Exponential Fourier Reconstruction of Piece-wise Smooth Functions

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Function Reconstruction from Fourier Data

- Assume $f : \mathbb{R} \to \mathbb{R}$ is a piece-wise smooth function:
  - supported on $[-1, 1]$
  - a few known jump discontinuities $\xi_1, \xi_2, \dots, \xi_p$

- We are given its Fourier data

$$\widehat{f}(\lambda_k) = \int_{-1}^{1} f(x) e^{-\pi i \lambda_k x} dx, \quad -N \leq k \leq N.$$

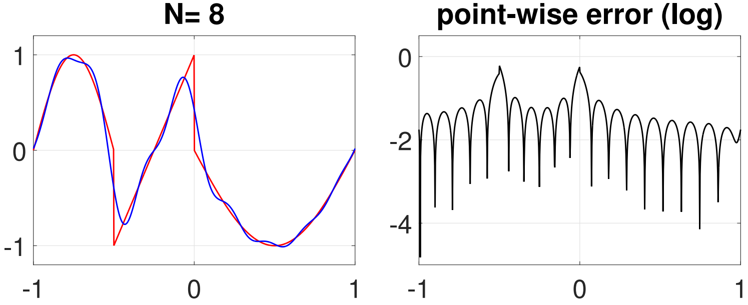
  - Uniform: $\lambda_k = k$
  - Jittered: $\lambda_k = k + \delta_k$, where δ_k is a small number

- We will try to approximate the unknown function f.

Uniform Fourier Reconstruction

- Fourier partial sum: $(S_N f)(x) = 1/2 \sum_{|j| \le N} \hat{f}(j) e^{\pi i j x}$.
- Filter/Mollifier [Fejér 1900; Vandeven 1991;. . .]
  - Filter: $\sum_{|j| \le N} \sigma_j \hat{f}(j) e^{\pi i j x}$
  - Mollifier: $S_N(\varphi * f)$
  - Adaptive [Tadmor, Tannner, 2001]
- Spectral re-projection [Gottlieb, Shu, Solomonoff, Vandeven, 1992]:
  - $P_M^{\mu} S_N f$: projection onto a subspace of Gegenbauer polynomials.
- Inverse polynomial reconstruction method [Jung and Shizgal, 2004]
- Generalized sampling [Adcock, Hansen, 2015]
  - $(P_M S_N)^{-1} P_M S_N f$

Fourier Partial Sum/Gibbs Phenomenon



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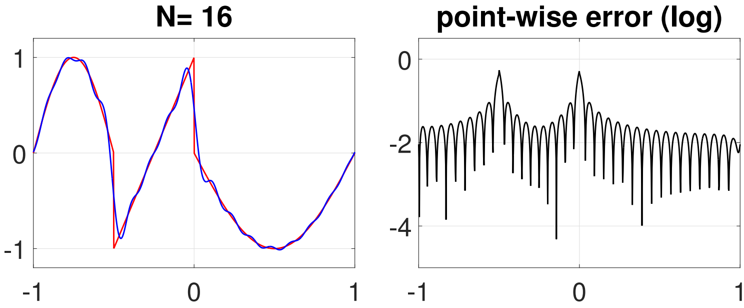

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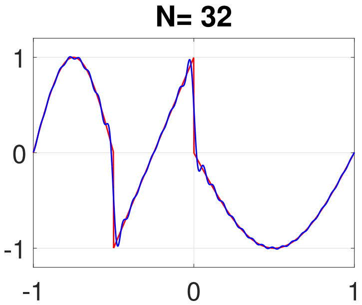

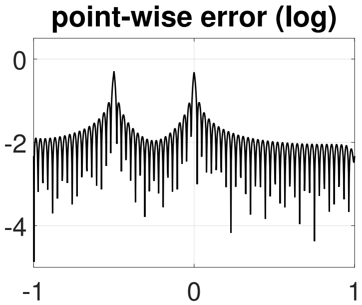

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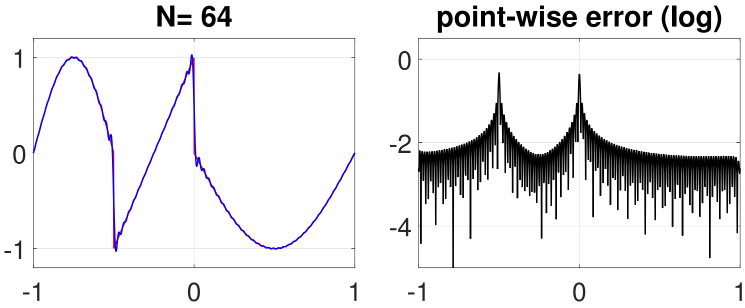

