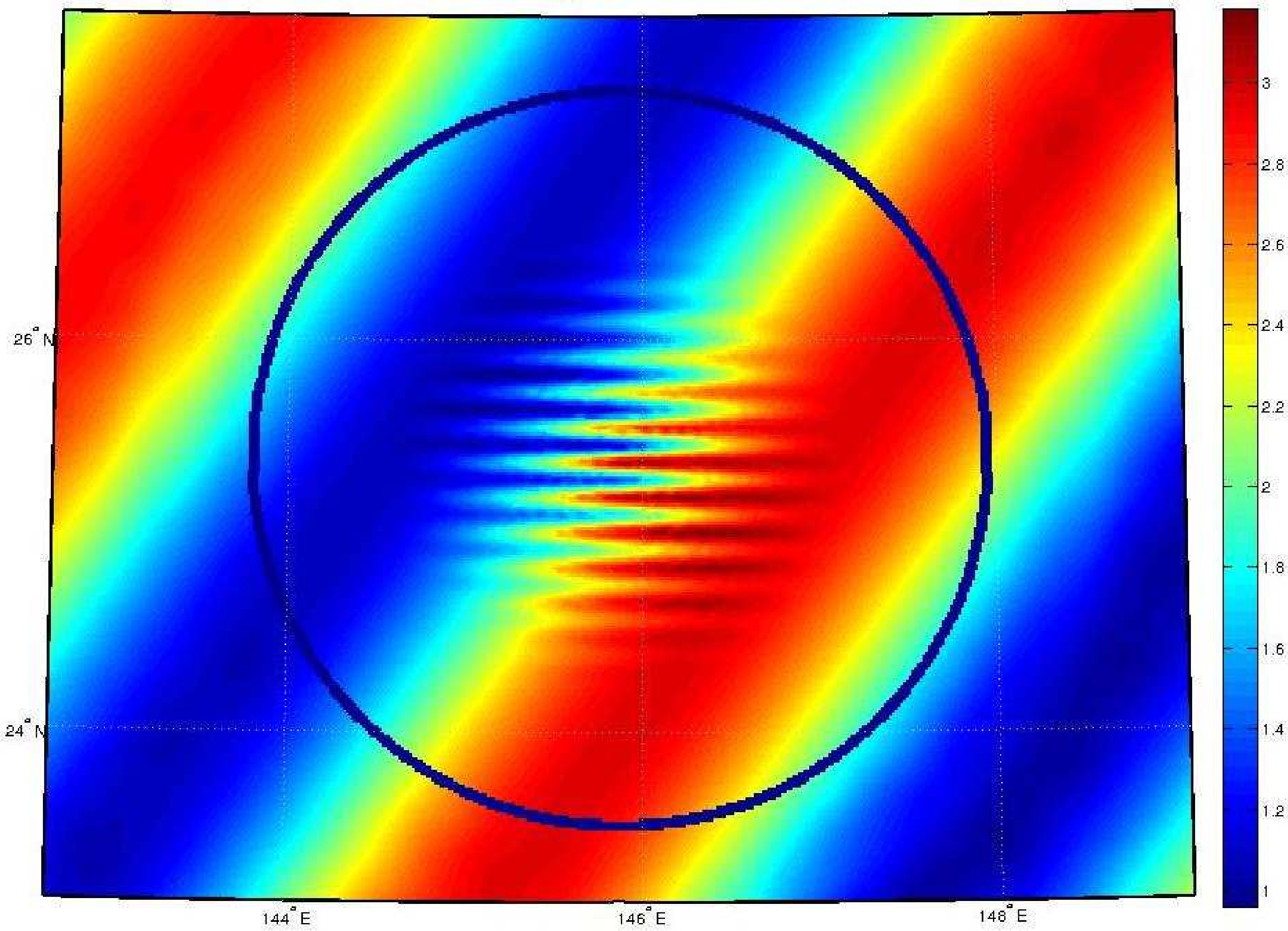
In the right picture we do one **single-scale interpolation** at the 9th level. The graininess is apparent!

In the left picture we do **three multiscale levels**, all on the superlocal cap (i.e. levels 7, 8, 9 of the 9-level calculation).





RBF approximation after 3 global + 3 local + 3 superlocal levels



Final L^2 errors

9-level: $\|e\|_{L^2(\Omega_2)} = \mathbf{8.0e{-}4}$

3-level: $\|e\|_{L^2(\Omega_2)} = \mathbf{9.2e{-}3}$

1-level: $\|e\|_{L^2(\Omega_2)} = \mathbf{2.0e{-}2}$

Final comments on “zooming in”

- The theory combines the ideas of the global theory for the sphere LeGia,S,Wendland 2010, AND the theory for bounded regions in Euclidean space Wendland, Numer Math 2010.
- The final approxn. uses the points and SBFs from all 9 levels.
- The condition number at the 9th level is less than 4!!

MATHEMATICS & STATISTICS
THE UNIVERSITY OF NEW SOUTH WALES

Final comments on "zooming in"

- The theory combines the ideas of the global theory for the sphere LeGia,S,Wendland 2010, AND the theory for bounded regions in Euclidean space Wendland, Numer Math 2010.
- The final approxn. uses the points and SBFs from all 9 levels.
- The condition number at the 9th level is less than 4!!

In both theory and practice it seems we could continue zooming to smaller and smaller regions, without limit.