Positive Definite Multi-Kernels for Scattered Data Interpolations

Qi Ye

School of Mathematical Sciences, South China Normal University, Guangzhou, Guangdong, China, 510631

Abstract

In this article, we use the knowledge of positive definite tensors to develop a concept of positive definite multi-kernels to construct the kernel-based interpolants of scattered data. By the theorems of reproducing kernel Banach spaces, the optimal recoveries and error analysis of the kernel-based interpolants are shown for a special class of strictly positive definite multi-kernels.

Keywords: Kernel-based approximation method, scattered data interpolation, positive definite multi-kernel, reproducing kernel Banach space, positive definite tensor, multi-linear system
2010 MSC: 65D05, 46E22.

1. Introduction

In many areas of practical applications, we often face a problem of reconstructing an unknown function f from the scattered data. The scattered data consist of high-dimension data points $\{\boldsymbol{x}_i\}_{i=1}^n$ and data values $\{y_i\}_{i=1}^n$ such that $y_i = f(\boldsymbol{x}_i)$ for all $i = 1, 2, \ldots, n$. The reconstruction is to find an estimate function s to approximate f. Generally, s is sought to interpolate the scattered data, that is, $s(\boldsymbol{x}_i) = y_i$ for all $i = 1, 2, \ldots, n$. It is well-known that the kernel-based approximation method is a fundamental approach of scattered data interpolations. The classical kernel-based approximation method is mainly dependent of the positive definite kernels. Here, we develop a concept of positive definite multi-kernels in Definition 3.1 which is a generalization of positive definite kernels. The positive definite multi-kernels can be also used to reconstruct f from the scattered data. For examples, we compare

Email address: `yeqi@m.scnu.edu.cn` (Qi Ye)

the interpolations by a positive definite kernel $K_2 : \otimes_{k=1}^{2}\mathbb{R}^d \to \mathbb{R}$ and a positive definite multi-kernel $K_4 : \otimes_{k=1}^{4}\mathbb{R}^d \to \mathbb{R}$. The classical interpolant s_2 is composed of a kernel basis

$$K_2(\cdot, \boldsymbol{x}_i), \quad \text{for } i = 1, 2, \ldots, n,$$

that is,

$$s_2(\boldsymbol{x}) := \sum_{i=1}^{n} c_i K_2(\boldsymbol{x}, \boldsymbol{x}_i), \quad \text{for } \boldsymbol{x} \in \mathbb{R}^d,$$

where the coefficients $c_1, c_2, \ldots, c_n$ are solved by a linear system

$$\sum_{i_2=1}^{n} K_2(\boldsymbol{x}_{i_1}, \boldsymbol{x}_{i_2}) c_{i_2} = y_{i_1}, \quad \text{for } i_1 = 1, 2, \ldots, n.$$

The new interpolant s_4 is composed of another kernel basis

$$K_4(\cdot, \boldsymbol{x}_{i_1}, \boldsymbol{x}_{i_2}, \boldsymbol{x}_{i_3}), \quad \text{for } i_1, i_2, i_3 = 1, 2, \ldots, n,$$

that is,

$$s_4(\boldsymbol{x}) := \sum_{i_1,i_2,i_3=1}^{n,n,n} c_{i_1} c_{i_2} c_{i_3} K_4(\boldsymbol{x}, \boldsymbol{x}_{i_1}, \boldsymbol{x}_{i_2}, \boldsymbol{x}_{i_3}), \quad \text{for } \boldsymbol{x} \in \mathbb{R}^d,$$

where the coefficients $c_1, c_2, \ldots, c_n$ are solved by a multi-linear system

$$\sum_{i_2,i_3,i_4=1}^{n,n,n} K_4(\boldsymbol{x}_{i_1}, \boldsymbol{x}_{i_2}, \boldsymbol{x}_{i_3}, \boldsymbol{x}_{i_4}) c_{i_2} c_{i_3} c_{i_4} = y_{i_1}, \quad \text{for } i_1 = 1, 2, \ldots, n.$$

In this article, we mainly discuss how to use a special class of strictly positive definite multi-kernel Φ_m in Equation (3.3) to construct the kernel-based interpolant s_m in Theorem 4.1 which shows that the related multi-linear system exists the unique solution. By the theorems of reproducing kernel Banach spaces in [5], we can obtain the advanced properties of s_m including optimal recoveries in Theorem 4.3 and error analysis in Theorems 4.5 and 4.6.

2. Positive Definite Tensors and Multi-Linear Systems

In this section, we review the theory of positive definite tensors which will be used to define the positive definite multi-kernels in Section 3. For convenience of the readers, the notations and operations of tensors are defined as in the book [2]. We say $T_{m,n}$ a collection of all mth order nth dimensional real tenors, for example,

$T_{3,n} = \mathbb{R}^{n\times n\times n}$, $T_{2,n} = \mathbb{R}^{n\times n}$, and $T_{1,n} = \mathbb{R}^n$. For $\mathsf{A}_m \in T_{m,n}$, we say A_m a symmetric tensor if all entries $a_{i_1\ldots i_m}$ of A_m are invariant under any permutation of the indices. For $\boldsymbol{v} \in \mathbb{R}^n$, the tensor outer product is defined as

$$\boldsymbol{v}^{\otimes m} = \boldsymbol{v} \otimes \cdots \otimes \boldsymbol{v} := (v_{i_1} \ldots v_{i_m}) \in T_{m,n}.$$

For $\boldsymbol{c} \in \mathbb{R}^n$, the $(m-1)$-mode and m-mode products are defined as

$$\mathsf{A}_m\boldsymbol{c}^{m-1} := \left(\sum_{i_2,\ldots,i_m=1}^{n,\ldots,n} a_{i_1 i_2\ldots i_m} c_{i_2} \ldots c_{i_m}\right)_{i_1=1}^{n} \in \mathbb{R}^n,$$

and

$$\mathsf{A}_m\boldsymbol{c}^{m} := \sum_{i_1,\ldots,i_m=1}^{n,\ldots,n} a_{i_1\ldots i_m} c_{i_1} \ldots c_{i_m} \in \mathbb{R},$$

respectively. For $\mathsf{A}_m \in T_{m,n}$, we say A_m a *semi-positive definite tensor* if

$$\mathsf{A}_m\boldsymbol{c}^m \geq 0, \quad \text{for all } \boldsymbol{c} \in \mathbb{R}^n.$$

If further $\mathsf{A}_m\boldsymbol{c}^m = 0$ if and only if $\boldsymbol{c} = 0$, then A_m is said a *positive definite tensor* same as in [3, Definition 4.1]. Specially, the positive definite tensor A_2 is a positive definite matrix.