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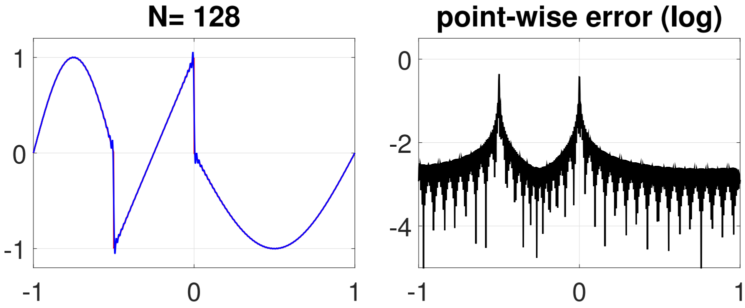

Filter/Mollifier

- Error decomposition:

$$\|f - S_N(\varphi * f)\| \le \underbrace{\boxed{\|f - \varphi * f\|}}_{\substack{\text{Regularization error} \\ \text{need } \varphi \approx \delta}} + \underbrace{\boxed{\|\varphi * f - S_N(\varphi * f)\|}}_{\substack{\text{Truncation error} \\ \text{need } \varphi \text{ smooth}}}$$

- Examples of mollifiers
  - Compactly supported $\quad \varphi(x) = e^{-\frac{cx^2}{1-x^2}} \mathbb{1}_{[-1,1]}(x)$
  - Hermite Distributed Approximating Functionals (HDAF)

$$\varphi(x) = e^{-x^2/2} \sum_{n=0}^{p} \frac{(-4)^{-n}}{n!} H_{2n}\left(\frac{x}{\sqrt{2}}\right)$$

- Adaptive mollifiers
  - $\varphi_\theta(x) = \frac{1}{\theta}\varphi\left(\frac{x}{\theta}\right)$
  - θ depends on N and $d(x)$ distance to the edges

Filter/Mollifier: Convergence

- Convergence
  - Compactly supported: root exponential up to the edges $e^{-c\sqrt{Nd(x)}}$
  - HDAF: exponential up to the edges $e^{-cNd(x)}$
- Numerics

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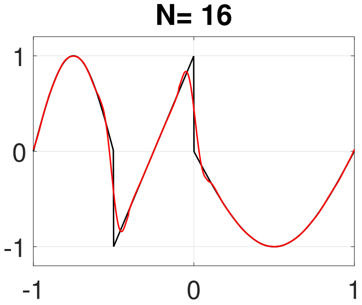

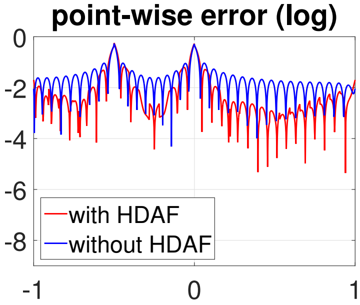

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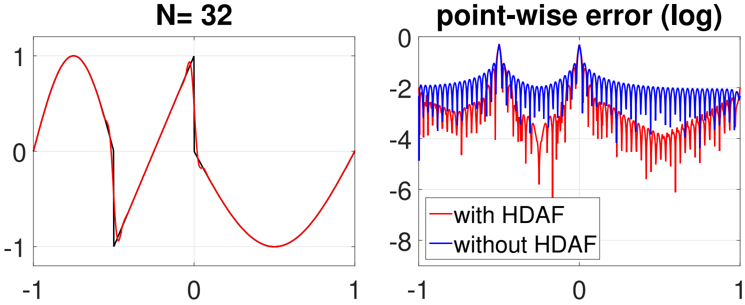

