Global-Local-~~Integration~~-Based Kernel Approximation Methods for Inverse Problems

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Outline

- Kernel-Based Approximation Method (KBA)
- KBA for solving Boundary Value Problems (BVPs)
    - Global (Radial Basis Functions RBFs, Meshfree Collocation)
    - Local Radial Basis Function Collocation Method (LRBFCM)
    - Convergence and Stability of KBA
- KBA for solving Inverse Problems
- Multiscale Support Vector Approach (MSVA)
- MSVA-KBA for solving Inverse Problems
- ~~Finite Integration Method (FIM) for Inverse Problem~~

Background

- E Kansa : Meshfree Global RBFs (Kernel)
- R Schaback : Everythings about RBFs
- C Micchelli : Proof for MQ-RBF Interpolant
- G Fasshauer : Benefits of PD Kernels
- I Sloan : Wendland's CS RBFs
- B Sarler : Local RBFs
- PH Wen : Integration-based RBFs
- Q Ye : RBFs for Stochastic PDEs
- YC Hon : KBA for Inverse Problems

Kernel-Based Approximation (KBA) Method

Kernels

- Given a Kernel $\Phi : \Omega \times \Omega \to \Re$
- Let *H* be a real Hilbert space of functions defined on $\Omega \in \Re^n$ with inner product $(\cdot,\cdot)_H$.
- A function Φ is called a reproducing kernel of *H* if

1. $\Phi(\cdot, x) \in H$ for all $x \in \Omega$.
2. $f(x) = (f, \Phi(\cdot, x))_H$ for all $f \in H$ and all $x \in \Omega$.

A Hilbert space of functions which admits a reproducing kernel is called a Reproducing Kernel Hilbert Space (RKHS).

Kernels Everywhere

- Interpolation
- Analysis: Transforms, Approximation
- Statistics: Spatial Stochastic Processes (Geostatistics); Density Estimation; Statistical Learning Theory
- Machine Learning: Kernel machines
- Boundary Value Problems: Radial Basis Functions(RBFs); Harmonic Functions; Fundamental Solutions; Green's functions; Particular Solutions; Meshfree Methods

KBA for multivariate interpolation

Given: Scattered data $(x_j, u(x_j)) \in \Omega \times \Re, 1 \le j \le N$

General linear reconstruction $\tilde{u}(x) := \sum_{j=1}^{n} L_j(x) u(x_j)$

Error estimate

$$|u(x) - \tilde{u}(x)|^2 = \left| u(x) - \sum_{j=1}^{n} L_j(x) u(x_j) \right|^2$$

Optimal *L* is Lagrange interpolant on a Kernel space

u belongs to Reproducing Kernel Hilbert Space

This implies *L* must depend on position of *x (Mairhuber-Curtis)*

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Mostly used Kernels: RBFs

- A *Basis Function* is a function such that $\phi : \mathbb{R}^d \to \mathbb{R}$.
- If ϕ is a basis function and $f(x) = f(y)$ whenever $\|x\| = \|y\|$, then ϕ is called a *Radial Basis Function* (RBF).
- A RBF is a weighted sum of translations of a radially symmetric basic function augmented by a polynomial term.

Some examples of RBFs ($r = \|x - y\|$),

$\phi(r) = r,$	linear,
$\phi(r) = r^3,$	cubic,
$\phi(r) = r^2 \log r,$	thin-plate spline,
$\phi(r) = e^{-\alpha r^2},$	Gaussian,
$\phi(r) = (r^2 + c^2)^{\frac{1}{2}},$	multiquadric,
$\phi(r) = (r^2 + c^2)^{-\frac{1}{2}},$	inverse multiquadric,

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RBFs for Multivariate Interpolation

Given N data $\{x_i, y_i\}_1^N$. The interpolation problem is to find a function f such that

$$f(x_i) = y_i, \quad \text{for} \quad i = 1, \dots, N.$$

The linear resultant system

$$\sum_{i=1}^{N} \lambda_i \phi(\|x_i - x_j\|) = y_j, \quad \text{for} \quad j = 1, \dots, N,$$

is well-posed if the interpolation matrix

$$A_\phi = \Big[\phi(\|x_i - x_j\|)\Big]_{1 \le i,j \le N}$$

is non-singular.

See: C. Micchelli, Conditional Positive Definiteness of RBFs interpolant, 1986

The best multivariate interpolant among 29 existing interpolation methods – Franke 1987

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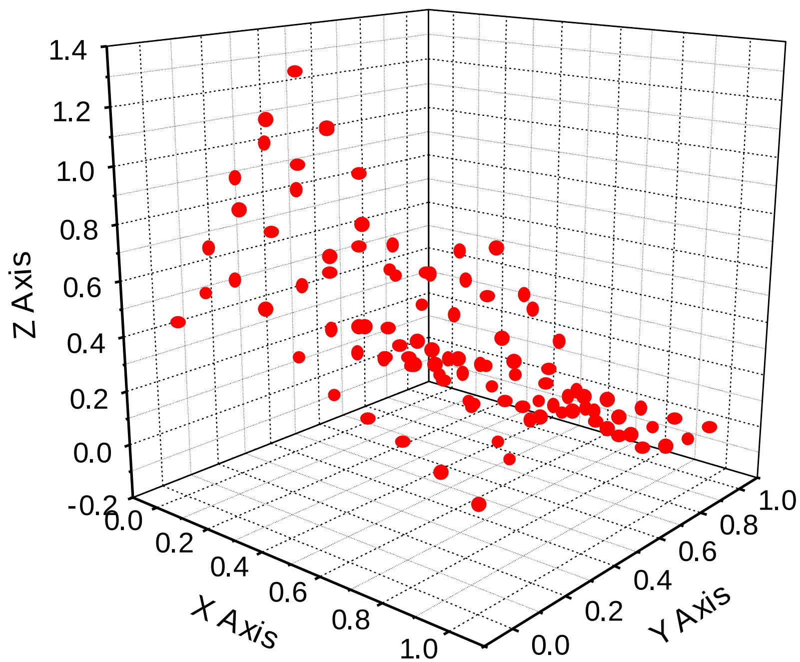
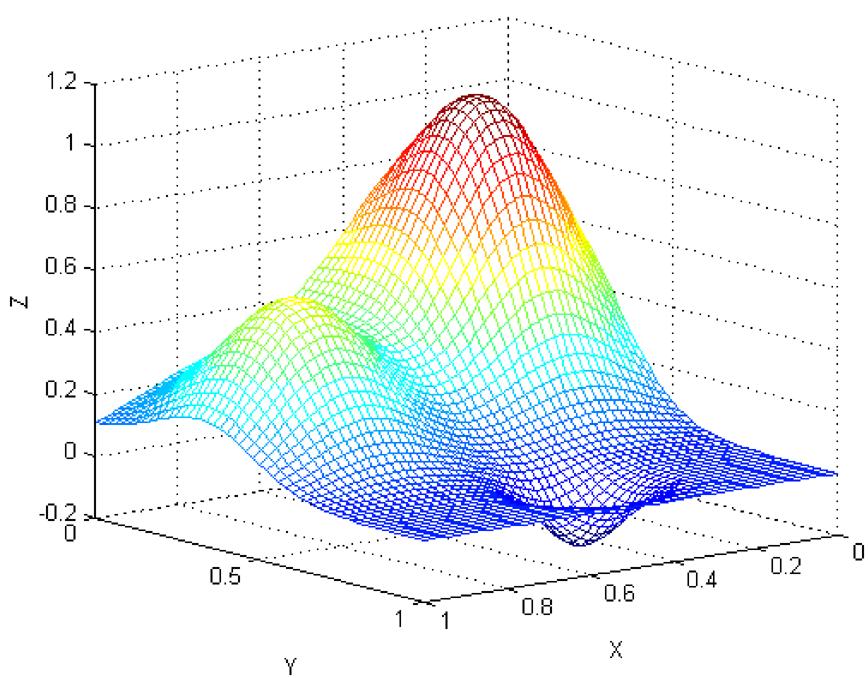
RBFs for Surface Interpolation

$$f(x,y)=\frac{3}{4}\exp\left(\frac{-1}{4}\left((9x-2)^2+(9y-2)^2\right)\right)+\frac{3}{4}\exp\left(\frac{-1}{49}(9x+1)^2-\frac{1}{10}(9y+1)^2\right)$$
$$+\frac{1}{2}\exp\left(\frac{-1}{4}(9x-7)^2-\frac{1}{4}(9y-3)^2\right)-\frac{1}{5}\exp\left(-(9x-4)^2-(9y-7)^2\right)$$





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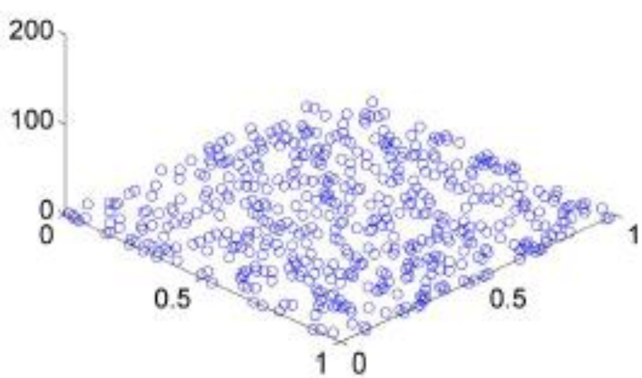
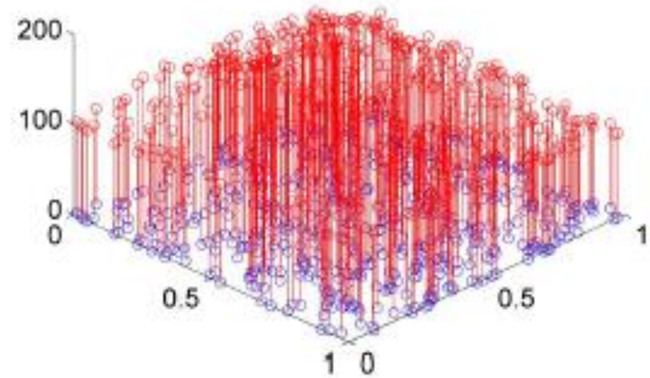
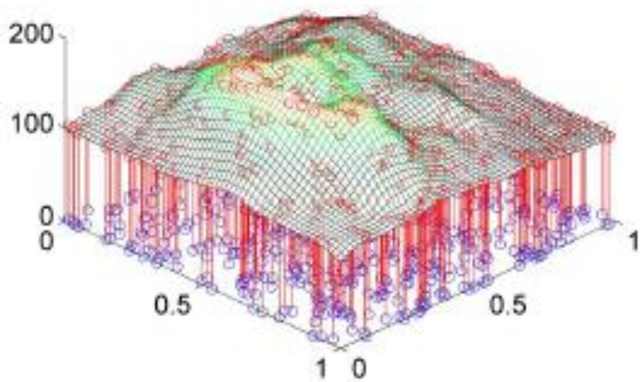


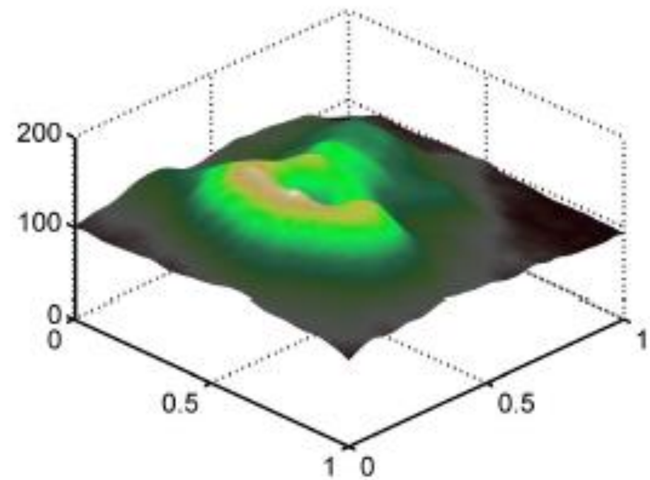
(a) Scattered measurement locations for volcano data (normalized to the unit square).



(b) Elevation measurements associated with the locations shown in (a).



(c) Volcano data fitted by a Gaussian kernel interpolant.



(d) Surface of (c) rendered with more realistic colors.

Figure 12: Illustration of the scattered data fitting problem.

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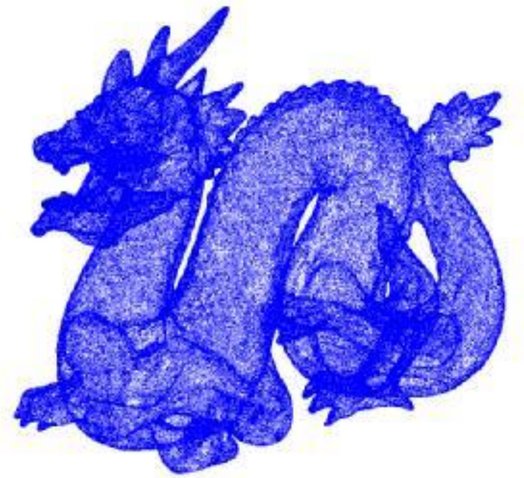
Recent Nice Example



- A dragon consisting of 473,000 vertices and 871,000 facets (left) is modelled with *ARANZ's FastRBF™* engine by a single function consisting of 32,000 terms (right) (Beason and Newsam, Fast evaluation of radial basis functions: Moment based methods, SIAM J. Sci. Comput., 1998.)

Theoretical works:

Wu and Schaback, Local error estimates for radial basis function interpolation of scattered data, IMA J. Numer. Anal., 1993.

Madych and Nelson, Multivariate interpolation and conditionally positive definite functions, Approximation Theory Appl., 1998.

Fasshauer, Hermite Interpolation with Radial Basis Functions on Spheres, *Adv. Comp. Math.,* 1999.

Yoon, Spectral approximation orders of radial basis function interpolation on the Sobolev space, SIAM J. Math. Anal., 2001.

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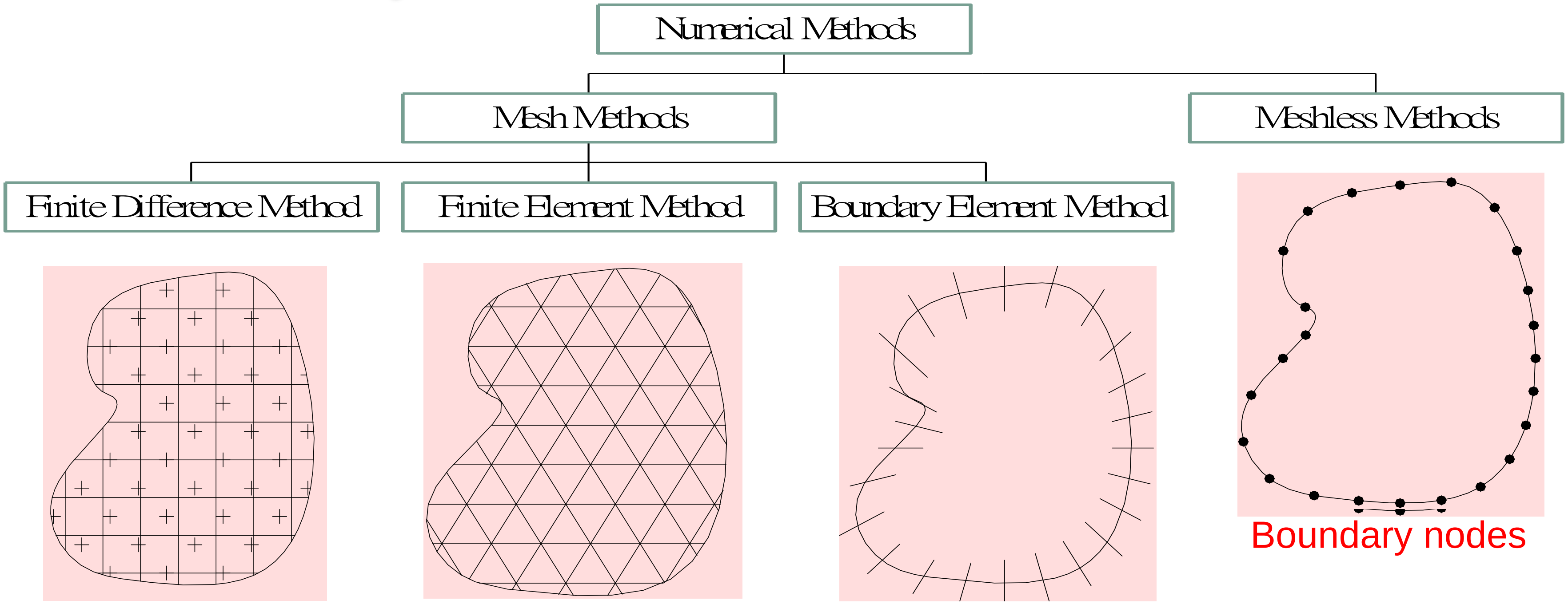
KBA for Boundary Value Problems