For any $\boldsymbol{b} \in \mathbb{R}^n$, we look at the multi-linear system

$$\mathsf{A}_m\boldsymbol{c}^{m-1} = \boldsymbol{b}. \tag{2.1}$$

If A_2 is a symmetric positive definite matrix, then Equation (2.1) has a unique solution $\boldsymbol{c} \in \mathbb{R}^n$. For the general cases, we will study the solutions of Equation (2.1) by the theorems of the tensor variational inequality (TVI) in [3]. Let

$$\mathcal{S}(\mathsf{A}_m, \boldsymbol{b}) := \left\{\boldsymbol{c} \in \mathbb{R}^n : (\boldsymbol{c} - \boldsymbol{d})^T \left(\mathsf{A}_m\boldsymbol{c}^{m-1} - \boldsymbol{b}\right) \geq 0, \text{ for all } \boldsymbol{d} \in \mathbb{R}^n\right\}.$$

Then $\mathcal{S}(\mathsf{A}_m, \boldsymbol{b})$ is said the solutions set of TVI with $\mathbb{R}^n$, A_m, and $-\boldsymbol{b}$. If A_m is a positive definite tensor, then [3, Theorem 4.2] guarantees that $\mathcal{S}(\mathsf{A}_m, \boldsymbol{b})$ is nonempty and compact.

Proposition 2.1. *If A_m is a positive definite tensor, then the solutions set of Equation* (2.1) *is nonempty and compact.*

Proof. If we verify that $\mathcal{S}(\mathsf{A}_m, \boldsymbol{b})$ is equal to the solutions set of Equation (2.1), then the proof is completed by [3, Theorem 4.2].

We first suppose that $\boldsymbol{c}$ is a solution of Equation (2.1). Thus, we conclude from $\mathsf{A}_m \boldsymbol{c}^{m-1} - \boldsymbol{b} = 0$ that $(\boldsymbol{c} - \boldsymbol{d})^T \left(\mathsf{A}_m \boldsymbol{c}^{m-1} - \boldsymbol{b}\right) = 0$ for all $\boldsymbol{d} \in \mathbb{R}^n$, hence that $\boldsymbol{c} \in \mathcal{S}(\mathsf{A}_m, \boldsymbol{b})$.

In the other side, we suppose that $\boldsymbol{c} \in \mathcal{S}(\mathsf{A}_m, \boldsymbol{b})$. Let $\boldsymbol{v} := \boldsymbol{c} - \boldsymbol{d}$ and $\boldsymbol{u} := \mathsf{A}_m \boldsymbol{c}^{m-1} - \boldsymbol{b}$. Thus, $\boldsymbol{v}^T \boldsymbol{u} \geq 0$ for any $\boldsymbol{v} \in \mathbb{R}^n$. Since the range of $\boldsymbol{v}$ is equal to $\mathbb{R}^n$, we take $\boldsymbol{v} := -\boldsymbol{u}$ such that $0 \geq -\|\boldsymbol{u}\|_2^2 = (-\boldsymbol{u})^T \boldsymbol{u} \geq 0$. This shows that $\mathsf{A}_m \boldsymbol{c}^{m-1} - \boldsymbol{b} = \boldsymbol{u} = 0$. Therefore, $\boldsymbol{c}$ is a solution of Equation (2.1). □

Same as in [3, Definition 4.1], we say A_m a *strictly positive definite tensor* if $(\boldsymbol{c} - \boldsymbol{d})^T \left(\mathsf{A}_m \boldsymbol{c}^{m-1} - \mathsf{A}_m \boldsymbol{d}^{m-1}\right) > 0$ for all $\boldsymbol{c}, \boldsymbol{d} \in \mathbb{R}^n$ with $\boldsymbol{c} \neq \boldsymbol{d}$. Obviously, if A_m is a strictly positive definite tensor, then A_m is a positive definite tensor. Specially, the positive definite matrix A_2 is a strictly positive definite tensor. If A_m is a strictly positive definite tensor, then [3, Theorem 4.2] guarantees that $\mathcal{S}(\mathsf{A}_m, \boldsymbol{b})$ is singleton.

Proposition 2.2. *If A_m is a strictly positive definite tensor, then Equation* (2.1) *exists a unique solution.*

Proof. As in the proof of Proposition 2.1, $\mathcal{S}(\mathsf{A}_m, \boldsymbol{b})$ is equal to the solutions set of Equation (2.1). Thus, the proof is completed by [3, Theorem 4.2]. □

In [1, 6], there are many kinds of multi-linear systems such as M-tensors, Z^+-tensors, and P-tensors to introduce the same results. In this article, we only discuss the positive definite tensors for the scattered data interpolations.

3. Positive Definite Multi-Kernels and Reproducing Kernel Banach Spaces

In this section, we develop a concept of positive definite multi-kernels. Let a domain $\Omega \subseteq \mathbb{R}^d$. By [4, Definition 6.24], a symmetric kernel $K_2 : \otimes_{k=1}^2 \Omega \to \mathbb{R}$ is said a positive definite kernel if, for all $n \in \mathbb{N}$ and all pairwise distinct points $\{\boldsymbol{x}_i\}_{i=1}^n \subseteq \Omega$, the quadratic form

$$\sum_{i_1, i_2 = 1}^{n,n} K_2(\boldsymbol{x}_{i_1}, \boldsymbol{x}_{i_2}) c_{i_1} c_{i_2},$$

for all $\boldsymbol{c} := (c_1, c_2, \cdots, c_n)^T \in \mathbb{R}^n \backslash \{0\}$. Let $\mathsf{A}_2 := (K_2(\boldsymbol{x}_{i_1}, \boldsymbol{x}_{i_2}))_{i_1, i_2 = 1}^{n,n} \in T_{2,n}$. Thus, A_2 is a symmetric positive definite matrix.