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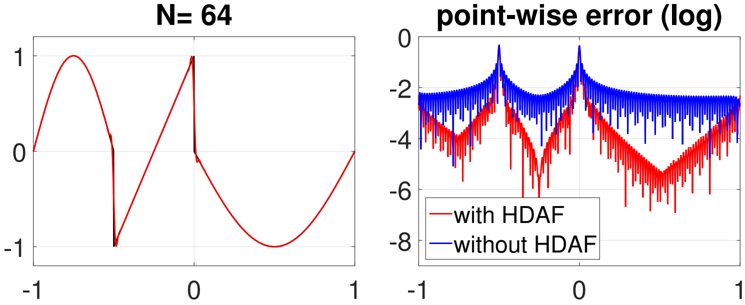

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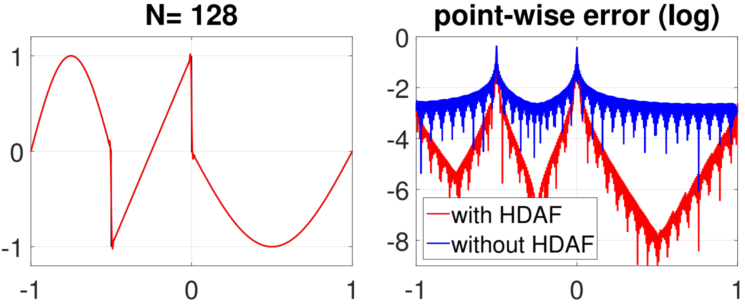

Spectral Re-projection

- Error decomposition

$$\|f - P_M^{\mu} S_N f\| \le \underbrace{\boxed{\|f - P_M^{\mu} f\|}}_{\text{Regularization error}} + \underbrace{\boxed{\|P_M^{\mu}(f - S_N f)\|}}_{\text{Truncation error}}$$

projection onto Gegenbauer polynomials up to degree M w.r.t weight $(1-x^2)^{\mu-1/2}$

$$\|P_M^{\mu}(f - S_N f)\| \propto \frac{\Gamma(\mu)(M+\mu)\Gamma(M+2\mu)}{(M-1)!\Gamma(2\mu)} N^{-(\mu-1)}$$

- Uniform exponential convergence.
  - need $\mu = \alpha N$ and $M = \beta N$
  - α and β need carefully chosen to avoid a Runge-type phenomenon
  - might be computationally inconvenient and unstable
  - does not work on non-uniform Fourier measurements

Inverse Polynomial Reconstruction Method

- Given Fourier measurements $\hat{f}(j), -N \leq j \leq N$
- Find the polynomial g with degree $2N$ such that

$$\hat{g}(j) = \hat{f}(j), \quad -N \leq j \leq N$$

- The interpolation matrix has an exponential growth condition number.

Generalized Sampling

- Generalized sampling: $(P_M S_N)^{-1} P_M S_N f$
- Root exponential convergence $\mathrm{e}^{-c\sqrt{N}}$
  - $M = c\sqrt{N}$

projection onto Legendre polynomials up to degree M

- Need to solve a least-square problem

$$\min_{g \in \mathcal{H}_M} \sum_{|j| \leq N} \left[\hat{g}(j) - \hat{f}(j) \right]^2,$$

where $\mathcal{H}_M$ is the space of piece-wise polynomials.

- Also works on non-uniform Fourier measurements

Goal

Find a Fourier reconstruction method:

- has uniform exponential convergence
- works with both uniform and non-uniform Fourier measurements
- is easy to compute

A Hybrid Method: Mollifier + Extrapolation

- Given uniform Fourier measurements
  - Use $S_N(\varphi * f)$ to approximate f away from the edge
  - Extrapolation based on the above approximated function values
- A "stable" extrapolation [Demanet and Townsend, 2016]
  - Assume f is analytic in a Bernstein ellipse with parameter $\rho > 1$
  - Given the samples of f on $[-1, 1]$ with perturbation level ϵ
  - Use least squares polynomial approximation with $M \propto \sqrt{N}$
  - It has error $\propto \epsilon^{r(x)}$ for $x \in \left[1, \frac{\rho + \rho^{-1}}{2}\right)$, where

$$r(x) = \log_{1/\rho} \frac{x + \sqrt{x^2 - 1}}{\rho}, \quad \text{decreasing from 1}$$