Numerical methods for PDEs with boundary conditions (BVPs)



Continuous partial Derivatives by Finite Quotient

PDE - weak form by variational formulation

Boundary Integral, Particular Solution

MFS,Trefftz method, MLS, EFG, RBF, KBA

Kernel-based Approximation (KBA) for solving BVPs

Given:

PDE data
$$\begin{cases} Lu(x) = f(x) & x \in \Omega \\ Bu(y) = g(y) & y \in \partial\Omega \end{cases}$$

Problem: Find *u* such that

Input data $\mapsto$ Reproducing kernel

This leads to Meshfree Computation

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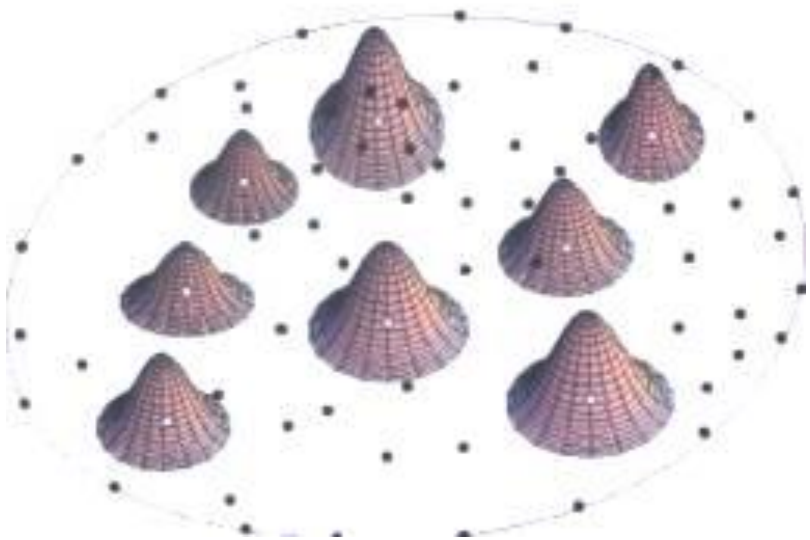
KBA by Weak-formulation



Distributed nodal points

Moving least square approximation

$$u^h(x) = \sum_{j=1}^{m} p_j(x) a_j(x)$$

$$\min \sum_{i=1}^{n} w_i(x)(p^T(x_i)a(x) - \hat{u}_i)^2$$

Gaussian weight functions w

Partition of Unity $\sum_{i=1}^{n} w_i(x) = 1$

Numerical integration techniques

Pros:

No meshing required

No extrapolation needed for computing derivatives

Allows larger trial spaces beyond piecewise polynomial in FEM

Meshfree: Constructing the approximation entirely in terms of nodes (Belytschko et al. 1996)

Cons:

Numerical integration errors can be serious

(Ciarlet, Basic error estimates for elliptic problems, Handbook of Numerical Analysis, 1991)

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Weak-formulation Meshfree Methods

- Element Free Galerkin (EFG)
- Generalized Finite Element (GFEM)
- Reproducing Kernel Particle Method (RKPM)
- Meshless Local Petrov-Galerkin (MLPG)

Atluri and Shen, The Meshless Local Petrov-Galerkin method, 2002.

Liu G.R., Mesh Free Methods Moving beyond finite element method, 2002.

Babuska, Banerjee and Osborn, Survey of meshless and generalized finite element, Acta Numerica, 2005.

Liu Y, Zhang X, Lu M W, A meshless method based on least-squares approach for steady- and unsteady-state heat conduction problems, Numerical Heat Transfer, 2005.

Schaback, Why does MLPG Work? ICCES, 2007.

Sladek et al and Hon, Inverse heat coefficient by MLPG, CMES, 2009.

KBA by Collocation for solving BVPs

Consider a PDE of the form

$$\begin{aligned} \mathcal{L}u &= f \quad \text{in } \Omega \subset \mathbb{R}^d, \\ \mathcal{B}u &= g \quad \text{in } \partial\Omega, \end{aligned}$$

The solution u is approximated by

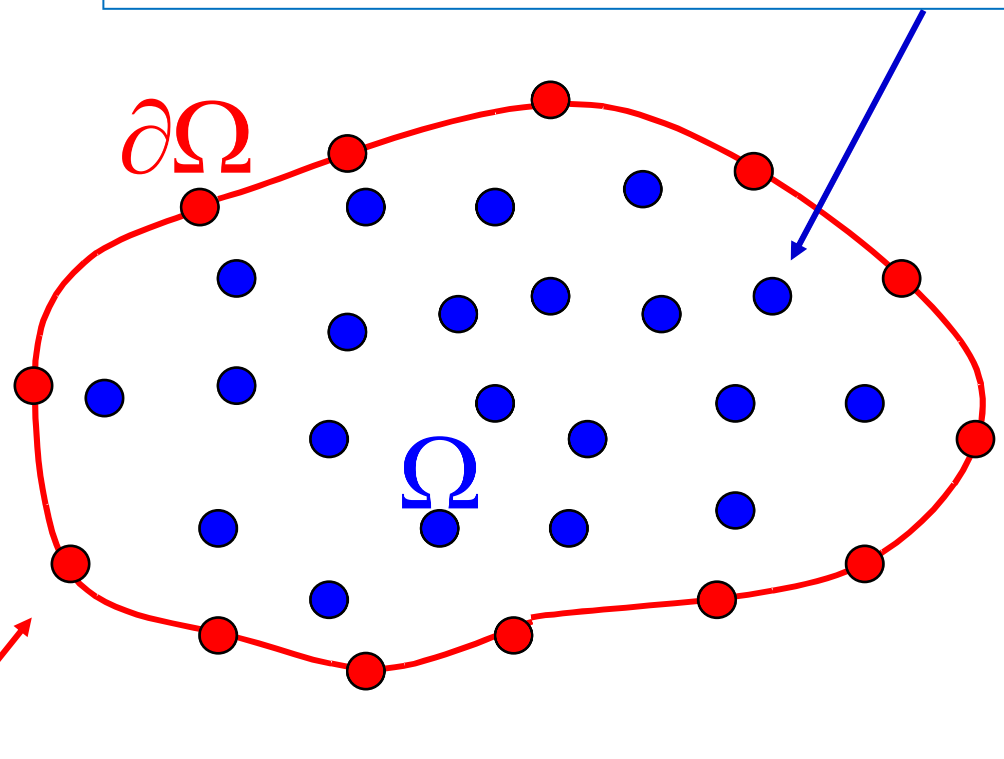
$$u(x) \approx U(x) = q(x) + \sum_{k=1}^{N} \lambda_k \phi(x - x_k),$$

Ensuring that U satisfies the PDE at the collocation points results in a good approximation of the solution,

$$\begin{aligned} \mathcal{L}q(x_i) + \sum_{k=1}^{N} \lambda_k \mathcal{L}\phi(x_i - x_k) &= f(x_i) \quad \text{for } x_i \subset X \cap \Omega, \\ \mathcal{B}q(x_i) + \sum_{k=1}^{N} \lambda_k \mathcal{B}\phi(x_i - x_k) &= g(x_i) \quad \text{for } x_i \subset X \cap \partial\Omega. \end{aligned}$$

Example 1: Poisson's equation

$$In\ \Omega: \sum \lambda_j \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right)(r_j) = f(x_i, y_i), \quad for\ i = 1, 2, \ldots, N_I.$$



$$\text{On } \partial\Omega: \quad \sum_{j=1}^{N} \lambda_j \phi_j(r_j) = g(x_i, y_i), \quad \text{for } i = N_I + 1, \ldots, N.$$

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KBA by Collocation for solving BVPs

The points are arranged in such a way that the first N_I points and the last N_B points are in Ω and $\partial\Omega$, respectively.

In matrix form,

$$\begin{bmatrix} A_\phi & P \\ P^T & 0 \end{bmatrix} \begin{bmatrix} \lambda \\ \gamma \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

where

$$A_\phi = \begin{bmatrix} A_{\mathcal{L}} \\ A_{\mathcal{B}} \end{bmatrix}, \; f = \begin{bmatrix} f(x_i) \\ g(x_i) \end{bmatrix},$$

and

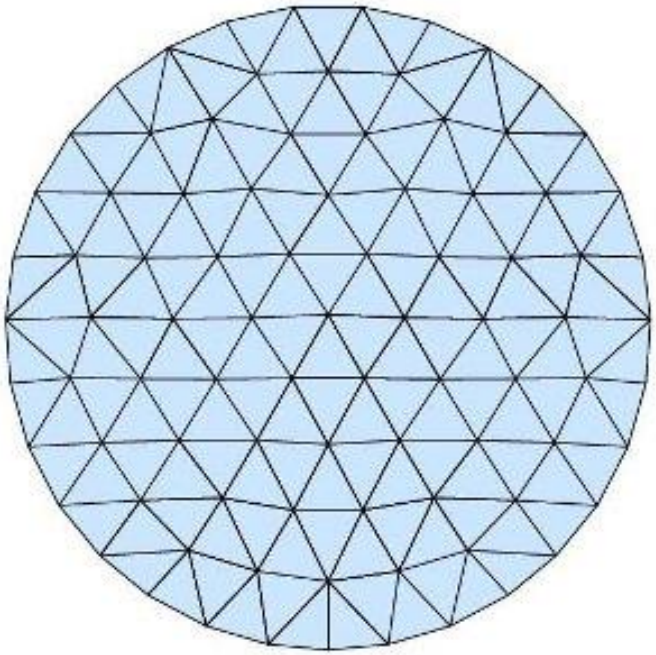
$$\begin{aligned} (A_{\mathcal{L}})_{ik} &= \mathcal{L}\phi(x_i - x_k), \quad x_i \subset \Omega, \quad\;\; x_k \in X, \\ (A_{\mathcal{B}})_{ik} &= \mathcal{B}\phi(x_i - x_k), \quad x_i \subset \partial\Omega, \quad x_k \in X. \end{aligned}$$

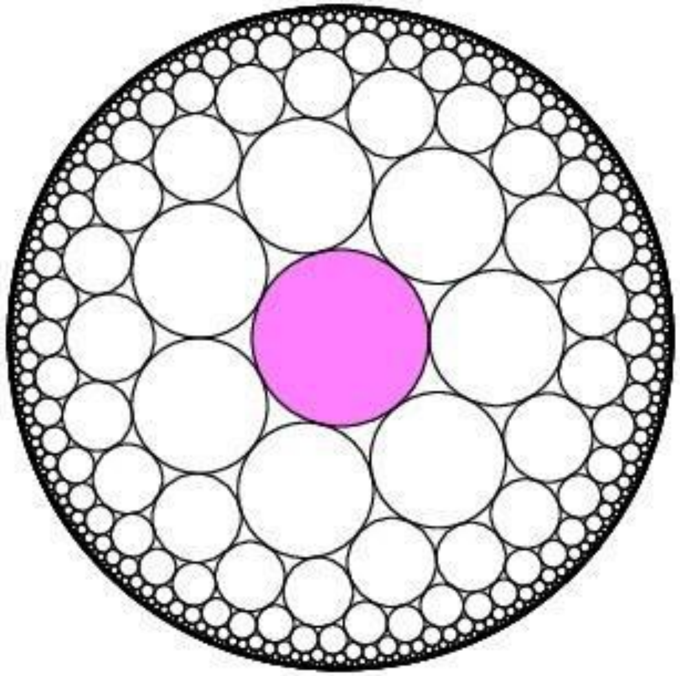
Unsymmetric collocation scheme firstly given by a physicist E.J. Kansa in Proc. Simulation Conference: Application of Hardy's multiquadric interpolation to hydrodynamics, 1986.

Singularity of the matrix A for some configuration given by: Hon & Schaback, On unsymmetric collocation by radial basis functions, Appl. Math. Comput., 2001.

KBA as Numerical PDE Solver

- Claim: If FDM, FEM, FVM, BEM work, then KBA should also work





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Example: Burger's equation with Shock Wave

$$u_t + uu_x = \frac{1}{R}u_{xx} \quad t > 0, R > 0, x \in (0,1)$$

$$u(0,t) = 0 = u(1,t)$$

$$u(x,0) = f(x) \quad x \in [0,1]$$

$$u^m + \Delta t(u^{m-1}u_x^m - \frac{1}{R}u_{xx}^m) = u^m$$

Total 7 points on [0, 1] :
$x_1, x_2, x_3, x_4, x_5, x_6, x_7$

x_4 **= peak = x* which can be computed by using Newton's iteration:**

$$x_{new}^* = x_{old}^* - \frac{\frac{d}{dx}u^m(x_{old}^*)}{\frac{d^2}{dx^2}u^m(x_{old}^*)}$$

Peak chasing

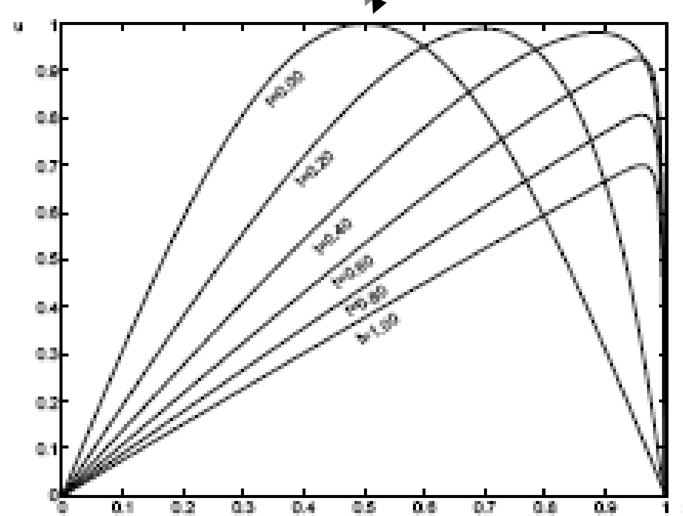

Figure 1e. R = 10000

Hon Y.C. and Mao X.Z., An Efficient Numerical Scheme for Burgers' Equations, AMC,1998.