A multi-kernel of order $m \in \mathbb{N}$ is defined as $K_m : \otimes_{k=1}^{m} \Omega \to \mathbb{R}$ for $m \in \mathbb{N}$. We say K_m a *symmetric multi-kernel* if $K_m(z_1, z_2, \cdots, z_m) = K_m(z_{i_1}, z_{i_2}, \cdots, z_{i_m})$ for all permutation of the pairwise distinct indices $i_1, i_2, \ldots, , i_m \in \{1, 2, \ldots, m\}$.

Definition 3.1. A symmetric multi-kernel K_m is said a *semi-positive definite multi-kernel* if, for all $n \in \mathbb{N}$ and all pairwise distinct points $\{\boldsymbol{x}_i\}_{i=1}^{n} \subseteq \Omega$, the multiple product form

$$\sum_{i_1, i_2, \ldots, i_m = 1}^{n, n, \ldots, n} K_m(\boldsymbol{x}_{i_1}, \boldsymbol{x}_{i_2}, \cdots, \boldsymbol{x}_{i_m}) c_{i_1} c_{i_2} \cdots c_{i_m} \geq 0,$$

for all $\boldsymbol{c} := (c_1, c_2, \cdots, c_n)^T \in \mathbb{R}^n$. If further the multiple product is equal to 0 if and only if $\boldsymbol{c} = 0$, then K_m is said a *positive definite multi-kernel*.

A symmetric multi-kernel K_m is said a *strictly positive definite multi-kernel* if, for all $n \in \mathbb{N}$ and all pairwise distinct points $\{\boldsymbol{x}_i\}_{i=1}^{n} \subseteq \Omega$, the multiple product form

$$\sum_{i_1, i_2, \ldots, i_m = 1}^{n, n, \ldots, n} K_m(\boldsymbol{x}_{i_1}, \boldsymbol{x}_{i_2}, \cdots, \boldsymbol{x}_{i_m}) \left(c_{i_1} - d_{i_1}\right) \left(c_{i_2} \cdots c_{i_m} - d_{i_2} \cdots d_{i_m}\right) > 0,$$

for all $\boldsymbol{c} := (c_1, c_2, \cdots, c_n)^T, \boldsymbol{d} := (d_1, d_2, \cdots, d_n)^T \in \mathbb{R}^n$ with $\boldsymbol{c} \neq \boldsymbol{d}$.

Obviously, the symmetric strictly positive definite multi-kernel K_m is a symmetric positive definite multi-kernel. Specially, the symmetric positive definite multi-kernel K_2 is a symmetric positive definite kernel and a symmetric strictly positive definite multi-kernel. Given any pairwise distinct points $\{\boldsymbol{x}_i\}_{i=1}^{n} \subseteq \Omega$, we define

$$\mathsf{A}_m := \left(K_m(\boldsymbol{x}_{i_1}, \boldsymbol{x}_{i_2}, \cdots, \boldsymbol{x}_{i_m})\right)_{i_1, i_2, \ldots, i_m = 1}^{n, n, \ldots, n} \in T_{m,n}. \tag{3.1}$$

Proposition 3.2. *If K_m is a symmetric (strictly) positive definite multi-kernel, then A_m in Equation* (3.1) *is a symmetric (strictly) positive definite tensor.*

Proof. The proof is straightforward by the definitions of (strictly) positive definite multi-kernels and (strictly) positive definite tensors. □

Now we look at a special class of multi-kernels related to the p-norm reproducing kernel Banach spaces in [4]. Same as in [4, Section 10.4], we suppose that the domain Ω is *compact* and the symmetric positive definite kernel Φ_2 is *continuous*. Thus, the Mercer theorem assures that Φ_2 possesses the absolutely uniformly convergent representation

$$\Phi_2(z_1, z_2) = \sum_{k=1}^{\infty} \lambda_k e_k(z_1) e_k(z_2), \quad \text{for } z_1, z_2 \in \Omega,$$

where $\{\lambda_k : k \in \mathbb{N}\}$ and $\{e_k : k \in \mathbb{N}\}$ are the related positive eigenvalues and continuous eigenfunctions of Φ_2, respectively. Let $\phi_k := \lambda_k^{1/2} e_k$ for all $k \in \mathbb{N}$. Same as [5, Section 4.2], we further suppose that

$$\sum_{k=1}^{\infty} |\phi_k(\boldsymbol{x})| < \infty, \quad \text{for all } \boldsymbol{x} \in \Omega. \tag{3.2}$$

Thus, by [5, Theorem 5.10], Equation (3.2) assures that the special multi-kernel

$$\Phi_m(z_1, z_2, \cdots, z_m) := \sum_{k=1}^{\infty} \prod_{i=1}^{m} \phi_k(z_i), \quad \text{for } z_1, z_2, \ldots, z_m \in \Omega, \tag{3.3}$$

is well-defined for all $m \geq 2$.

Remark 3.3. In this article, the positive definite multi-kernels are consistent with the positive definite tensors in [2, 3]. The semi-positive definite kernels discussed here have the same meaning as the positive definite kernels in [5]. For convenience, we will *always* suppose that K_2 is a continuous symmetric positive definite kernel on the compact domain Ω. Moreover, the eigenvalues and eigenfunctions of Φ_2 are fixed and satisfy the condition in Equation (3.2) such that Φ_m is fixed and well-defined in the following discussions.