Convergence: Mollifier + Extrapolation

- Assume $\xi_1, \xi_2, \ldots, \xi_p$ are jumps.
- On each $[\xi_j, \xi_{j+1}]$, choose $[a_j, b_j] \subset [\xi_j, \xi_{j+1}]$.
- In the interval $[a_j, b_j]$, use the mollifier method $S_N(\varphi * f)$.
- Extrapolation on $[\xi_j, a_j] \cup [b_j, \xi_{j+1}]$: $\quad \epsilon \sim \mathrm{e}^{-\mathrm{rdN}}$.
- Convergence
  - The error on $[a_j, b_j]$: $\epsilon_j \sim \mathrm{e}^{-\mathrm{d_j N}}$, where d_j is decreasing on $w_j = b_j - a_j$
  - The extrapolation error $\propto \epsilon^{r_j}$, where r_j is increasing on w_j
  - Need to choose an optimal w_j to obtain a uniform exponential convergence.

Numerics: Mollifier + Extrapolation

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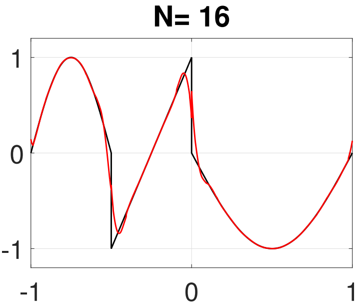

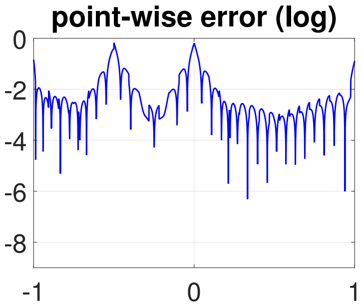

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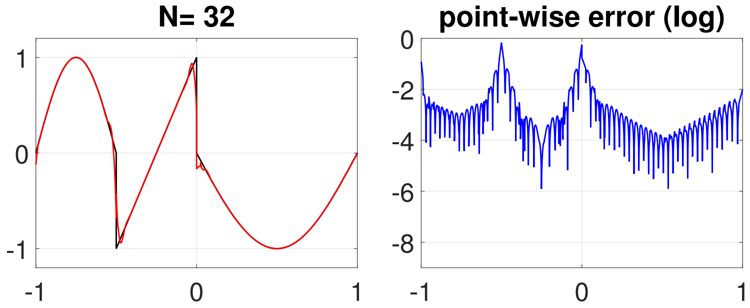

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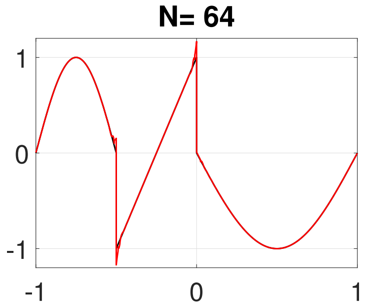

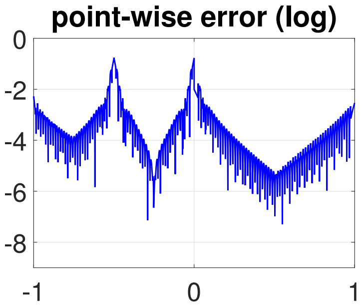

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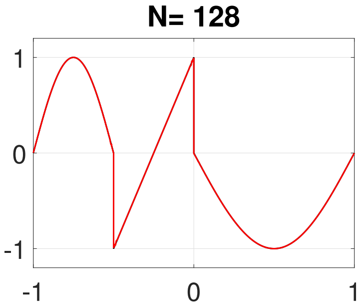

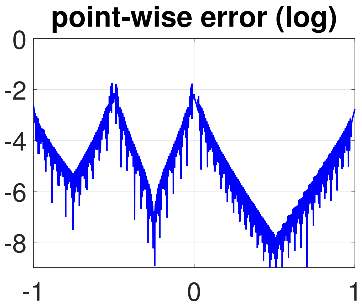

Non-uniform Fourier Reconstruction

- Consider $\hat{f}(\lambda_k) = \int_{-1}^{1} f(x) \mathrm{e}^{-\pi \mathrm{i} \lambda_k \mathrm{x}} \mathrm{dx}$, where $\{\mathrm{e}^{-\pi \mathrm{i} \lambda_k \mathrm{x}}\}$ gives a frame.
- Approximate $\varphi * f$ from $\{\hat{f}(\lambda_k) : -N \leq k \leq N\}$.
  - Need to approximate the dual frame.
  - Admissible frame approach [Gelb, Song, 2013]