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It is not easy to get recognised

Minisymposium on

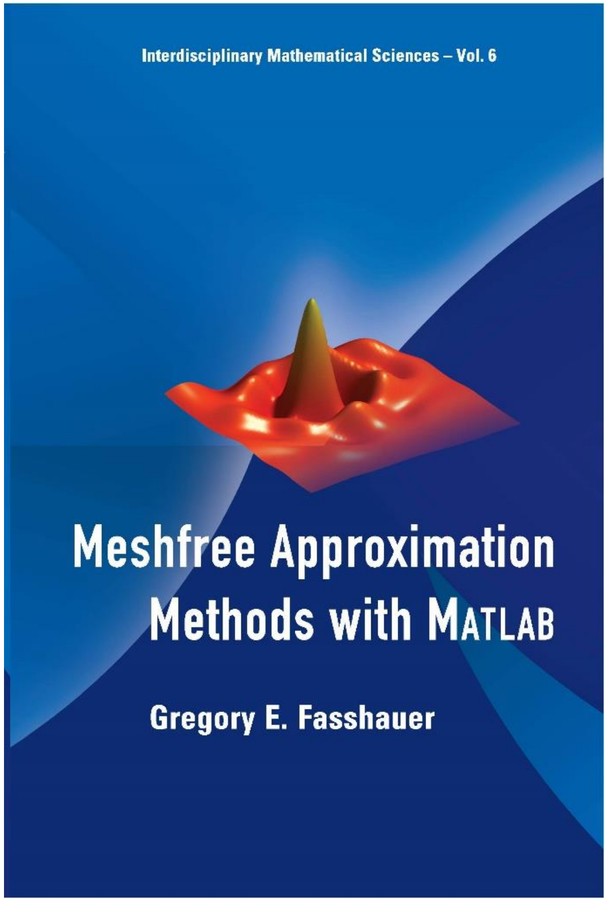
WHAT MESHFREE PARTICLE METHODS CAN DO THAT TRADITIONAL FEA CANNOT

11th World Congress on Computational Mechanics (WCCM XI)

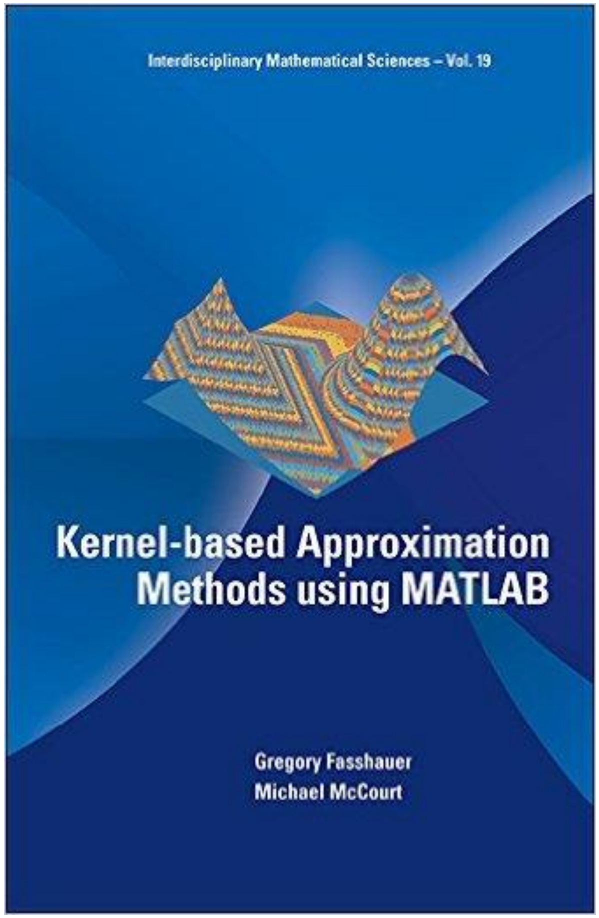
Barcelona, Spain, July 20-25, 2014

http://www.wccm-eccm-ecfd2014.org/frontal/default.asp

Meshfree RBFs to KBA



2007

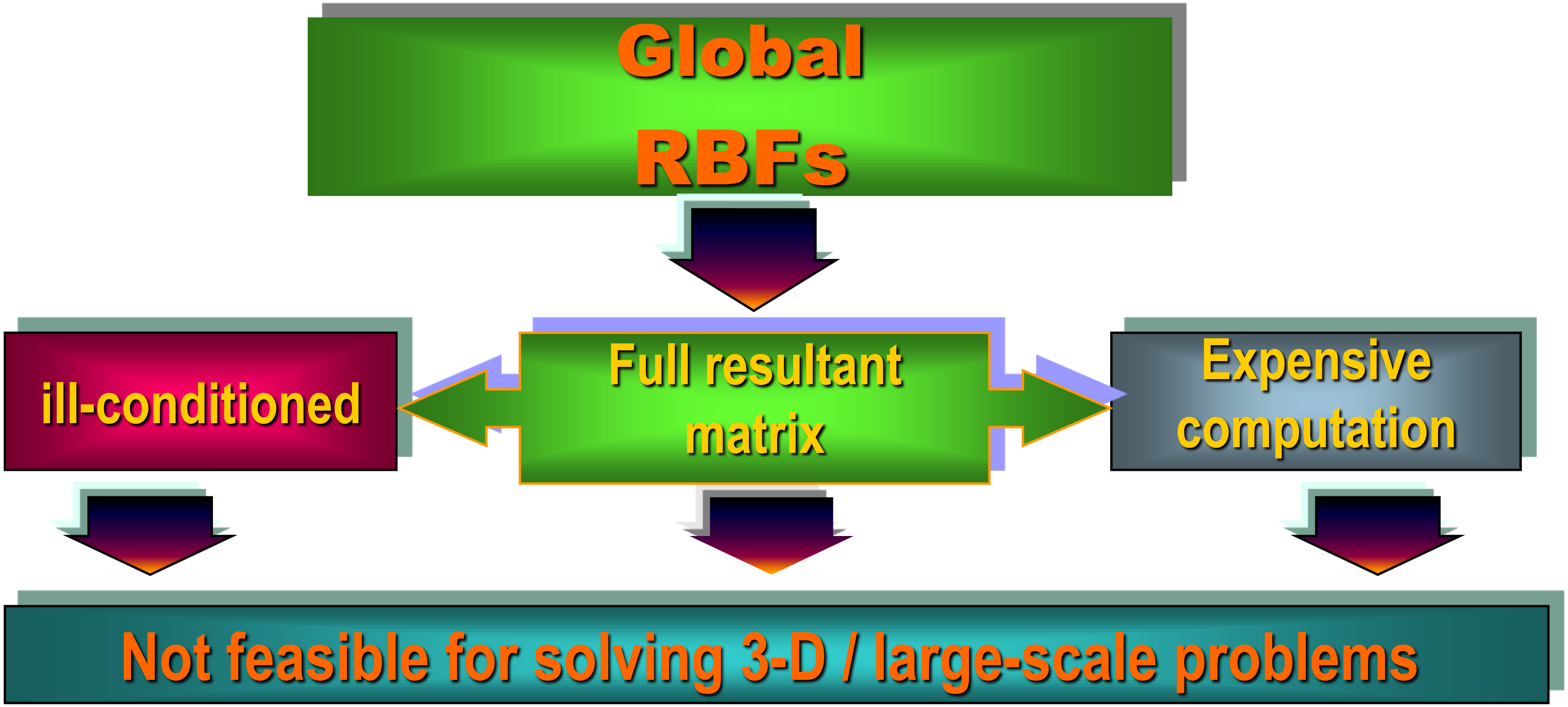


2015

FEM compared with Global KBA

	FEM	KBA
Trial function	Piecewise low-order polynomial	smooth kernel
Domain discretization	Triangular mesh	Points/nodes
Domain operator	Weak formulation	Weak formulation Strong collocation
Resultant matrix	Sparse/banded	Full for strong collocation (ill-cond)
Convergence	Linear	Spectral for problems with smooth solution and boundary

Identified Problems



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Feasible Solutions

Domain decomposition	Compactly Supported functions	Matrix-free algorithm

Improve the ill-conditioning problem, more efficient

Feasible for 3-D larger-scale problems

Solving larger scale PDEs

Compactly supported RBFs ➡ Banded matrix

$$\phi(r) = \left(1 - \frac{r}{\rho}\right)_+^n p\left(\frac{r}{\rho}\right) \quad \text{where} \quad \left(1 - \frac{r}{\rho}\right)_+^n = \begin{cases} \left(1 - r/\rho\right)^n & \text{if } 0 \le \frac{r}{\rho} \le 1, \\ 0 & \text{if } \frac{r}{\rho} > 1 \end{cases}$$

=> Local RBF collcation method by Bozidar SARLAR

Adaptive greedy algorithm

Iteratively adding nodes that minimize residual errors (Hon and Schaback, Numerical Algorithm, 2003)

=> Greedy RBFs by Leevan LING of HKBU

Domain decomposition method

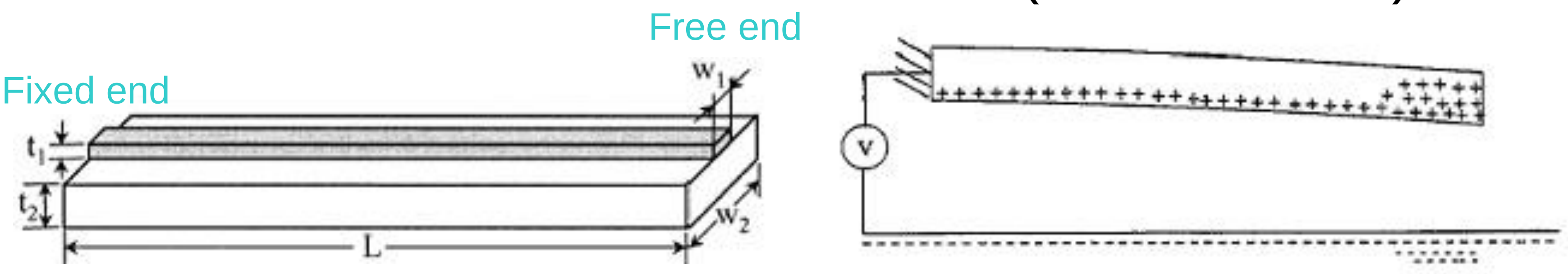
Overlapping and non-overlapping iterative Algorithms both works well

=> Adaptive DDM with Chebyshev tau method

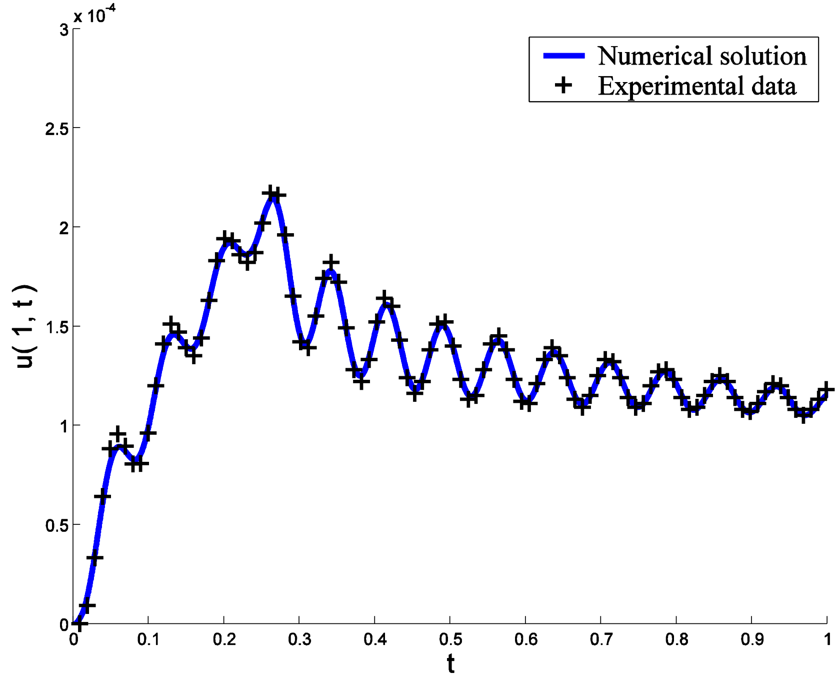
Larger Scale Problems

Example : CSRBF for MEM Model

- Micro-Electrical-Mechanical Models (Beam & Plate)



Control of Ink from bubble jet printer: real data from Kodak



Governing equation

$$a_1 \frac{\partial^2 u(x,t)}{\partial t^2} + a_2 \frac{\partial u(x,t)}{\partial t} + a_3 \frac{\partial^4 u(x,t)}{\partial x^4} = 0$$

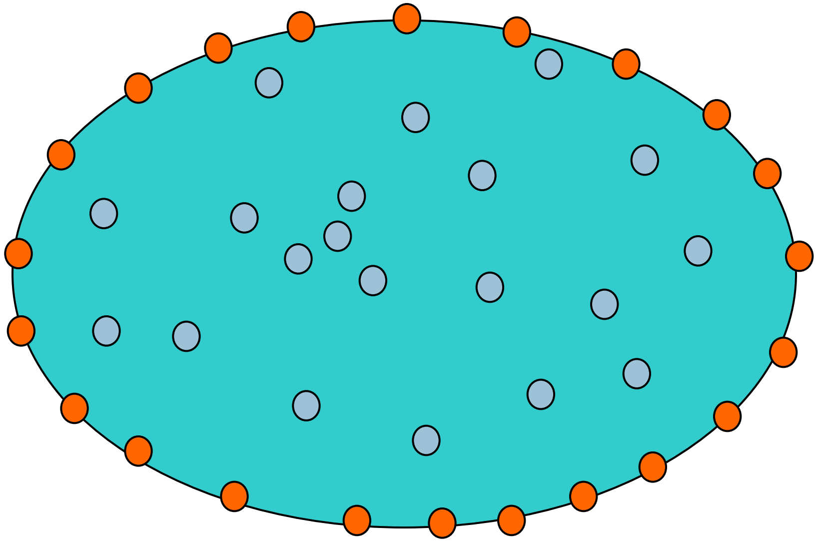
Liu, Hon and Liew, A meshfree Hermite-type radial point interpolation method for Kirchhoff plate problems, International Journal for Numerical Methods in Engineering, 2006.

Liu, Liew, Hon and Zhang, Numerical simulation and analysis of an electroactuated beam using a radial basis function, Smart Materials and Structures, 2005.

Hon, Ling and Liew, Numerical analysis of parameters in a laminated beam model by radial basis functions, Computers, Materials and Continua, 2005.

Local KBA for Large Scale Problems

Idea: Representation of field variables by kernels in a set of non-uniformly spaced set of points