Proposition 3.4. *If m is a positive even integer and $K_m := \Phi_m$, then A_m in Equation* (3.1) *is a symmetric strictly positive definite tensor.*

Proof. Equation (3.3) shows that Φ_m is a symmetric multi-kernel. Thus, A_m is a symmetric tensor. We take any $\boldsymbol{c}, \boldsymbol{d} \in \mathbb{R}^n$ with $\boldsymbol{c} \neq \boldsymbol{d}$. By the definition of strictly positive definite tensors, if we prove that $(\boldsymbol{c} - \boldsymbol{d})^T \left(\mathsf{A}_m \boldsymbol{c}^{m-1} - \mathsf{A}_m \boldsymbol{d}^{m-1}\right) > 0$, then the proof is complete.

Let $\boldsymbol{v}_k := (\phi_k(\boldsymbol{x}_1), \phi_k(\boldsymbol{x}_2), \cdots, \phi_k(\boldsymbol{x}_n))^T$ for all $k \in \mathbb{N}$. We conclude from Equation (3.3) that $\mathsf{A}_m = \sum_{k=1}^{\infty} \boldsymbol{v}_k^{\otimes m}$, hence that $\mathsf{A}_m \boldsymbol{c}^{m-1} = \sum_{k=1}^{\infty} (\boldsymbol{v}_k^T \boldsymbol{c})^{m-1} \boldsymbol{v}_k$ and $\mathsf{A}_m \boldsymbol{d}^{m-1} = \sum_{k=1}^{\infty} (\boldsymbol{v}_k^T \boldsymbol{d})^{m-1} \boldsymbol{v}_k$. Let $\alpha_k := \boldsymbol{v}_k^T \boldsymbol{c}$ and $\beta_k := \boldsymbol{v}_k^T \boldsymbol{d}$ for all $k \in \mathbb{N}$. Thus,

$$(\boldsymbol{c} - \boldsymbol{d})^T \left(\mathsf{A}_m \boldsymbol{c}^{m-1} - \mathsf{A}_m \boldsymbol{d}^{m-1}\right) = \sum_{k=1}^{\infty} (\alpha_k - \beta_k) \left(\alpha_k^{m-1} - \beta_k^{m-1}\right).$$

Now we verify that $(\alpha_k - \beta_k)\left(\alpha_k^{m-1} - \beta_k^{m-1}\right)$ is nonnegative for all $k \in \mathbb{N}$. Since m is even, $m - 1$ is odd. If $\alpha_k \geq 0$ and $\beta_k \geq 0$, then $\alpha_k - \beta_k$ and $\alpha_k^{m-1} - \beta_k^{m-1}$ have the same sign. If $\alpha_k \geq 0$ and $\beta_k \leq 0$, then $\alpha_k - \beta_k \geq 0$ and $\alpha_k^{m-1} - \beta_k^{m-1} \geq$

0. Thus, $(\alpha_k - \beta_k)\left(\alpha_k^{m-1} - \beta_k^{m-1}\right) \geq 0$. Moreover, since $(\alpha_k - \beta_k)\left(\alpha_k^{m-1} - \beta_k^{m-1}\right) = (\beta_k - \alpha_k)\left(\beta_k^{m-1} - \alpha_k^{m-1}\right)$, the assertion also holds. Therefore, we conclude that $(\boldsymbol{c} - \boldsymbol{d})^T\left(\mathsf{A}_m\boldsymbol{c}^{m-1} - \mathsf{A}_m\boldsymbol{d}^{m-1}\right) \geq 0$.

Since Φ_2 is a symmetric positive definite kernel, A_2 is a symmetric positive definite matrix. Thus, we conclude from $\sum_{k=1}^{\infty}(\alpha_k - \beta_k)^2 = (\boldsymbol{c}-\boldsymbol{d})^T(\mathsf{A}_2\boldsymbol{c} - \mathsf{A}_2\boldsymbol{d}) = \mathsf{A}_2(\boldsymbol{c}-\boldsymbol{d})^2 > 0$ that $\alpha_k - \beta_k \neq 0$ such that $(\alpha_k - \beta_k)\left(\alpha_k^{m-1} - \beta_k^{m-1}\right) \neq 0$ for at least one k, hence that $(\boldsymbol{c} - \boldsymbol{d})^T\left(\mathsf{A}_m\boldsymbol{c}^{m-1} - \mathsf{A}_m\boldsymbol{d}^{m-1}\right)$ is positive. □

Proposition 3.5. *If m is a positive even integer, then Φ_m is a symmetric strictly positive definite multi-kernel.*

Proof. Proposition 3.4 guarantees that A_m is a symmetric strictly positive definite tensor. Thus, the proof is straightforward by Definition 3.1. □

Example 3.6. Let

$$\Phi_2(\boldsymbol{z}_1, \boldsymbol{z}_2) := \prod_{j=1}^{d} R(z_{1j}, z_{2j}),$$

for $\boldsymbol{z}_1 := (z_{11}, z_{12}, \cdots, z_{1d})^T, \boldsymbol{z}_2 := (z_{21}, z_{22}, \cdots, z_{2d})^T \in (0,1)^d$, where

$$R(z_1, z_2) := \begin{cases} -z_1^3 + z_1^3 z_2 + z_1 z_2^3 - 3z_1 z_2^2 + 2z_1 z_2, & 0 < z_1 \leq z_2 < 1, \\ -z_2^3 + z_1 z_2^3 + z_1^3 z_2 - 3z_1^2 z_2 + 2z_1 z_2, & 0 < z_2 \leq z_1 < 1. \end{cases}$$