    $$\|f - T_N f\| \leq c\|f - S_N f\|$$

    where $T_N f$ is an approximation based on $\{\hat{f}(\lambda_k) : -N \leq k \leq N\}$.
  - Use $T_N(\varphi * f)$ as an approximation on the interval away from the edges

    $$\|\varphi * f - T_N(\varphi * f)\| \leq c\|\varphi * f - S_N(\varphi * f)\|.$$

- Apply the same extrapolation on the interval around the edges.

Numerics: Non-uniform Fourier Measurements

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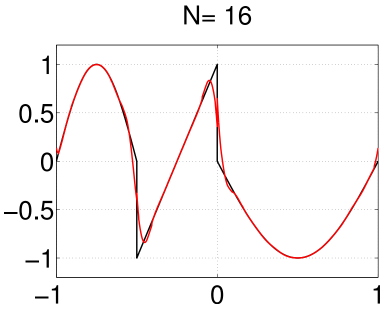

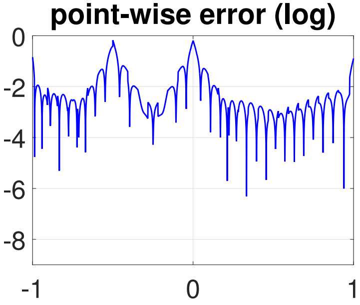

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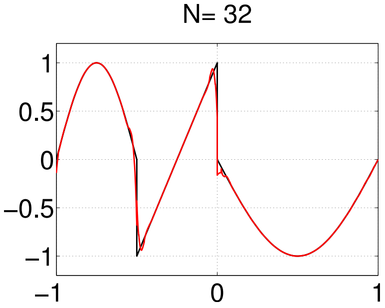

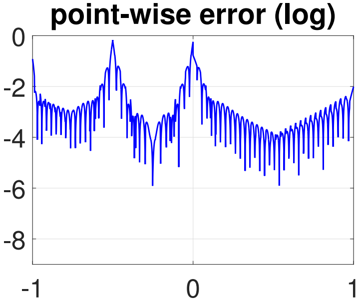

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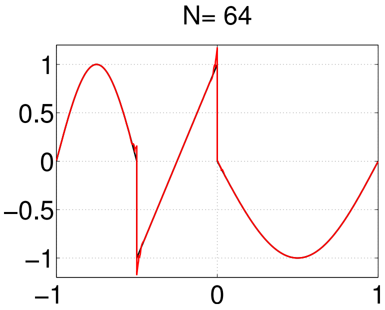

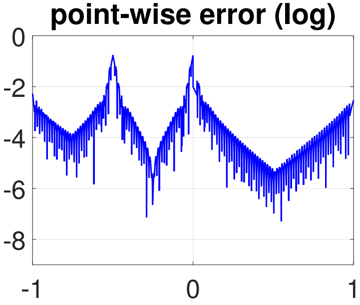

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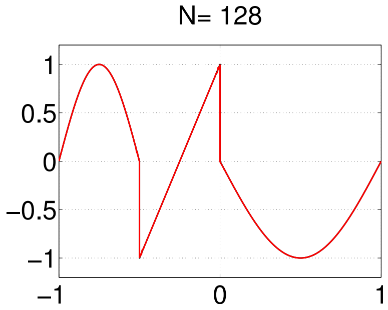

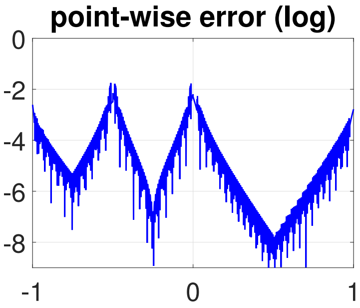

Discussion

- Summary
  - The mollifier method could obtain exponential convergence up to the edges for both uniform and non-uniform Fourier measurements.
  - A stable extrapolation could improve the accuracy close to the edges.
- Questions
  - Other extrapolation methods?
  - 2D mollifier method
  - 2D extrapolation method

Thank you !