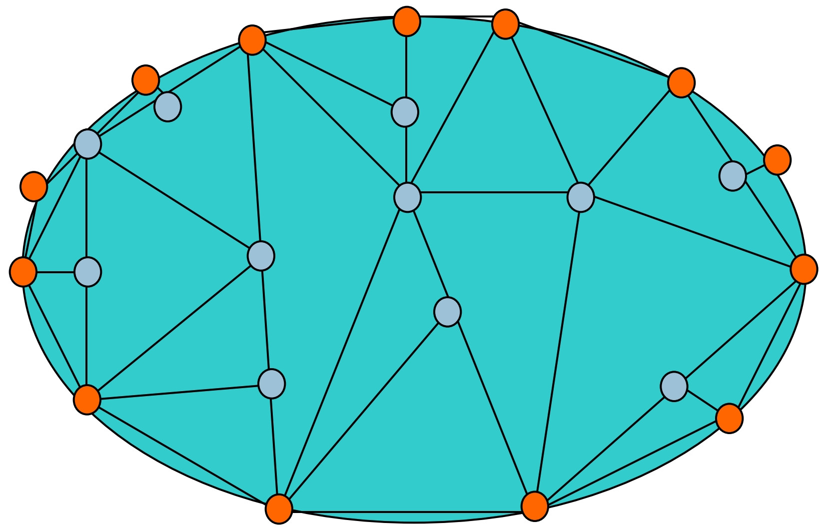
 BOUNDARY NODES

 DOMAIN NODES

CONCEPT OF CONTIGUOUS SUBDOMAINS

MESH
METHODS

FINITE
ELEMENTS

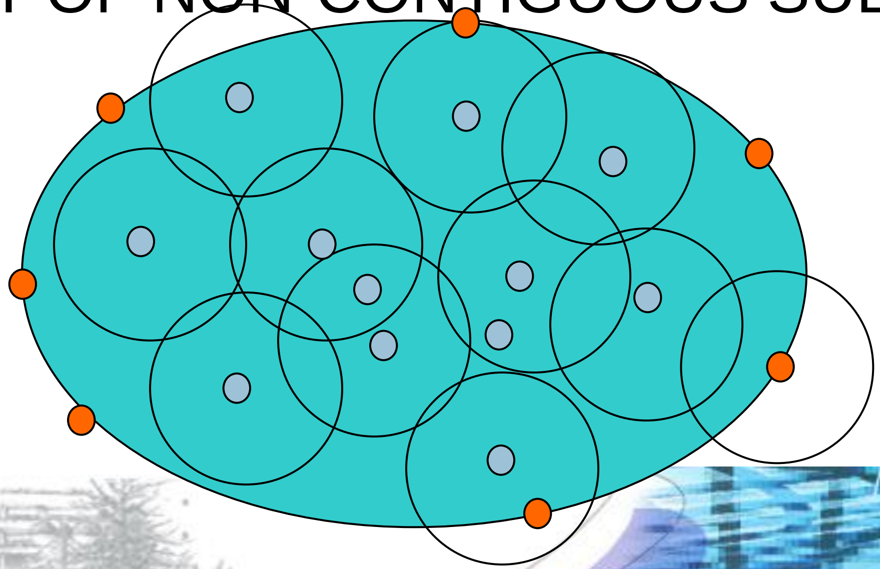


FVM, FEM, BEM

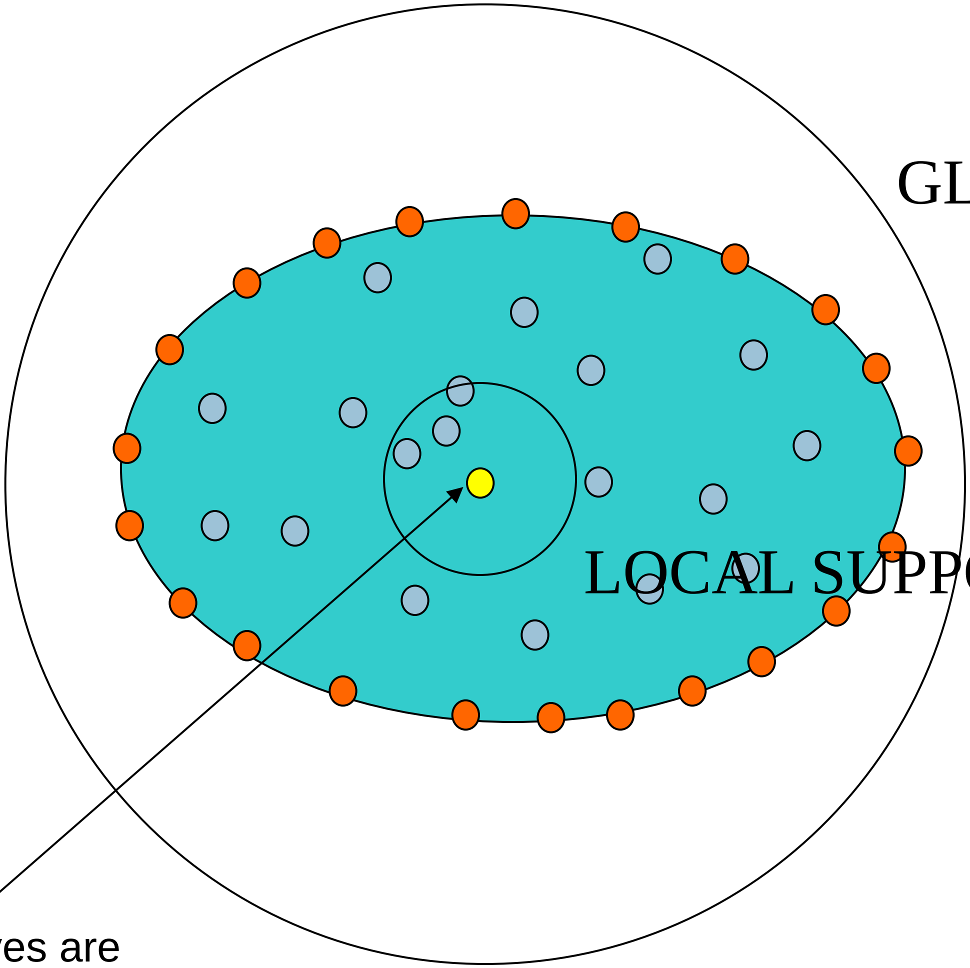
CONCEPT OF NON-CONTIGUOUS SUBDOMAINS

MESHFREE
METHODS

SUPPORT
DOMAINS



LOCAL & GLOBAL REPRESENTATIONS



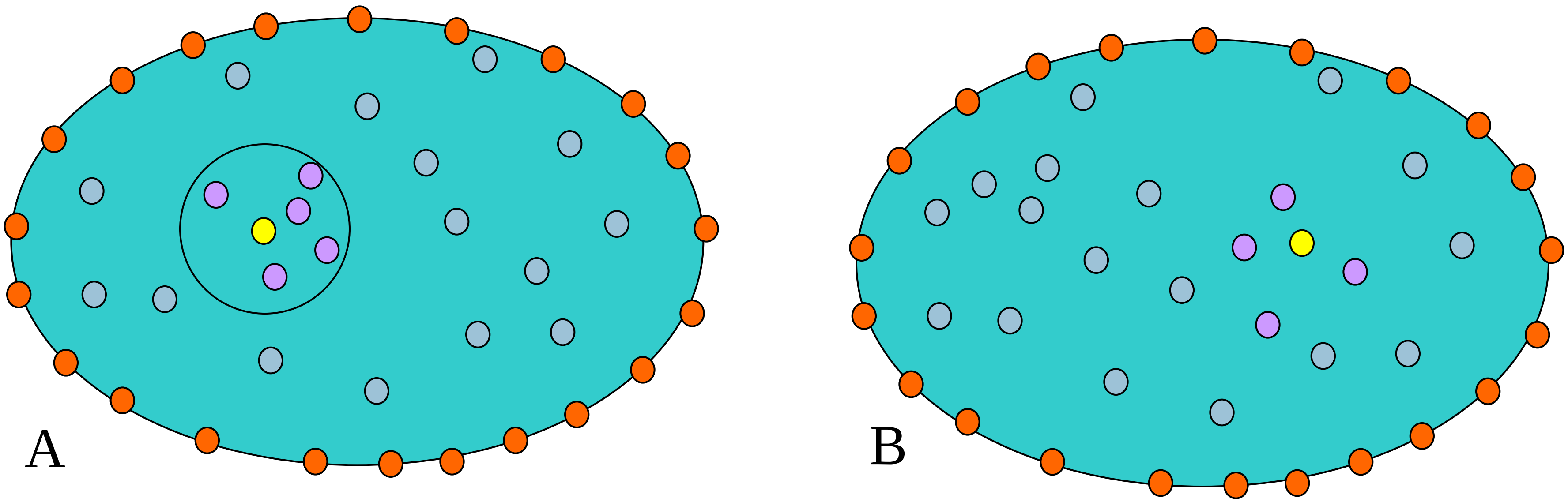
GLOBAL SUPPORT

$$\Phi(\mathbf{p}_n) = \sum_{k=1}^{N} \psi_k(\mathbf{p}_n)\alpha_k$$

$$\Phi'(\mathbf{p}_n) = \sum_{k=1}^{N} \psi_k{}'(\mathbf{p}_n)\alpha_k$$

node where derivatives are calculated

LOCAL & GLOBAL REPRESENTATIONS



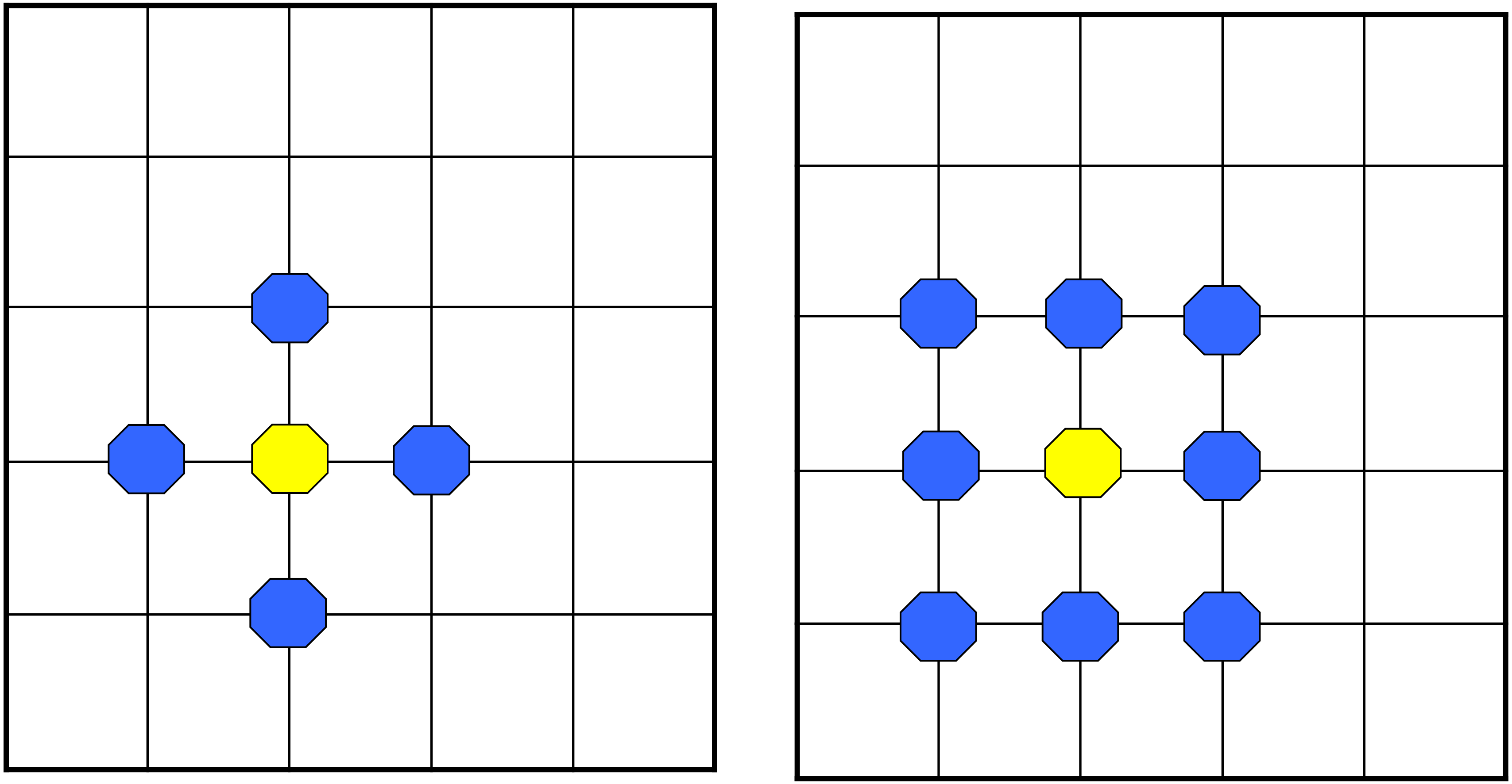
point p_i support is defined by:

A: the points that fall into support circle with radius r^i_{ref}

B: the nearest $\mathrm{N}^{\mathrm{i}}_{\mathrm{ref}}$ points

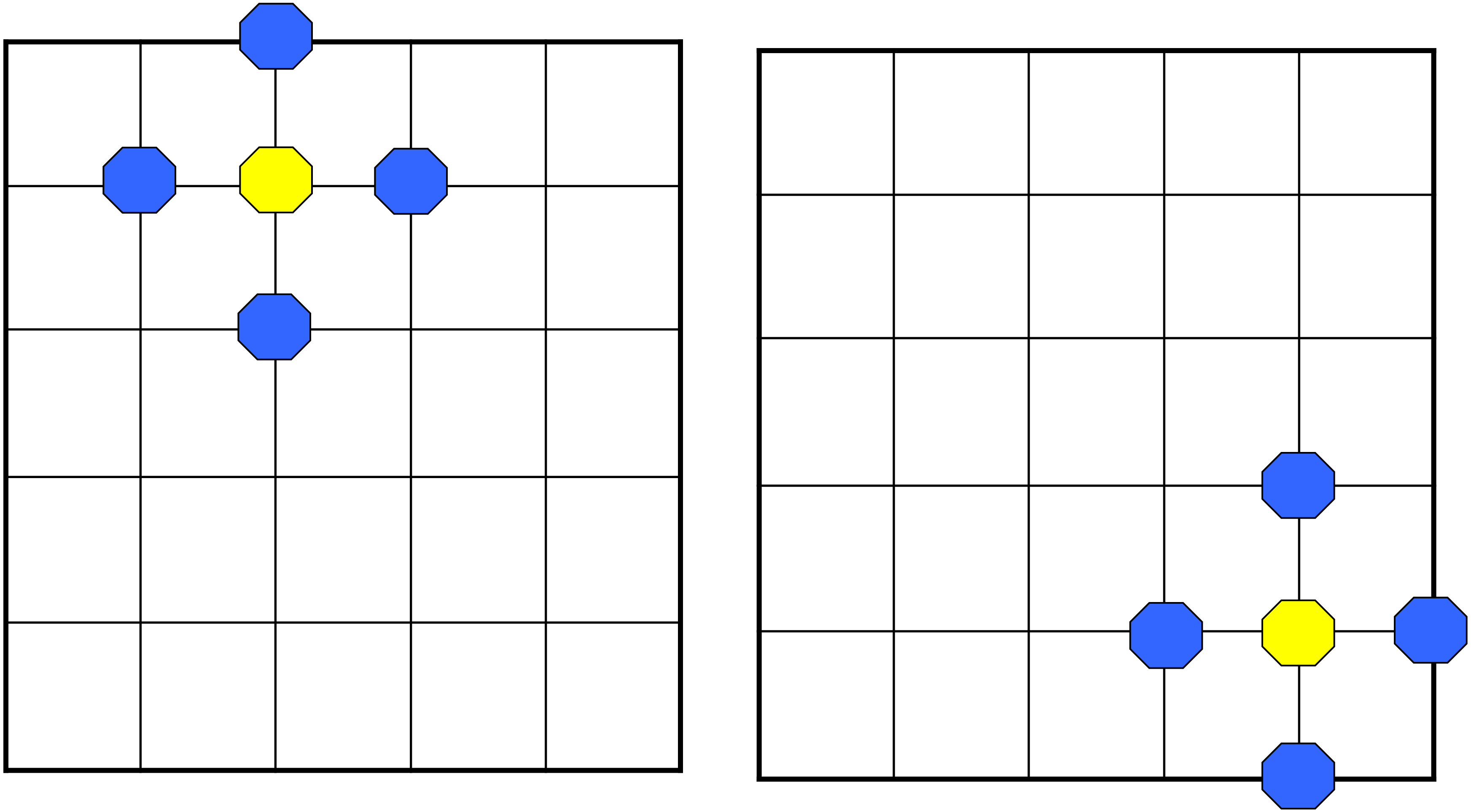
Global support: all points are included in the support