Thus, Φ_2 is a positive definite kernel on $(0,1)^d$. Let

$$\lambda_{\boldsymbol{k}} := \prod_{j=1}^{d} \frac{6}{k_j^4 \pi^4}, \quad e_{\boldsymbol{k}}(\boldsymbol{x}) := 2^{d/2} \prod_{j=1}^{d} \sin(k_j \pi x_j),$$

for all $\boldsymbol{k} := (k_1, k_2, \cdots, k_d)^T \in \mathbb{N}^d$. Thus, $\lambda_{\boldsymbol{k}}$ and $e_{\boldsymbol{k}}$ are the eigenvalues and eigenfunctions of K_2 and $\phi_{\boldsymbol{k}} := \lambda_{\boldsymbol{k}}^{1/2} e_{\boldsymbol{k}}$ satisfy the condition in Equation (3.2). By Proposition 3.5, we obtain the strictly positive definite multi-kernel

$$\Phi_m(\boldsymbol{z}_1, \boldsymbol{z}_2, \cdots, \boldsymbol{z}_m) := \sum_{\boldsymbol{k} \in \mathbb{N}^d} \prod_{i=1}^{m} \phi_{\boldsymbol{k}}(\boldsymbol{z}_i), \quad \text{for } \boldsymbol{z}_1, \boldsymbol{z}_2, \cdots, \boldsymbol{z}_m \in (0,1)^d.$$

Another examples of strictly positive definite multi-kernels can be constructed by Gaussian kernels and power series kernels such as in [5, Sections 4.4 and 4.5].

Finally, we introduce the reproducing kernel Banach spaces defined by the positive definite kernels such as in [5, Chapter 4]. Since Φ_2 is a symmetric positive definite kernel, the eigenfunctions $\{e_k : k \in \mathbb{N}\}$ of Φ_2 can be an orthonormal basis of $\mathrm{L}_2(\Omega)$. Thus, $\{\phi_k : k \in \mathbb{N}\}$ is a basis of $\mathrm{L}_2(\Omega)$. Let $\boldsymbol{\phi} := (\phi_1, \phi_2, \cdots, \phi_k, \cdots)^T$. We define a normed space

$$\mathcal{B}^p := \left\{ f := \boldsymbol{\alpha}^T \boldsymbol{\phi} : \boldsymbol{\alpha} \in \mathbb{R}^{\mathbb{N}} \text{ and } \|\boldsymbol{\alpha}\|_p < \infty \right\},$$

equipped with the norm $\|f\|_{\mathcal{B}^p} := \|\boldsymbol{\alpha}\|_p$, where $1 < p < \infty$. By Equation (3.2), [5, Theorem 4.3] guarantees that $\mathcal{B}^p$ is a two-sided reproducing kernel Banach space and Φ_2 is the two-sided reproducing kernel of $\mathcal{B}^p$, more precisely, (i) $\Phi_2(z_1, \cdot) \in \mathcal{B}^q$, (ii) $\langle f, \Phi_2(z_1, \cdot)\rangle = f(z_1)$, (iii) $\Phi_2(\cdot, z_2) \in \mathcal{B}^p$, and (iv) $\langle \Phi_2(\cdot, z_2), g\rangle = g(z_2)$, for all $z_1, z_2 \in \Omega$, all $f \in \mathcal{B}^p$, and all $g \in \mathcal{B}^q$, where $q := p/(p-1)$, the dual space of $\mathcal{B}^p$ is isometrically equivalent to $\mathcal{B}^q$, and $\langle \cdot, \cdot \rangle$ represents the dual bilinear product of $\mathcal{B}^p \times \mathcal{B}^q$, that is, $\langle f, g\rangle = \boldsymbol{\alpha}^T \boldsymbol{\beta}$ for $f = \boldsymbol{\alpha}^T \boldsymbol{\phi}$ and $g = \boldsymbol{\beta}^T \boldsymbol{\phi}$. Moreover, [5, Proposition 4.4] guarantees that $\mathcal{B}^p \subseteq \mathrm{C}(\Omega)$. The Gâteaux derivative of $\|\cdot\|_{\mathcal{B}^p}$ at $f \neq 0$ has the form

$$d_G \|\cdot\|_{\mathcal{B}^p}(f) = \sum_{k=1}^{\infty} \left(\frac{\alpha_k |\alpha_k|^{p-2}}{\|\boldsymbol{\alpha}\|_p^{p-1}} \right) \phi_k, \quad \text{for } f = \boldsymbol{\alpha}^T \boldsymbol{\phi} \in \mathcal{B}^p. \tag{3.4}$$

In Section 4, we will use the positive definite multi-kernel Φ_m to construct the interpolant and analyze its properties in the reproducing kernel Banach space $\mathcal{B}^{m/(m-1)}$ such as optimal recoveries and error analysis when m is a positive even integer.

4. Interpolations, Optimal Recoveries, and Error Analysis

In this section, we discuss how to construct the interpolant s_m from the scattered data by the strictly positive definite multi-kernel Φ_m in Equation (3.3). Suppose that the data $(\boldsymbol{x}_1, y_1), (\boldsymbol{x}_2, y_2), \ldots, (\boldsymbol{x}_n, y_n)$ compose of the pairwise distinct points $\{\boldsymbol{x}_i\}_{i=1}^n \subseteq \Omega \subseteq \mathbb{R}^d$ and the values $\{y_i\}_{i=1}^n \subseteq \mathbb{R}$ evaluated by some function $f \in \mathrm{C}(\Omega)$, that is,

$$y_1 := f(\boldsymbol{x}_1), y_2 := f(\boldsymbol{x}_2), \ldots, y_n := f(\boldsymbol{x}_n). \tag{4.1}$$

Let $\boldsymbol{y} := (y_1, y_2, \ldots, y_n)^T \in \mathbb{R}^n$. Let A_m be a tensor defined in Equation (3.1) by the multi-kernel $K_m := \Phi_m$ and the data points $\{\boldsymbol{x}_i\}_{i=1}^n$. Different from the classical kernel-based interpolations, the basis of s_m composes of $\Phi_m(\cdot, \boldsymbol{x}_{i_1}, \boldsymbol{x}_{i_2}, \cdots, \boldsymbol{x}_{i_{m-1}})$ for $i_1, i_2, \ldots, i_{m-1} = 1, 2, \ldots, n$. Let a tensor function

$$\mathsf{B}_m(\boldsymbol{x}) := \left(\Phi_m(\boldsymbol{x}, \boldsymbol{x}_{i_1}, \boldsymbol{x}_{i_2}, \ldots, \boldsymbol{x}_{i_{m-1}})\right)_{i_1, i_2, \ldots, i_{m-1}=1}^{n,n,\ldots,n} \in T_{m-1,n}, \quad \text{for } \boldsymbol{x} \in \Omega.$$

Thus, $\mathsf{A}_m = (\mathsf{B}_m(\boldsymbol{x}_i))_{i=1}^n$.

Theorem 4.1. *If m is a positive even integer, then s_m has the from*

$$s_m(\boldsymbol{x}) := \mathsf{B}_m(\boldsymbol{x})\boldsymbol{c}^{m-1}, \quad \textit{for } \boldsymbol{x} \in \Omega,$$

such that

$$s_m(\boldsymbol{x}_1) = y_1, s_m(\boldsymbol{x}_2) = y_2, \ldots, s_m(\boldsymbol{x}_n) = y_n,$$

where the coefficients $\boldsymbol{c} \in \mathbb{R}^n$ are uniquely solved by the multi-linear system

$$\mathsf{A}_m\boldsymbol{c}^{m-1} = \boldsymbol{y}. \tag{4.2}$$

Proof. Proposition 3.4 guarantees that A_m is a symmetric strictly positive definite tensor. Thus, by Proposition 2.2, multi-linear system (4.2) has the unique solution $\boldsymbol{c}$. This assures that s_m satisfies the interpolation conditions. □

Remark 4.2. If s_m is constructed by the general positive definite multi-kernel K_m, then A_m is a symmetric positive definite tensor such that multi-linear system (4.2) still has the solution $\boldsymbol{c}$. But, the solution $\boldsymbol{c}$ may not be unique by Proposition 2.1 such that the interpolant s_m may not be unique.

Equation (3.2) shows that $\Phi_m(\cdot, \boldsymbol{x}_{i_1}, \boldsymbol{x}_{i_2}, \cdots, \boldsymbol{x}_{i_{m-1}}) \in \mathcal{B}^{m/(m-1)}$; hence $s_m \in \mathcal{B}^{m/(m-1)}$. Now we show the optimal recovery of s_m in $\mathcal{B}^{m/(m-1)}$ by the theorems in [5].

Theorem 4.3. *If m is a positive even integer, then s_m is the minimizer of the norm-minimal interpolation*

$$\min_{f \in \mathcal{B}^{m/(m-1)}} \|f\|_{\mathcal{B}^{m/(m-1)}} \textit{ subjected to } f(\boldsymbol{x}_i) = y_i \textit{ for all } i = 1, 2, \ldots, n. \tag{4.3}$$

Proof. If $\boldsymbol{y} = 0$, then $s_m = 0$ and s_m is the unique minimizer of minimization (4.3). If $\boldsymbol{y} \neq 0$, then the minimizer of minimization (4.3) is nonzero. For convenience, we suppose that $\boldsymbol{y} \neq 0$. [5, Lemma 2.22] guarantees that minimization (4.3) has the unique minimizer s such that its Gateaux derivative $d_G \|\cdot\|_{\mathcal{B}^{m/(m-1)}}(s) = \sum_{i=1}^n \beta_i \Phi_2(\boldsymbol{x}_i, \cdot)$, where the constants $\beta_1, \ldots, \beta_n \in \mathbb{R}$. By the expansions of Φ_2, the Gateaux derivative can be rewritten as

$$d_G \|\cdot\|_{\mathcal{B}^{m/(m-1)}}(s) = \sum_{k=1}^{\infty} \left(\sum_{i=1}^{n} \beta_i \phi_k(\boldsymbol{x}_i) \right) \phi_k. \tag{4.4}$$

Combining Equations (3.4) and (4.4), the coefficients $\boldsymbol{\alpha}$ of s has the form

$$\frac{\alpha_k |\alpha_k|^{m/(m-1)-2}}{\|\boldsymbol{\alpha}\|_{m/(m-1)}^{m/(m-1)-1}} = \sum_{i=1}^{n} \beta_i \phi_k(\boldsymbol{x}_i), \quad \text{for } k \in \mathbb{N}. \tag{4.5}$$

Let $c_i := \|\boldsymbol{\alpha}\|_{m/(m-1)}^{m/(m-1)-1}\beta_i$ for all $i = 1, 2, \ldots, n$. Thus, Equation (4.5) shows that

$$s(\boldsymbol{x}) = \sum_{k=1}^{\infty}\left(\sum_{i=1}^{n} c_i\phi_k(\boldsymbol{x}_i)\left|\sum_{j=1}^{n} c_j\phi_k(\boldsymbol{x}_j)\right|^{m-2}\right)\phi_k(\boldsymbol{x}), \quad \text{for } \boldsymbol{x} \in \Omega. \tag{4.6}$$

Expanding of Equation (4.6), we have

$$\begin{aligned} s(\boldsymbol{x}) &= \sum_{k=1}^{\infty}\sum_{i_1,\ldots,i_{m-1}=1}^{n,\ldots,n}\phi_k(\boldsymbol{x})\prod_{j=1}^{m-1}c_{i_j}\phi_n(\boldsymbol{x}_{i_j}) = \sum_{i_1,\ldots,i_{m-1}=1}^{n,\ldots,n}\prod_{j=1}^{m-1}c_{i_j}\sum_{k=1}^{\infty}\phi_k(\boldsymbol{x})\prod_{l=1}^{m-1}\phi_n(\boldsymbol{x}_{i_l}) \\ &= \sum_{i_1,\ldots,i_{m-1}=1}^{n,\ldots,n} c_{i_1}c_{i_2}\cdots c_{i_{m-1}}\Phi_m(\boldsymbol{x}, \boldsymbol{x}_{i_1}, \boldsymbol{x}_{i_2}, \cdots, \boldsymbol{x}_{i_{m-1}}) = \mathsf{B}_m(\boldsymbol{x})\boldsymbol{c}^{m-1}. \end{aligned}$$