SUBDOMAIN SCHEMATICS – UNIFORM NODE POSITIONS



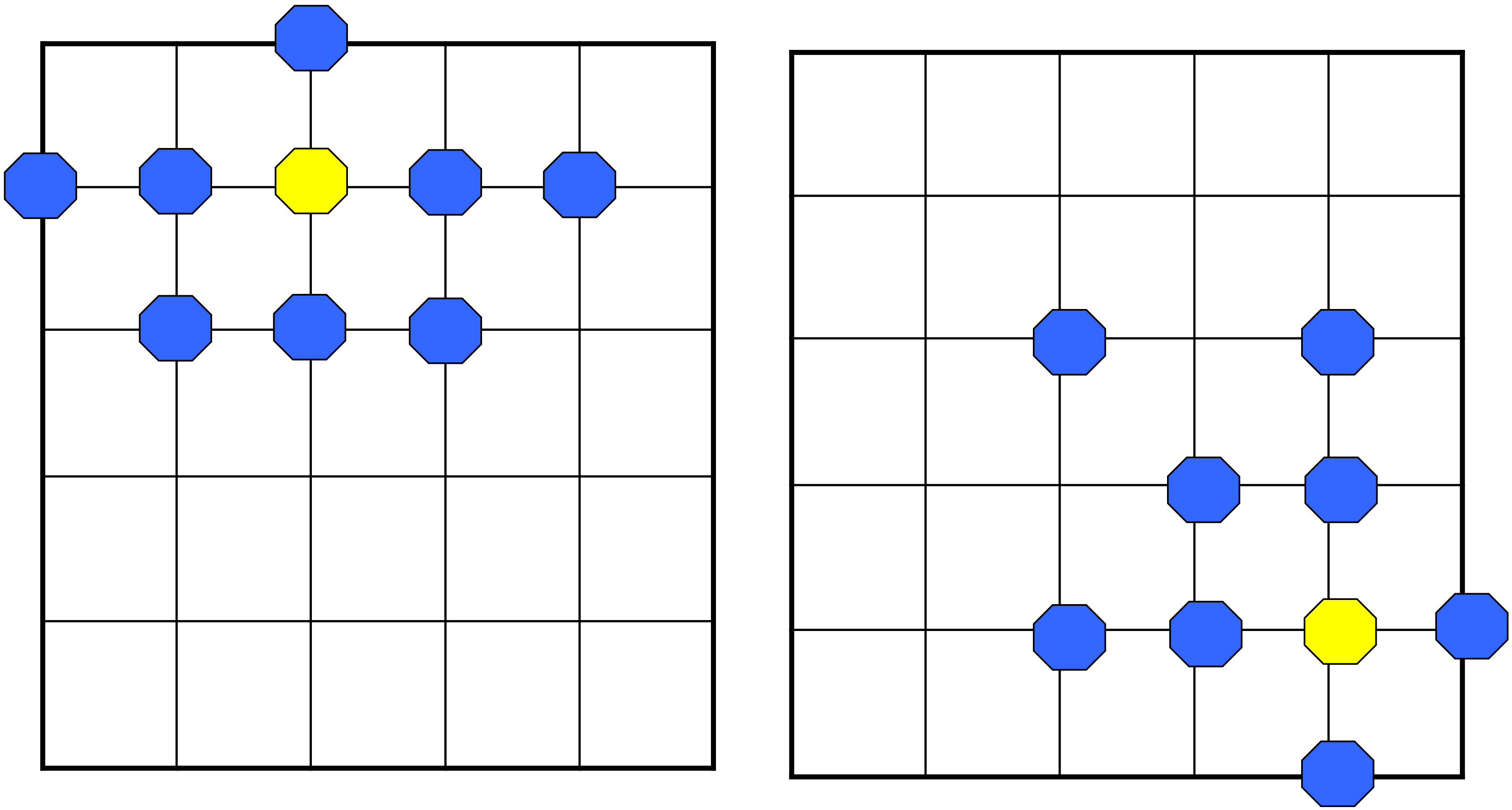
5 and 9 noded influence domains

SUBDOMAIN SCHEMATICS – UNIFORM NODE POSITIONS



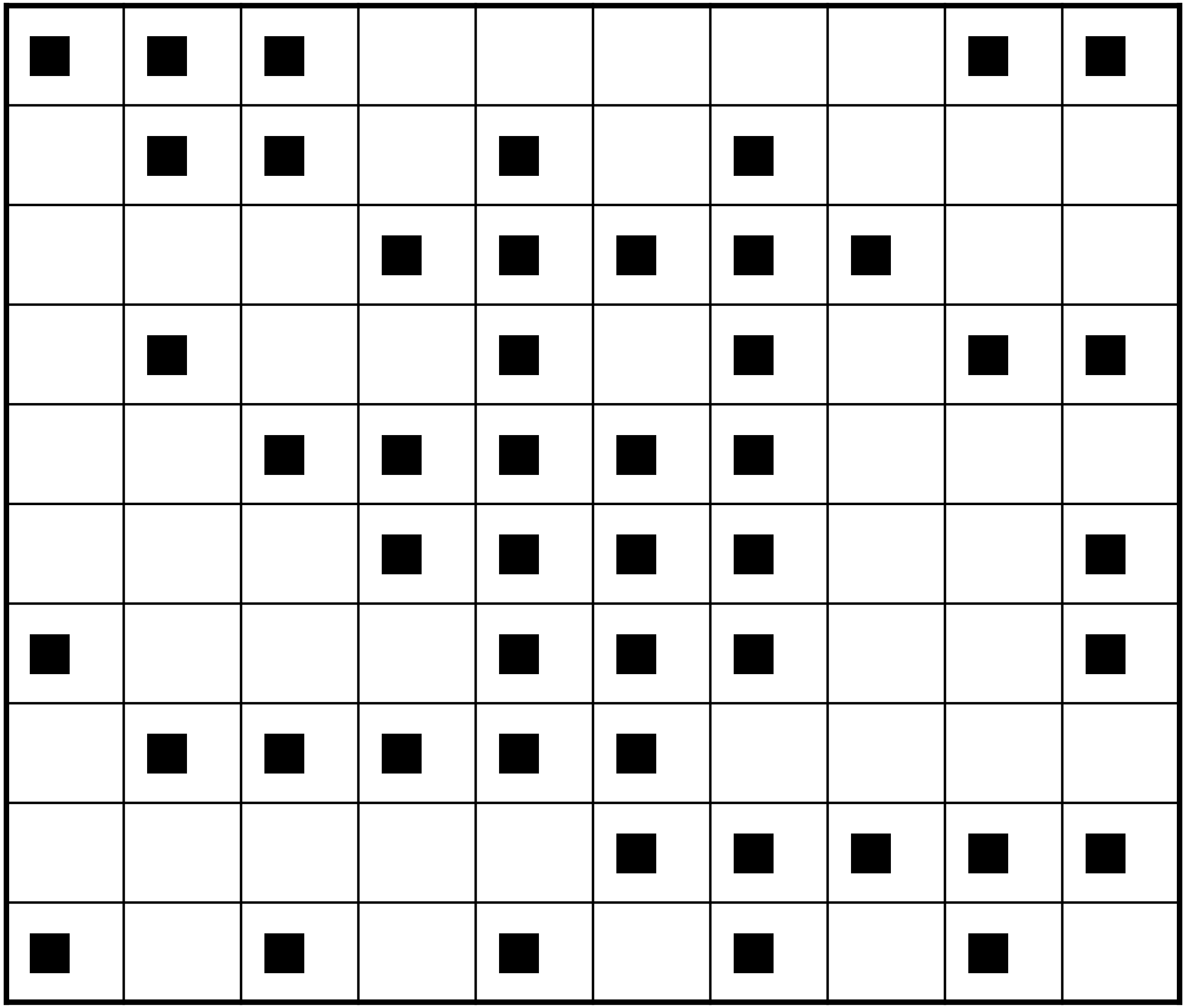
5 noded influence: boundary and corner influence domain

SUBDOMAIN SCHEMATICS – UNIFORM NODE POSITIONS



9 noded influence: boundary and corner influence domain

Resultant sparse block matrix



Example : High Dimensional Problem

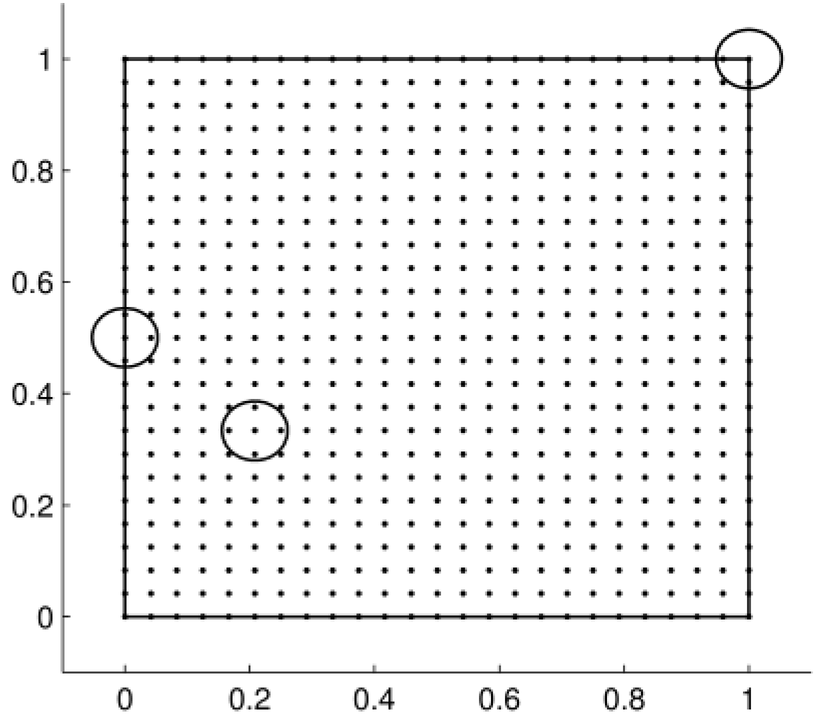


Figure 6.1: Domain Ω, collocation points$(\cdot)$ and the neighborhoods.

$$\Delta u = f \ in \ \Omega \subseteq \Re^d$$

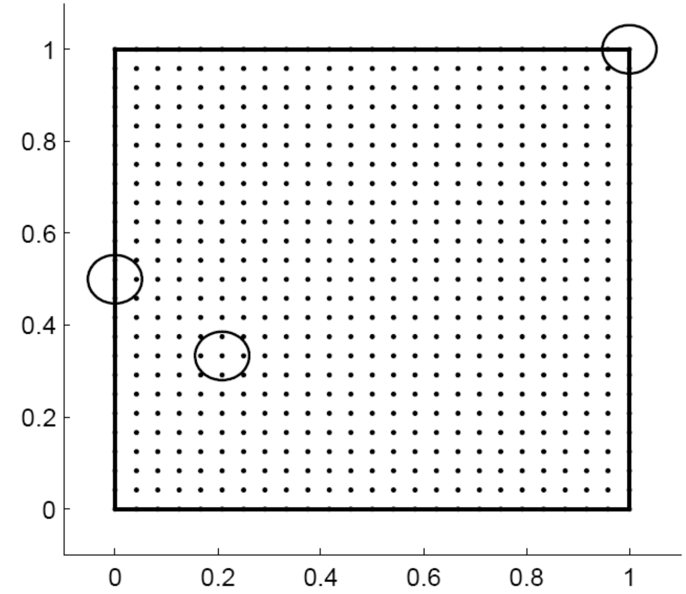
d = dimension of space

Localized KBA for Large Scale Problems

$$B^{\mathbf{x}}\lambda^{\mathbf{x}} = \mathbf{u}^{\mathbf{x}}$$

$$\mathcal{L}\tilde{u}(\mathbf{x}) = \Psi^{\mathbf{x}}(B^{\mathbf{x}})^{-1}\mathbf{u}^{\mathbf{x}}$$

$$\triangle u(\mathbf{x}) + du(\mathbf{x}) = 0 \qquad x \in \Re^d$$



d	n	$RMSE$	Max	$Conditional\ Number$
2	10^2	5.2320×10^{-4}	1.2123×10^{-3}	2.2857×10^{3}
2	20^2	9.9244×10^{-5}	2.7306×10^{-4}	1.9954×10^{4}
3	30^3	4.8270×10^{-5}	1.3209×10^{-4}	8.7465×10^{4}
4	10^4	4.2875×10^{-4}	2.3321×10^{-3}	3.5608×10^{3}
4	15^4	7.3184×10^{-5}	1.3154×10^{-3}	9.8454×10^{4}
5	9^5	2.8904×10^{-4}	2.8123×10^{-3}	3.0512×10^{3}
6	6^6	3.5873×10^{-4}	7.7243×10^{-3}	941.42

NUMERICAL VERIFICATIONS

NAFEMS benchmark test 15 (all types of BC)

Dirichlet jump problem (transient response)

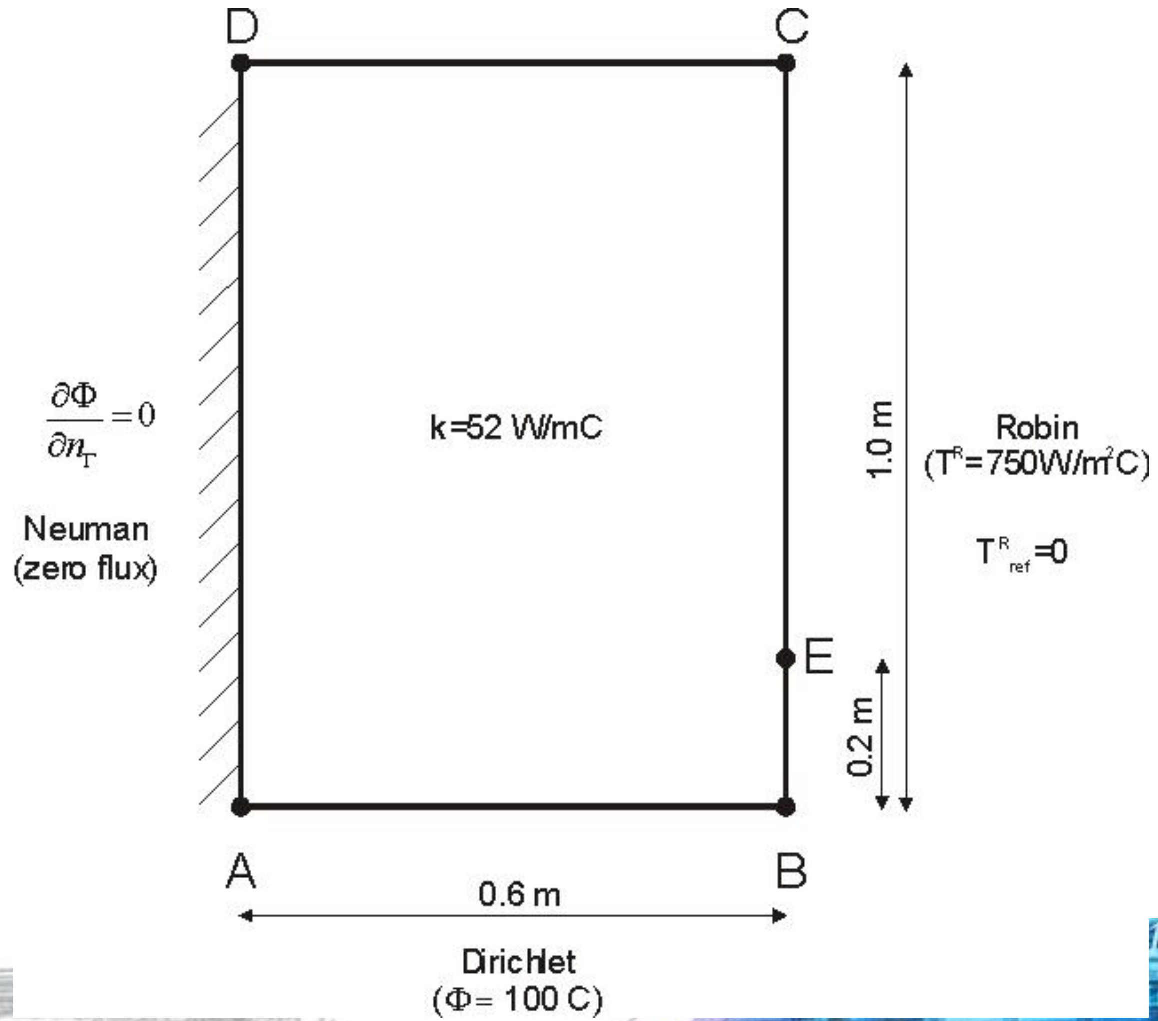
Steady convective-diffusive problems

De Vahl Davis Natural convection

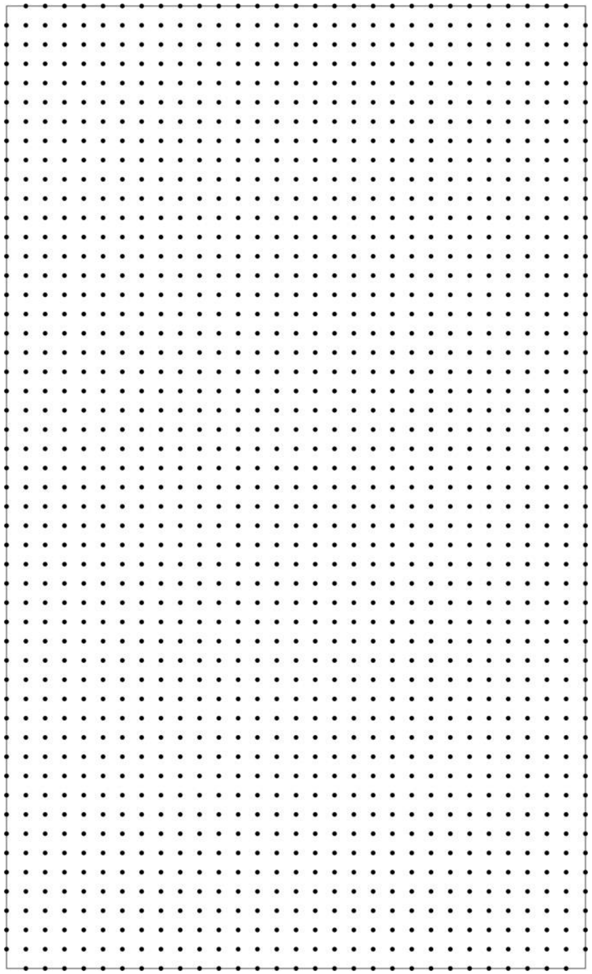
Gobin - Le Quéré (melting)

K-Epsilon turbulence modeling

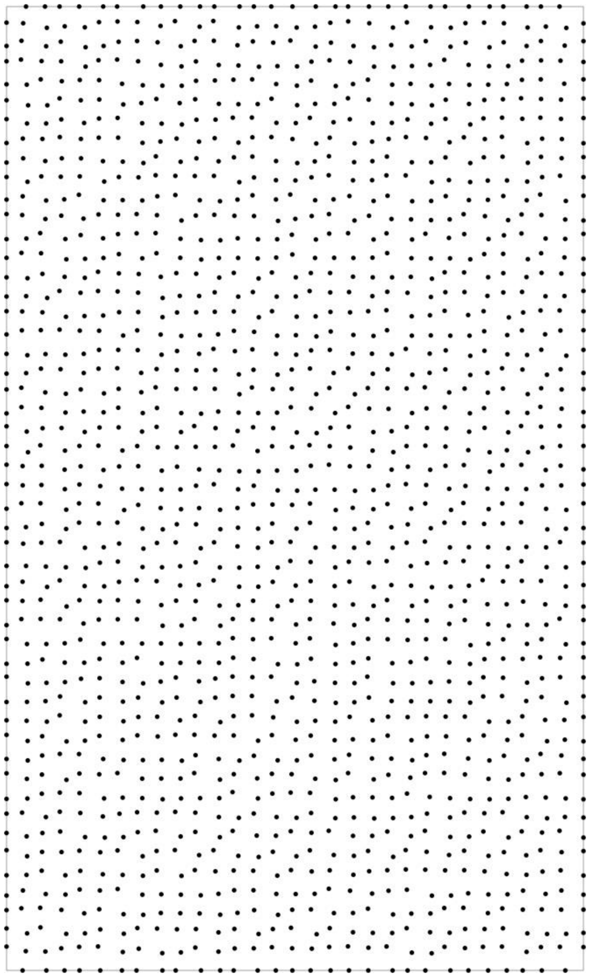
EXAMPLE : TWO-DIMENSIONAL THERMAL TEST PROPOSED BY NAFEMS



NODE POSITIONS



uniform



non-uniform

Test_3,_MQ,_Greedy.avi

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Local radial basis function collocation method for solving thermo-driven fluid-flow problems with free surface

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ABSTRACT

This paper explores the application of the meshless Local Radial Basis Function Collocation Method (LRBFCM) for the solution of coupled heat transfer and fluid flow problems with a free surface. The method employs the representation of temperature, velocity and pressure fields on overlapping five-noded sub-domains through collocation by using Radial Basis Functions (RBFs). This simple representation is then used to compute the first and second derivatives of the fields from the respective derivatives of the RBFs. The energy and momentum equations are solved through explicit time integration scheme. For numerical efficiency, the Artificial Compressibility Method (ACM) with Characteristic Based Split (CBS) technique is firstly adopted to solve the pressure–velocity coupled equations. The performance of the method is assessed based on solving the classical two-dimensional De Vahl Davis steady natural convection benchmark problem with an upper free surface for Rayleigh number ranged from 10^3 to 10^5 and Prandtl number equals to 0.71.

KBA for 3D Laplace Equation
- with convergence order

Idea

- It is tempting to use harmonic kernels *s* for approximating the unknown solution *u* due to the Maximum Principle:

$$\|u-s\|_{\infty,\overline{\Omega}} \leq \|u-s\|_{\infty,\partial\Omega}$$

Hon & Wu, A numerical computation for inverse boundary determination problems, Eng. Anal. Bound. Elem., Vol. 24, pp. 599-606, 2000,

Harmonic Kernels

$$P_c(x,y) = \left(1 - 2x^T y c^{-2} + c^{-4}\|x\|_2^2\|y\|_2^2\right)^{-1/2},$$

$$B_c(x,y) = e^{x^T y c^{-2}} J_0\left(c^{-2}\sqrt{\|x\|_2^2\|y\|_2^2 - (x^T y)^2}\right)$$

Poisson and Bessel kernels which are:

1. Symmetric and harmonic in both arguments;
2. Positive definite;
3. Scalable by the parameter *c*.

Well known Fact

A good approximation *s* to the boundary data by interpolation

$$s(x) := \sum_{j=1}^{n} \alpha_j K(x, x_j),\ x \in \overline{\Omega}$$

can be obtained by solving the following linear system of equations involving kernels *K*

$$\sum_{j=1}^{n} \alpha_j K(x_k, x_j) = u(x_k),\ 1 \le k \le n$$

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Domains for the 3D Laplace

Star-shaped domains

$$(r, \theta, \varphi) \in \mathbb{R}_{\geq 0} \times [0, 2\pi] \times [0, \pi]$$

$$\overline{\Omega} = \{(r, \theta, \varphi)\ :\ 0 \leq r \leq R(\theta, \varphi)\}$$

Theorem 4 *Then there is a constant C, dependent on $\tau > 3/2, K, F$, and Ω, such that for all sets $Y = F(X)$ of scattered points on Γ as images of sets X on the sphere $\mathbb{S}^2$ and all functions f on Γ such that $g := f \circ F$ is in H_τ, the interpolant $I_{Y,K,f}$ to f on Γ using Y–translates of K satisfies*

$$\|f - I_{Y,K,f}\|_{\infty,\Gamma} = \|g - I_{X,L,g}\|_{\infty,\mathbb{S}^2} \leq C h_X^{\tau-3/2} \|g\|_\tau$$

holds on the boundary. By the maximum principle, this error bound extends to Ω for the harmonic function u having boundary values of f, namely

$$\|u - I_{Y,K,f}\|_{\infty,\Omega} \leq \|f - I_{Y,K,f}\|_{\infty,\Gamma} \leq C h_X^{\tau-3/2} \|g\|_\tau.$$

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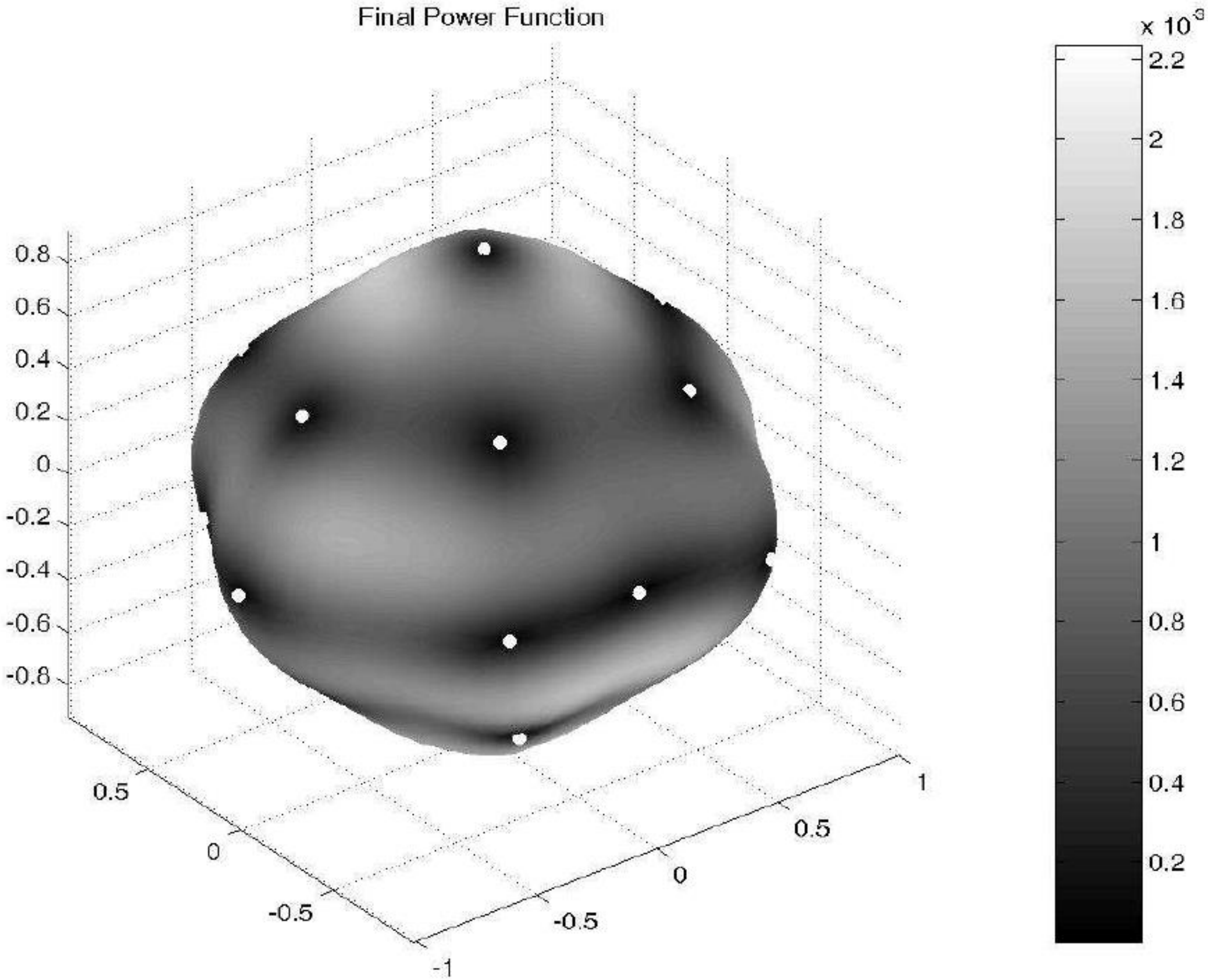
Numerical Example

$$r = (2 + 2\cos^2(2\phi)) \cdot (1 + 0.2\cos^2(3\theta)\cos^2(\phi))/2.5$$ UFO-Domain



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Results

Function	Sph Bes	Sph Poi	UFO Bes	UFO Poi
$x^2 - 2y^2 + z^2$	13.70	12.31	14.33	13.96
FS	13.78	12.43	13.92	11.06
z^2	13.85	12.39	2.67	2.54
$\|x\|^5$	5.34	4.99	3.07	2.92
$K_5(\cdot, u)$	13.26	9.72	6.18	3.30

. Estimated convergence orders

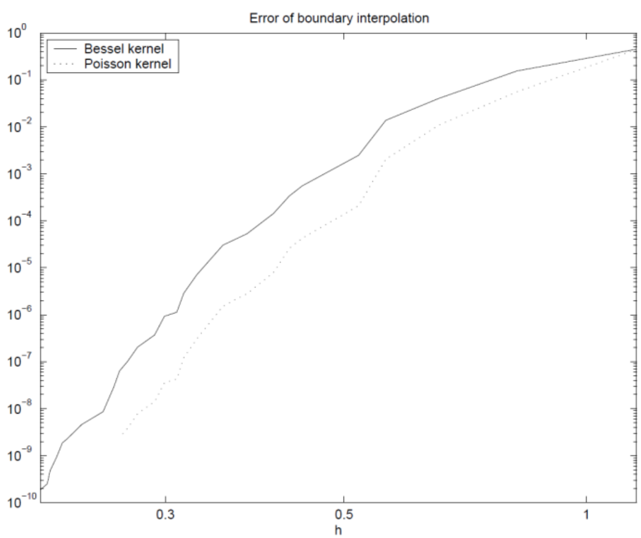

Fig. 10 Error for data from harmonic function

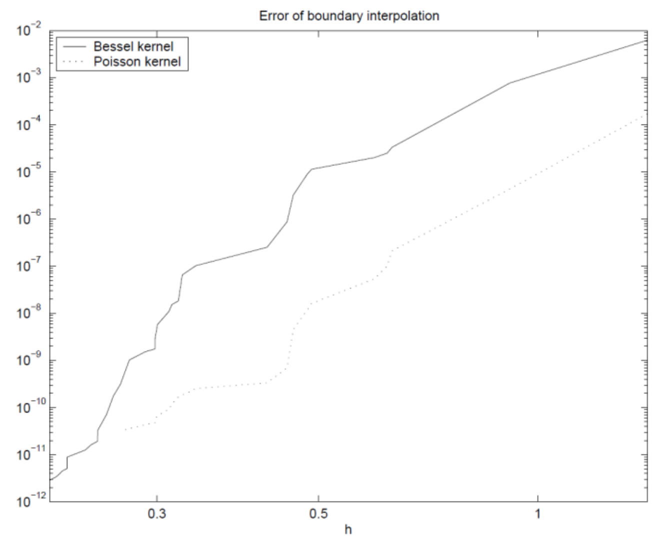

Fig. 11 Error for K_5 data on sphere

Hon Y.C. and Schaback R., Solving the 3D Laplace equation by meshless collocation via harmonic kernels, Advances in Computational Mathematics, Vol. 38, pp. 1-19, 2013.

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Theoretical Justifications

Symmetric Hermite (Wu 1992, Franke & Schaback 1998)

Hon & Schaback, On unsymmetric collocation by radial basis functions, Applied Mathematics and Computations, Vol. 119, 2001.

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Duan, Hon & Zhao, Stability estimate on meshless unsymmetric collocation method for solving boundary value problems, Engineering Analysis with Boundary Elements, Vol. 37, pp. 666-672, 2013.

Hon, Schaback & Zhong, The meshless kernel-based method of lines for parabolic equations, Computers and Mathematics with Application, Vol. 68, pp. 2057-2067, 2014.

Zhong, Hon & Lu, Multiscale support vector approach for solving ill-posed problems, J. Scientific Computing (JOMP), Vol. 64, pp. 317-340, 2015.

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RKHS for Inverse (ill-posed) Problems

Adv Comput Math (2011) 34:167–183
DOI 10.1007/s10444-010-9148-1

Discretized Tikhonov regularization by reproducing kernel Hilbert space for backward heat conduction problem

Y. C. Hon · Tomoya Takeuchi

Backward Time-Fractional Heat Conduction Problem

$$\frac{\partial^{\beta} u(\mathbf{x},t)}{\partial t^{\beta}} = \nabla^2 u, \quad \mathbf{x} \in \Omega \subset \mathbb{R}^n, \ t \in (0,T),$$

$$\frac{\partial^{\beta}\varphi(t)}{\partial t^{\beta}} = \begin{cases} \dfrac{\partial^n \varphi(t)}{\partial t^n}, & \beta = n, \\ \dfrac{1}{\Gamma(n-\beta)} \int_0^t \frac{\varphi^{(n)}(\tau)}{(t-\tau)^{\beta-n+1}}\,\mathrm{d}\tau, & n-1 < \beta \le n. \end{cases}$$

$$u(\mathbf{x},t) = f(\mathbf{x},t), \quad \mathbf{x} \in \partial\Omega, \quad 0 < t < T,$$

and the final condition:

$$u(\mathbf{x},T) = g(\mathbf{x}), \quad \mathbf{x} \in \Omega,$$

Semi-Discrete Tikhonov Analysis

Operator equation

$$Af = g \quad A : \mathscr{X} \to \mathscr{Y};$$
$$Af(x) = \int_{\Omega} k(x,t) f(t) dt.$$

Semi-discrete Tikhonov regularization

$$\min\{\|Af - \bar{g}\|^2_{l_2(X)} + \varepsilon \|f\|^2_{\mathscr{X}} : \quad f \in \mathscr{X}\};$$
$$\min\{\|Af - \bar{g}^\delta\|^2_{l_2(X)} + \varepsilon \|f\|^2_{\mathscr{X}} : \quad f \in \mathscr{X}\}.$$

where X is a set of data sites $X = \{x_1, \ldots, x_N\}$; $\bar{g} = (g_1, \ldots, g_N)^T$ with $g_j = g(x_j)$ and $\bar{g}^\delta = (g_1^\delta, \ldots, g_N^\delta)^T$.

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RKHS with Radial Basis Function

Reproducing Kernel Hilbert Spaces (RKHS)

Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain. $H(\Omega) \subseteq C(\Omega)$ be a Hilbert space of continuous functions $f : \Omega \to \mathbb{R}$ and let H^* be its dual. We call $H(\Omega)$ is a reproducing kernel Hilbert space (RKHS) if there exists a unique kernel $\Phi : \Omega \times \Omega \to \mathbb{R}$ satisfying

1. $\Phi(\cdot, x) \in H(\Omega)$ for all $x \in \Omega$.
2. $f(x) = \langle f(\cdot), \Phi(\cdot, x) \rangle_H$ for all $x \in \Omega$ and $f \in H(\Omega)$.

RKHS Examples: $H^\tau(\Omega)$ with $\tau > d/2$. Ω can be extended to $\mathbb{R}^d$.

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Semi-Discrete Regularization

Lemma [Wahba 1990]

Let H be a RKHS with kernel Φ. $\{\lambda_j\}_{j=1}^N \in H^*$ with $\lambda_j = \delta_{x_j} \circ A$ are assumed to be linearly independent functionals over H. Then, the solution f^ε is given by

$$f^\varepsilon = \sum_{j=1}^N \alpha_j^\varepsilon \lambda_j^y \Phi(\cdot, y),$$

where the coefficient $\alpha^\varepsilon \in \mathscr{R}^N$ is the solution of linear system $(A_{\Phi,X} + \varepsilon I)\,\alpha^\varepsilon = \bar{g}$ with $(A_{\Phi,X})_{ij} = \lambda_i^x \lambda_j^y \Phi(x,y)$.