Since $s(\boldsymbol{x}_i) = y_i$ for all $i = 1, 2, \ldots, n$, we have $\mathsf{A}_m\boldsymbol{c}^{m-1} = \boldsymbol{y}$. Moreover, Proposition 4.1 guarantees that the solution of multi-linear system (4.2) is unique. This assures that $s(\boldsymbol{x}) = s_m(\boldsymbol{x})$. □

Theorem 4.4. *If m is a positive even integer, then*

$$\|s_m\|_{\mathcal{B}^{m/(m-1)}} = \left(\mathsf{A}_m\boldsymbol{c}^m\right)^{1-1/m}.$$

Proof. If $s_m = 0$, then the proof is straightforward. For convenience, we suppose that $s_m \neq 0$. As in the proof of Theorem 4.3, we have

$$\|s_m\|_{\mathcal{B}^{m/(m-1)}} = \langle s_m, d_G\,\|\cdot\|_{\mathcal{B}^{m/(m-1)}}(s_m)\rangle = \sum_{i=1}^{n}\beta_i\langle s_m, \Phi_2(\boldsymbol{x}_i, \cdot)\rangle,$$

and

$$\beta_i = \|\boldsymbol{\alpha}\|_{m/(m-1)}^{1-m/(m-1)}c_i = \|s_m\|_{\mathcal{B}^{m/(m-1)}}^{-1/(m-1)}c_i, \quad \text{for } i = 1, \ldots, n.$$

By the reproducing properties of $\mathcal{B}^{m/(m-1)}$, we compute the norm

$$\|s_m\|_{\mathcal{B}^{m/(m-1)}} = \sum_{i=1}^{n}\beta_i s_m(\boldsymbol{x}_i) = \|s_m\|_{\mathcal{B}^{m/(m-1)}}^{-1/(m-1)}\sum_{i=1}^{n}c_i\mathsf{B}_m(\boldsymbol{x}_i)\boldsymbol{c}^{m-1}.$$

Therefore,

$$\|s_m\|_{\mathcal{B}^{m/(m-1)}}^{m/(m-1)} = \mathsf{A}_m\boldsymbol{c}^m.$$

□

For an approximate problem of $f(\boldsymbol{x}_1) \approx y_1, \dots, f(\boldsymbol{x}_n) \approx y_n$, the estimate function s is solved by the regularization

$$\min_{f \in \mathcal{B}^{m/(m-1)}} \sum_{i=1}^{n} (f(\boldsymbol{x}_i) - y_i)^2 + \sigma \|f\|_{\mathcal{B}^{m/(m-1)}}^{m/(m-1)}, \tag{4.7}$$

where σ is a positive parameter. In the same manner of Theorems 4.3 and 4.4, the minimizer of optimization (4.7) has the form as $s = B_m \boldsymbol{c}^{m-1}$, where the coefficients $\boldsymbol{c}$ are uniquely solved by the minimization

$$\min_{\boldsymbol{c} \in \mathbb{R}^n} \left\| \mathsf{A}_m \boldsymbol{c}^{m-1} - \boldsymbol{y} \right\|_2^2 + \sigma \mathsf{A}_m \boldsymbol{c}^m.$$

Finally, we study the error analysis of $|s_m(\boldsymbol{x}) - f(\boldsymbol{x})|$ when $f \in \mathcal{B}^{m/(m-1)}$. We define the generalized power function

$$P_m(\boldsymbol{x}) := \min_{\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}} \left\| \Phi_2(\boldsymbol{x}, \cdot) - \sum_{i=1}^{n} \lambda_i \Phi_2(\boldsymbol{x}_i, \cdot) \right\|_{\mathcal{B}^m}, \quad \text{for } \boldsymbol{x} \in \Omega. \tag{4.8}$$

Let $\boldsymbol{x}_0 := \boldsymbol{x}$ and $\tilde{\mathsf{A}}_m := (\Phi_m(\boldsymbol{x}_{i_1}, \boldsymbol{x}_{i_2}, \cdots, \boldsymbol{x}_{i_m}))_{i_1, i_2, \dots, i_m = 0}^{n, n, \dots, n} \in T_{m, n+1}$. As in the proof of Theorems 4.3 and 4.4, we can compute

$$P_m(\boldsymbol{x}) = \min_{\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}} \left(\tilde{\mathsf{A}}_m \tilde{\boldsymbol{\lambda}}^m \right)^{1/m},$$

where $\tilde{\boldsymbol{\lambda}} := (1, -\lambda_1, -\lambda_2, \dots, -\lambda_n)^T \in \mathbb{R}^{n+1}$. Since $\mathcal{B}^2$ is a reproducing kernel Hilbert space with the reproducing kernel Φ_2, P_2 is equal to the classical power function in [4, Section 11.1] such that $P_2(\boldsymbol{x}) = \left(\Phi_2(\boldsymbol{x}, \boldsymbol{x}) - \mathsf{A}_2^{-1} \mathsf{B}_2(\boldsymbol{x})^2 \right)^{1/2}$.

Theorem 4.5. *If m is a positive even integer and $f \in \mathcal{B}^{m/(m-1)}$ satisfies interpolation condition* (4.1)*, then*

$$|s_m(\boldsymbol{x}) - f(\boldsymbol{x})| \leq 2 \|f\|_{\mathcal{B}^{m/(m-1)}} P_m(\boldsymbol{x}), \quad \textit{for } \boldsymbol{x} \in \Omega.$$