The Gramian matrix $A_{\Phi,X}$ is called *symmetric collocation matrix*, for instance, it consists of the entries

$$(A_{\Phi,X})_{ij} = \int_\Omega \int_\Omega k(x_i, s) k(x_j, t) \Phi(s,t)\, ds dt.$$

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RKHS with Radial Basis Functions

Radial basis functions

$\Phi(x,y) = \Phi(\|x-y\|_2)$ (Radial even functions). Let $\tau > d/2$, suppose the Fourier transform of an integrable function $\Phi : \mathbb{R}^d \to \mathbb{R}$ satisfies

$$c_1 \left(1+\|\omega\|_2^2\right)^{-\tau} \leq \widehat{\Phi}(\omega) \leq c_2 \left(1+\|\omega\|_2^2\right)^{-\tau}, \quad \omega \in \mathbb{R}^d,$$

with fixed constants $0 < c_1 \leq c_2$. Then, the kernel Φ is also a reproducing kernel of $H^\tau(\mathbb{R}^d)$ with the inner-product

$$(f,g) = \int_{\mathbb{R}^d} \frac{\widehat{f}(\omega)\overline{\widehat{g}(\omega)}}{\widehat{\Phi}(\omega)} d\omega,$$

which is equivalent to the usual inner product on $H^\tau(\mathbb{R}^d)$.

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RKHS with Radial Basis Functions

Sobolev norm equivalence

$\mathcal{N}_\Phi(\mathbb{R}^d)$: RKHS generated by the kernel Φ with the norm

$$\|f\|_{\mathcal{N}_\Phi(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} \frac{|\widehat{f}(\omega)|^2}{\widehat{\Phi}(\omega)} d\omega \right)^{1/2}.$$

It is straightforward to know $\mathcal{N}_\Phi(\mathbb{R}^d)$ is norm-equivalent to the Sobolev space $H^\tau(\mathbb{R}^d)$, i.e.

$$c_1^{1/2} \|f\|_{\mathcal{N}_\Phi(\mathbb{R}^d)} \leq \|f\|_{H^\tau(\mathbb{R}^d)} \leq c_2^{1/2} \|f\|_{\mathcal{N}_\Phi(\mathbb{R}^d)}.$$

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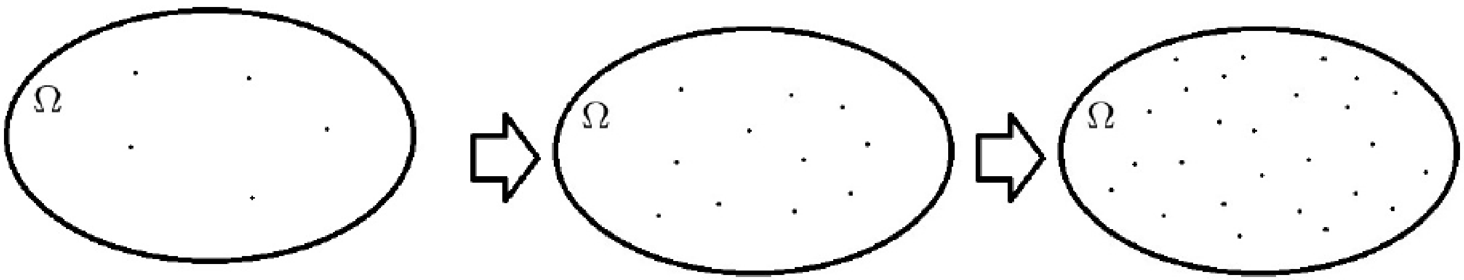
Scaled Radial Basis Functions

$X_1, X_2, \cdots$ with $X_k = \left\{x_1^{(k)}, x_2^{(k)}, \cdots, x_{N_k}^{(k)}\right\}$.



$\Phi : \mathbb{R}^d \to \mathbb{R}$ with a compact support in the unit ball $B(0,1)$. At each level, there is a scaling parameter η_k such that the kernel will be given by the scaled version,

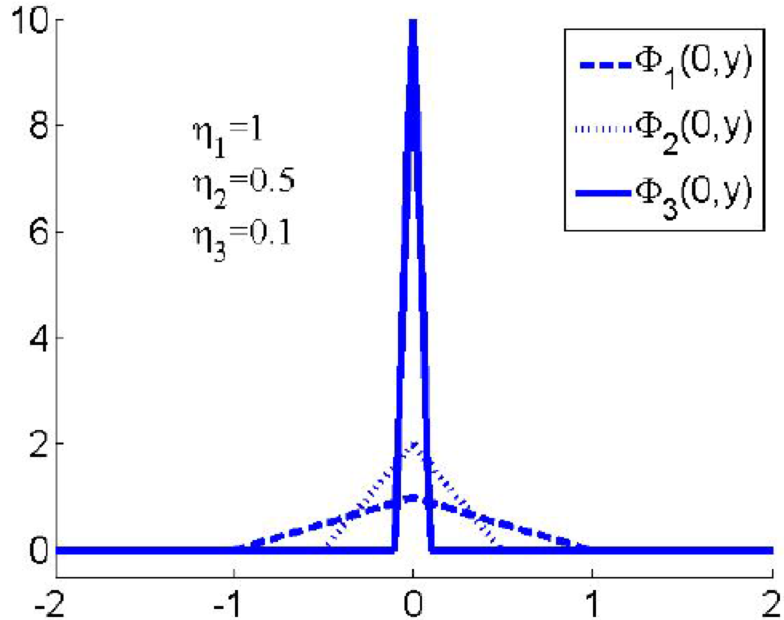
$$\Phi_k(x,y) = \eta_k^{-d} \Phi\left(\frac{x-y}{\eta_k}\right).$$

Wendland H. (2010): *Multiscale analysis in Sobolev spaces on bounded domains*, Numer. Math. 116, 493–517.

Scaled RBFs - Example

Choose a kernel function $\Phi(x,y) = \max(0, 1-|x-y|)$ for $H^1(\mathbb{R})$, defined the scaled version

$$\Phi_k(x,y) = \eta_k^{-1} \max\left(0, 1 - \frac{|x-y|}{\eta_k}\right) \quad \text{with} \quad \eta_k = 1, 0.5, 0.1.$$



Schaback R. (2011): *The missing Wendland functions*, Adv. Comput. Math. 34, 67–81.

Scaled Radial Basis Functions

Scaling Lemma [Wendland 2010]

For every fixed k with a scaling parameter $\eta_k \in (0,1)$, we have the following norm inequality

$$c_1^{1/2}\|f\|_{\Phi_k} \leq \|f\|_{H^\tau(\mathbb{R}^d)} \leq c_2^{1/2}\eta_k^{-\tau}\|f\|_{\Phi_k},$$

where the constants $0 < c_1 \leq c_2$ are the same as those in single-level radial basis functions.

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RKHS for Multiscale Reconstruction

Zhong M, Lu S and Cheng J (2012)

Multiscale analysis for ill-posed problems with semi-discrete Tikhonov regularization

Inverse Problems, 28 065019 (19pp).

Multiscale Reconstruction Algorithm

Algorithm 1 Multiscale reconstructed solution algorithm (Iterative Tikhonov form)

Input: Righthand side $\bar{g}$, Number of levels n.
Output: Reconstructed solution $f_n \in V_n = W_1 + \cdots + W_n$.

1: Set $f_0 = s_0 = 0$, $e_0 = g$.
2: **For** $k = 1, 2, \ldots, n$
3: Determine s_k^ε satisfying $\min \left(\|e_{k-1} - As\|^2_{l^2(X_k)} + \varepsilon_k \|s\|^2_{\Phi_k} : s \in H^\tau(\mathbb{R}^d) \right)$.
4: Set $f_k^\varepsilon = f_{k-1}^\varepsilon + s_k^\varepsilon$.
5: Set $e_k = e_{k-1} - As_k^\varepsilon$.
6: **End**.

$W_k = \mathrm{span}\{\lambda^{y,(k)} \Phi_k(\cdot, y), y \in X_k\}$ where $\lambda^{y,(k)}$ are linearly independent functionals at level k with

$$\lambda_j^{(k)}(f) = Af(x_j^{(k)}), \quad 1 \le j \le N_k, \quad x_j \in X_k.$$

Further more, we have the direct sums

$$\bigoplus_{k=1}^{\infty} W_k = L^2(\Omega).$$

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Extension of the RKHS-RBFs

Extension operator [Stein 1971]

Suppose $\Omega \subseteq \mathbb{R}^d$ is an open domain with a Lipschitz boundary. Let $\tau \geq 0$. Then, there exists a linear continuous operator $E : H^\tau(\Omega) \to H^\tau(\mathbb{R}^d)$, such that for all $f \in H^\tau(\Omega)$,

1. $Ef|_\Omega = f$,
2. $\|f\|_{H^\tau(\Omega)} \leq \|Ef\|_{H^\tau(\mathbb{R}^d)} \leq C_\tau \|f\|_{H^\tau(\Omega)}$,

where C_τ is a constant only depending on τ.

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Assumptions on the Ill-posed Problem

1. $\Omega \subseteq \mathbb{R}^d$ is a bounded domain with a Lipschitz boundary.
2. The considered problem is moderately ill-posed of degree $\alpha > 0$ in the sense that there exists two constants $c_{22} \geq c_{11} > 0$ such that

$$c_{11}\|f\|_{H^\theta(\Omega)} \leq \|Af\|_{H^{\theta+\alpha}(\Omega)} \leq c_{22}\|f\|_{H^\theta(\Omega)}, \quad \forall \theta \in \mathbb{R}.$$

3. The problem is solvable in $H^\tau(\Omega)$ for $\tau > d/2$. This means that $g \in \mathscr{R}(A) \subseteq H^{\tau+\alpha}(\Omega)$, and the exact reconstructed solution f^* exists. Furthermore, the local exact reconstructed solution s_k^* exists at level k.
4. The absolute accuracy of measurement is δ_k at level k, i.e.,

$$|g^\delta(x_j^{(k)}) - g(x_j^{(k)})| \leq \delta_k, \quad \text{for } j = 1, \cdots, N_k.$$

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Notations on the Reconstruction Algorithm

Recall on each scale we find $s_k^\varepsilon \in H^\tau(\mathbb{R}^d)$ s.t.

$$\min\left(\|e_{k-1} - As\|^2_{l^2(X_k)} + \varepsilon_k \|s\|^2_{\Phi_k} : s \in H^\tau(\mathbb{R}^d)\right).$$

We define following elements

- $s_k^\varepsilon \in H^\tau(\mathbb{R}^d)$: the local reconstructed solution at level k,
- $f_k^\varepsilon = s_1^\varepsilon + \cdots + s_k^\varepsilon$: the reconstructed solution,
- $e_k \in H^{\tau+\alpha}(\Omega)$: the residual at level k with $e_k = e_{k-1} - As_k^\varepsilon$,
- $s_k^* \in H^\tau(\Omega)$: the local exact solution at level k with $As_k^* = e_{k-1}$,
- ε_k: the Tikhonov regularization parameter at each level k.

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Convergence: Noisy free data case

Lemma 1 (Estimation on fixed level)

Under above Assumption and notations, we have the following estimations:

$$\|e_k\|_{l^2(X_k)} \leq \sqrt{\varepsilon_k}\|Es_k^*\|_{\Phi_k},$$

$$\|s_k^\varepsilon\|_{\Phi_k} \leq \|Es_k^*\|_{\Phi_k},$$

$$\|e_k\|_{H^{\tau+\alpha}(\Omega)} \leq 2c_2^{1/2}c_{22}\eta_k^{-\tau}\|Es_k^*\|_{\Phi_k},$$

where $E : H^\tau(\Omega) \to H^\tau(\mathbb{R}^d)$ is the extension operator.

Key information

Notice the fact that $As_k^* = e_{k-1}$. The information at each level is thus connected.

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Convergence: Noisy free data case

Lemma 2 (Monotone decreasing)

Under the proposed assumption, assume that there exists a constant $T > 0$ such that $T/\mu \le \nu \le 1/h_1$ with $\eta_k = \nu h_k$. Then, we have the following estimation

$$\|Es_k^*\|_{\Phi_k} \le \frac{2^{\tau/2} C_\tau}{c_1^{1/2} c_{11}} \left[C_d \left(2 c_2^{1/2} c_{22} T^{-\tau} \mu^\tau + \sqrt{\varepsilon_{k-1}} h_{k-1}^{d/2-\alpha} \right) + 2 c_{22} c_2^{1/2} \mu^\tau \right] \|Es_{k-1}^*\|_{\Phi_{k-1}}.$$

In particular, if we choose the regularization parameter ε_k at each level k satisfying

$$\sqrt{\varepsilon_k} h_k^{d/2-\alpha} \le \kappa \left(\frac{h_k}{\eta_k} \right)^\tau$$

with a fixed constant $\kappa > 0$. Then, there exists a constant C_1 such that, if $p = C_1 \mu^\tau$ smaller than 1, $\|Es_k^*\|_{\Phi_k}$ will sequentially decrease, i.e.

$$\|Es_k^*\|_{\Phi_k} \le p \|Es_{k-1}^*\|_{\Phi_{k-1}}.$$

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Convergence: Noisy free data case

Theorem 1 (Convergence result)

Under the proposed assumption, choose the regularization parameter ε_k at each level k appropriately. Then there exists, at the final level n, the following error estimation in the Hilbert space $L^2(\Omega)$

$$\|f^* - f_n^\varepsilon\|_{L^2(\Omega)} \leq C p^n \|f^*\|_{H^\tau(\Omega)},$$

where the constant $C = \frac{C_d\left(2c_2^{1/2}c_{22}+\kappa\right)T^{-\tau}C_\tau}{c_1^{1/2}c_{11}C_1}$. Thus, the multiscale reconstructed solution f_n^ε converges linearly to the exact solution f^* under the $L^2(\Omega)$ norm if $p = C_1\mu^\tau$ smaller than 1.

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Error Bound: Noisy data case

Parameter choice rule

we establish an *a posteriori* parameter choice rule based on the classic Morozov discrepancy principle, which assumes that the regularization parameter ε_k^{dis} satisfies

$$\bar{c}_1 \sqrt{N_k}\delta_k \leq \|e_{k-1}^{\delta} - As(\varepsilon_k^{dis})\|_{l^2(X_k)} \leq \bar{c}_2 \sqrt{N_k}\delta_k$$

where the two constants fulfill $2 < \bar{c}_1 \leq \bar{c}_2$ and $s(\varepsilon_k^{dis})$ is the minimizer of the functional

$$\min\left(\|e_{k-1}^{\delta} - As\|_{l^2(X_k)}^2 + \varepsilon_k \|s\|_{\Phi_k}^2, \quad s \in H^{\tau}(\mathbb{R}^d)\right).$$

with $\varepsilon_k = \varepsilon_k^{dis}$.

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Error Bound: Noisy data case

Theorem 2 (Error estimation for noisy data)

Under the proposed assumption, the regularization parameter ε_k at each level k is chosen to satisfy the discrepancy principle. Then there exists, at the final level n, the following error estimation in the Hilbert space $L^2(\Omega)$

$$\|f^* - f_n^{\varepsilon,\delta}\|_{L^2(\Omega)} \leq \overline{C}_1 \widetilde{p}^n \|f^*\|_{H^\tau(\Omega)} + \overline{C}_2 \sum_{j=1}^{n-1} \left(\widetilde{p}^j h_{n-j}^{d/2-\alpha} \sqrt{N_{n-j}} \delta_{n-j} \right) + \overline{C}_3 h_n^{d/2-\alpha} \sqrt{N_n} \delta_n,$$

with appropriate constants.

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Numerical Example

Operator equation [KLW 2009]

This example involves a simple Volterra operator

$$Af(x) = \int_0^x f(t)dt$$

defined on $\Omega = [0,2]$, which is moderately ill-posed problem with degree $\alpha = 1$. The given right hand side

$$g(x) = \begin{cases} x - 0.5x^2 & x \leq 1 \\ 0.5 & x > 1 \end{cases}$$

yields the exact solution

$$f^*(x) = \max(0, 1-x) \in H^1(\Omega).$$

In the multiscale reconstruction scheme, the original reproducing kernel

$$\Phi(x,y) = \max(0, 1-|x-y|)$$

is chosen.