Proof. By the reproducing properties of $\mathcal{B}^{m/(m-1)}$, the interpolation conditions of f and s_m show that

$$\langle f - s_m, \Phi_2(\boldsymbol{x}_i, \cdot) \rangle = f(\boldsymbol{x}_i) - s_m(\boldsymbol{x}_i) = y_i - y_i = 0,$$

for $i = 1, 2, \ldots, n$. Thus, we have

$$\begin{aligned} f(\boldsymbol{x}) - s_m(\boldsymbol{x}) &= \langle f - s_m, \Phi_2(\boldsymbol{x}, \cdot) \rangle \\ &= \langle f - s_m, \Phi_2(\boldsymbol{x}, \cdot) \rangle - \sum_{i=1}^{n} \lambda_i \langle f - s_m, \Phi_2(\boldsymbol{x}_i, \cdot) \rangle \\ &= \langle f - s_m, \Phi_2(\boldsymbol{x}, \cdot) - \sum_{i=1}^{n} \lambda_i \Phi_2(\boldsymbol{x}_i, \cdot) \rangle, \end{aligned} \tag{4.9}$$

for any $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$. Since $\frac{m-1}{m} + \frac{1}{m} = 1$, Equation (4.9) shows that

$$|f(\boldsymbol{x}) - s_m(\boldsymbol{x})| \leq \|f - s_m\|_{\mathcal{B}^{m/(m-1)}} \left\| \Phi_2(\boldsymbol{x}, \cdot) - \sum_{i=1}^{n} \lambda_i \Phi_2(\boldsymbol{x}_i, \cdot) \right\|_{\mathcal{B}^m}. \tag{4.10}$$

Moreover, since Theorem 4.3 guarantees that $\|f\|_{\mathcal{B}^{m/(m-1)}} \geq \|s_m\|_{\mathcal{B}^{m/(m-1)}}$, we have

$$\|f - s_m\|_{\mathcal{B}^{m/(m-1)}} \leq \|f\|_{\mathcal{B}^{m/(m-1)}} + \|s_m\|_{\mathcal{B}^{m/(m-1)}} \leq 2 \|f\|_{\mathcal{B}^{m/(m-1)}}. \tag{4.11}$$

We conclude from Equations (4.10) and (4.11) that

$$|f(\boldsymbol{x}) - s_m(\boldsymbol{x})| \leq 2 \|f\|_{\mathcal{B}^{m/(m-1)}} \min_{\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}} \left\| \Phi_2(\boldsymbol{x}, \cdot) - \sum_{i=1}^{n} \lambda_i \Phi_2(\boldsymbol{x}_i, \cdot) \right\|_{\mathcal{B}^m},$$

hence that the proof is completed by Equation (4.8). □

Let the fill distance $h := \sup_{\boldsymbol{x} \in \Omega} \min_{i=1,2,\ldots,n} \|\boldsymbol{x} - \boldsymbol{x}_i\|_2$. If $\Phi_2 \in \mathrm{C}^{2\gamma}(\Omega \times \Omega)$ for $\gamma \in \mathbb{N}$, then [4, Theorem 11.13] guarantees that

$$P_2(\boldsymbol{x}) \leq C_\Omega h^\gamma, \quad \text{for } \boldsymbol{x} \in \Omega, \tag{4.12}$$

where C_Ω is a positive constant independent of h and $\boldsymbol{x}$.

Theorem 4.6. *If m is a positive even integer, $f \in \mathcal{B}^{m/(m-1)}$ satisfies interpolation condition* (4.1)*, and $\Phi_2 \in \mathrm{C}^{2\gamma}(\Omega \times \Omega)$ for $\gamma \in \mathbb{N}$, then*

$$|f(\boldsymbol{x}) - s_m(\boldsymbol{x})| \leq C h^\gamma, \quad \text{for } \boldsymbol{x} \in \Omega,$$

where C is a positive constant independent of h and $\boldsymbol{x}$.

Proof. The main idea of the proof is to compute the upper bound of $P_m(\boldsymbol{x})$ by the fill distance h. The constructions of $\mathcal{B}^m$ and $\mathcal{B}^2$ shows that $\|f\|_{\mathcal{B}^m} \leq \|f\|_{\mathcal{B}^2}$ for $f \in \mathcal{B}^m \subseteq \mathcal{B}^2$; hence

$$\left\| \Phi_2(\boldsymbol{x}, \cdot) - \sum_{i=1}^{n} \lambda_i \Phi_2(\boldsymbol{x}_i, \cdot) \right\|_{\mathcal{B}^m} \leq \left\| \Phi_2(\boldsymbol{x}, \cdot) - \sum_{i=1}^{n} \lambda_i \Phi_2(\boldsymbol{x}_i, \cdot) \right\|_{\mathcal{B}^2}, \tag{4.13}$$

for any $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$. Substituting Equations (4.13) into (4.8), we have

$$P_m(\boldsymbol{x}) \leq P_2(\boldsymbol{x}). \tag{4.14}$$

Combining Equations (4.12) and (4.14), we also have

$$P_m(\boldsymbol{x}) \leq C_\Omega h^\gamma.$$

Let $C := 2 \|f\|_{\mathcal{B}^{m/(m-1)}} C_\Omega$. Therefore, the proof is completed by Theorem 4.5. □

Theorem 4.6 guarantees that $s_m(\boldsymbol{x}) \to f(\boldsymbol{x})$ when $h \to 0$ for $\boldsymbol{x} \in \Omega$, and moreover, $s_m \to f$ uniformly when $h \to 0$.

Acknowledgments

The author would like to acknowledge support for this project from the Natural Science Foundation of China (12071157 and 11931003) and the Natural Science Foundation of Guangdong Province (2019A1515011995 and 2020B1515310013).

References

[1] W. Ding and Y. Wei. Solving multi-linear systems with M-tensors. *J. Sci. Comput.*, 68:689–715, 2016.

[2] L. Qi, H. Chen, and Y. Chen. *Tensor Eigenvalues and Their Applications*. Springer-Verlag, 2019.

[3] Y. Wang, Z.-H. Huang, and L. Qi. Global uniqueness and solvability of tensor variational inequalities. *J. Optim. Theory Appl.*, 177:137–152, 2018.

[4] H. Wendland. *Scattered Data Approximation*, volume 17. Cambridge University Press, Cambridge, 2005.

[5] Y. Xu and Q. Ye. Generalized Mercer kernels and reproducing kernel Banach spaces. *Memoirs of the AMS*, 258(1243):1–122, 2019.

[6] W. Yan, C. Ling, L. Ling, and H. He. Generalized tensor equations with leading structured tensors. *Appl. Math. Comput.*, 361:311–324, 2019.