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Accuracy: Noisy free data case

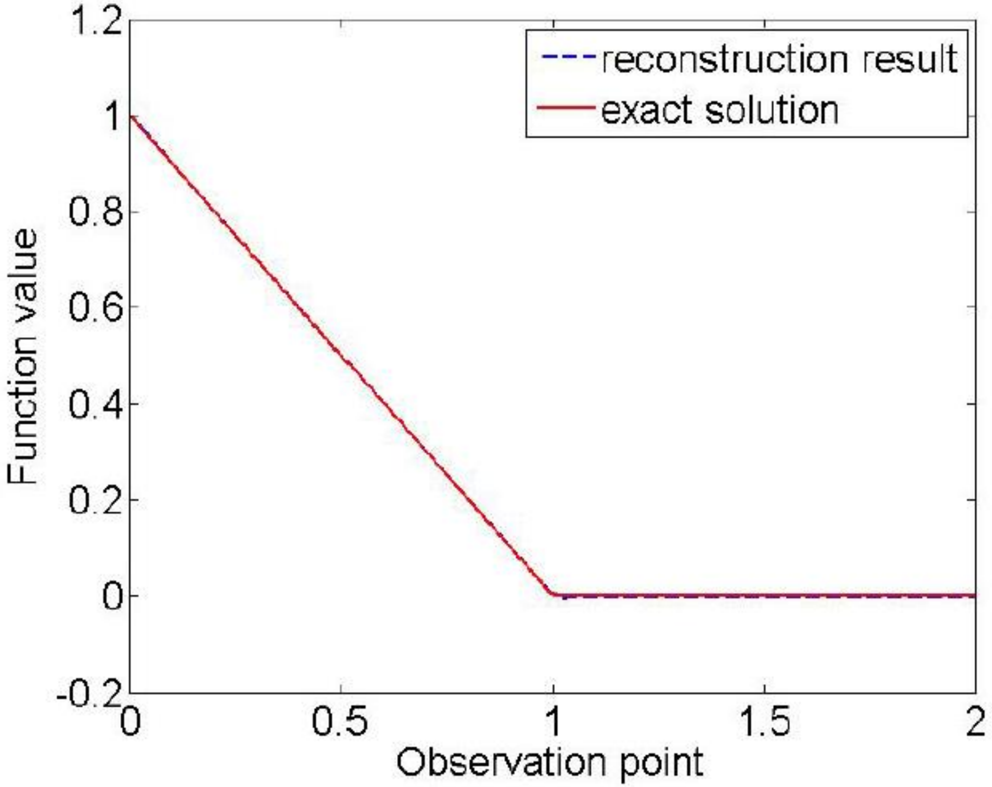

Figure: The reconstructed solution for noise-free data at the sixth level with $\varepsilon_k = 0.01 h_k$.

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Accuracy: Noisy free data case

Level	N_k	h_k	$l^2(X_k)$ error	$l^\infty(X_k)$ error
1	12	0.17	0.521327	0.122327
2	25	0.08	0.160746	0.054597
3	50	0.04	0.060095	0.026490
4	100	0.02	0.023453	0.011096
5	200	0.01	0.011395	0.003396
6	400	0.005	0.008442	0.001832

Table: Quantity information about the reconstructed solution at each level k for noise-free data. N_k represents the number of discretization points at level k. $l^2(X_k)$ error and $l^\infty(X_k)$ error represent the corresponding error between the reconstructed solution f_n^ε and the exact solution f^*.

Accuracy: Noisy data case

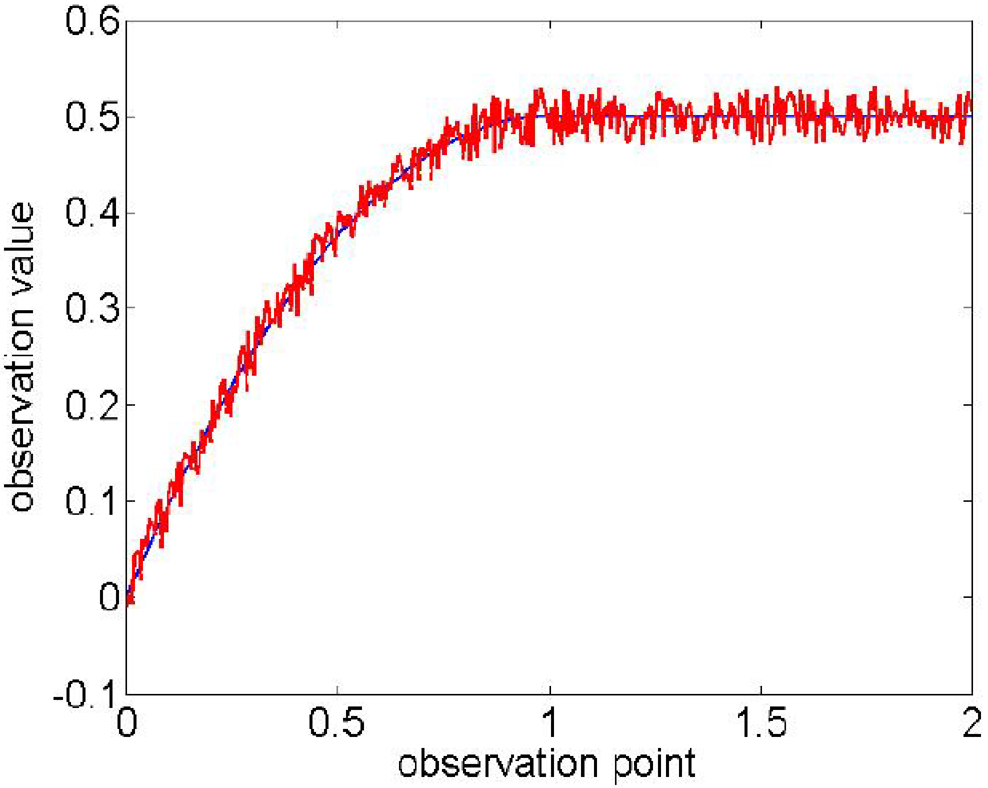

Figure: The exact observation data and the noisy one with a uniform noise level $\delta = 0.03$ at level 6.

Accuracy: Noisy data case

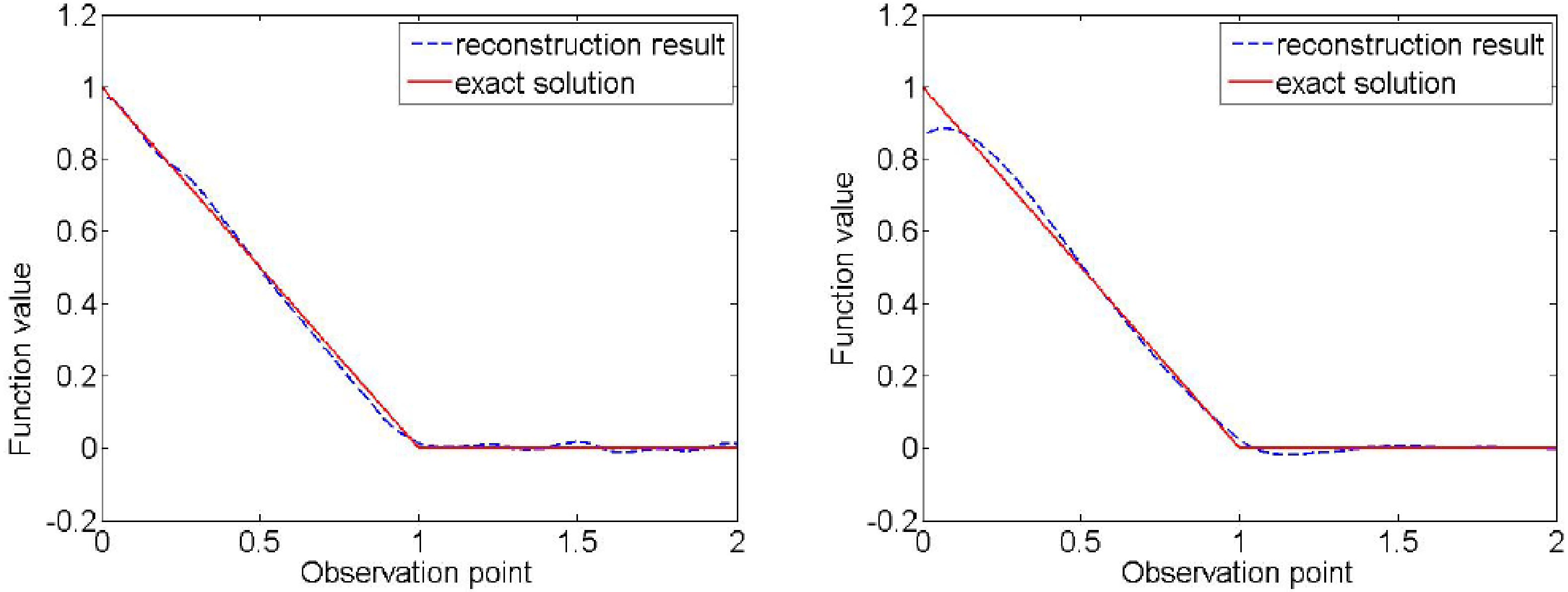

Figure: Left: The final reconstructed solution compares with the exact solution with l^2 and l^∞ error 0.292367 and 0.027605. Right: The single level reconstructed solution with 400 data value of numerical example in [KLW 2009] with l^2 and l^∞ error 0.502820 and 0.132031, respectively.

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Multiscale Support Vector Approach

Consider the following SVA minimization problem

$$\min_{f \in H} \left(\sum_{j=1}^{n} |g_j^\delta - (Af)(x_j)|_\epsilon + \gamma \|f\|_H^2 \right)$$

Algorithm 1 Multiscale Support Vector Approach

Input: pre-specified iterative step n, righthand side $g^\delta|_{X_1}, \cdots, g^\delta|_{X_n}$,

Output: n-th approximate solution $f_n^{\epsilon,\gamma,\delta} = s_1^{\epsilon,\gamma,\delta} + \cdots + s_n^{\epsilon,\gamma,\delta}$.

1: Set $f_0^{\epsilon,\gamma,\delta} = s_0^{\epsilon,\gamma,\delta} = 0$.
2: **For** $k = 1, 2, \ldots, n$
3: $\quad e_{k-1}^\delta = g^\delta - A^{(k)}(s_1^{\epsilon,\gamma,\delta} + \cdots + s_{k-1}^{\epsilon,\gamma,\delta}) = g^\delta - A^{(k)} f_{k-1}^{\epsilon,\gamma,\delta}$;
4: $\quad s_k^{\epsilon,\gamma,\delta} = \arg\min_{s \in H^\tau(\mathbb{R}^d)} \left(\sum_{j=1}^{N_k} |e_{k-1}^\delta(x_{k,j}) - A^{(k)} s(x_{k,j})|_{\epsilon_k} + \gamma_k \|s\|_{\Phi_k}^2 \right)$;
5: $\quad$ Set $f_k^{\epsilon,\gamma,\delta} = f_{k-1}^{\epsilon,\gamma,\delta} + s_k^{\epsilon,\gamma,\delta}$;
6: **End.**

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Assumptions

[A1] $\Omega \subseteq \mathbb{R}^d$ *is a bounded domain with a Lipschitz boundary ;*

[A2] The mesh norm of the sampling grid satisfies $h_{k+1} = \mu h_k$ *with a fixed* $\mu \in (0, 1)$ *;*

[A3] The considered problem is solvable in $H^\tau(\Omega)$ *with* $\tau > d/2$*, which guarantees the existence of an exact solution* $f^* \in H^\tau(\Omega)$*, set* $\varrho := \|f^*\|_{H^\tau(\Omega)}$*;*

[A4] The considered problem is moderately ill-posed with a degree $\alpha > 0$*, i.e., there exists constants* $c_4 \geq c_3 > 0$ *such that:*

$$c_3\|f\|_{H^\theta(\Omega)} \leq \|Af\|_{H^{\theta+\alpha}(\Omega)} \leq c_4\|f\|_{H^\theta(\Omega)}, \quad \text{for all } \theta \in \mathbb{R}\,;$$

[A5] There exists a constant $K = K(A, \Omega) > 0$*, such that for all* $f \in H^\tau(\Omega)$ *,*

$$\|(A^{(k)} - A)f\|_{L^\infty(\Omega)} \leq K h_k^r \|f\|_{H^\tau(\Omega)}\,,$$

where the convergence rate $r > 0$ *depends on the specified discretization scheme. In practice, the convergence order* r *can be chosen as* $r > \alpha$*;*

[A6] The scaling parameter η_k *is chosen in the form of* $\eta_k = \nu h_k^\beta$ *with fixed constants* $\nu > 0$ *and a positive constant* $\beta = \min\{1, \frac{r-\alpha}{\tau}\}$ *where constants* τ*,* α *and* r *comes from [A3]-[A5] respectively ;*

[A7] The noise level at the k-th sampling grid is measured by $|g^\delta - g|_{l^\infty(X_k)} \leq \delta_k$ *.*

MSVA-RKHS for Ill-posed problems

Theorem 1 *Let Assumption 1 [A1]-[A6] and the parameter choices in Lemma 6 hold. Furthermore, assume* $p + C_3 h_1^{r-\alpha} \leq 1$ *and re-scale the problem in the sense that* $\varrho \leq \frac{c_1^{1/2}}{C_\tau}$. *The error between the exact solution* f^* *and the reconstructed solution* $f_n^{\epsilon,\gamma}$ *can be estimated as*

$$\|f^* - f_n^{\epsilon,\gamma}\|_{L^2(\Omega)} \leq \frac{C_\tau}{c_1^{1/2}} \hat{C} p^n \|f^*\|_{H^\tau(\Omega)} + \widehat{C} C_3 \sum_{j=1}^{n-1} p^j h_{n-j}^{r-\alpha} + \frac{2C_d}{c_3} K h_n^{r-\alpha} \|f^*\|_{H^\tau(\Omega)} \tag{20}$$

where the constant $\widehat{C}$ *takes the form*

$$\widehat{C} = \frac{C_d}{c_3(C_1 + C_2)} \cdot \left(\frac{2c_2^{1/2}(c_4 + K) + \kappa}{T^\tau} \right).$$

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Error Estimate: Noisy data case

Lemma 11 *Let Assumption 1 hold. At each level k, choose the cut-off parameter* $\epsilon_k = \delta_k + K\varrho h_k^r$ *and choose* γ_k *satisfying (17). Assuming* $T/\mu^\beta \le \nu \le 1/h_1$ *with appropriate constants T and* μ*, there exists a semi-monotonic inequality*

$$\|Es_{k+1}^{\delta,*}\|_{\Phi_{k+1}} \le C_1\mu^{\beta\tau}\|Es_k^{\delta,*}\|_{\Phi_k} + C_2\mu^{\beta\tau}\|Es_k^{\delta,*}\|_{\Phi_k}^2 + C_3 h_k^{r-\alpha} + C_4 h_k^{-\alpha}\delta_k\,,$$

Theorem 4 *Let Assumption 1 and the parameter choices in Lemma 11 hold. Furthermore, assume* $p + C_3 h_1^{r-\alpha} + C_4 h_k^{-\alpha}\delta_k \le 1$ *and re-scale the problem in the sense that* $\varrho \le \frac{c_1^{1/2}}{C_\tau}$. *Then, the error between the exact solution* f^* *and reconstructed one* $f_n^{\epsilon,\gamma,\delta}$ *can be estimated as*

$$\|f^* - f_n^{\epsilon,\gamma,\delta}\|_{L^2(\Omega)} \le \frac{C_\tau}{c_1^{1/2}}\hat{C}p^n\|f^*\|_{H^\tau(\Omega)} + \hat{C}C_3\sum_{j=1}^{n-1} p^j h_{n-j}^{r-\alpha} + \frac{2C_d}{c_3}Kh_n^{r-\alpha}\|f^*\|_{H^\tau(\Omega)}$$
$$+ \hat{C}C_4\sum_{j=1}^{n-1} p^j h_{n-j}^{-\alpha}\delta_{n-j} + \frac{2C_d}{c_3}h_n^{-\alpha}\delta_n \qquad (27)$$

with the same constant $\hat{C}$ *from Theorem 1.*

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