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The solutions of the strongly nonlocal spatial solitons with several types of nonlocal response functions*

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The fundamental and second order strongly nonlocal solitons of the nonlocal nonlinear Schrödinger equation for several types of nonlocal responses are calculated by Ritz's variational method. For a specific type of nonlocal response, the solutions of the strongly nonlocal solitons with the same beam width but different degrees of nonlocality are identical except for an amplitude factor. For a nonlocal case where the nonlocal response function decays in direct proportion to the mth power of the distance near the source point, the power and the phase constant of the strongly nonlocal soliton are in inverse proportion to the $(m+2)$th power of its beam width.

Keywords: nonlocal nonlinear Schrödinger equation, strong nonlocality, spatial optical soliton
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1. Introduction

Solitary waves/solitons are found in a variety of nonlinear systems.[1−13] Recently the spatial solitons propagating in the nonlocal nonlinear media have attracted much attention.[1−9] According to the ratio of the characteristic nonlocal length to the beam width, the degree of the nonlocality is mainly sorted into four categories: local, weakly nonlocal, generally nonlocal and strongly nonlocal.[1,7] In particular, the nonlocal case is called a strongly nonlocal case when the characteristic nonlocal length is much larger than the beam width. In the present work attention is paid to the strongly nonlocal issue. It has been experimentally indicated that the spatial solitons in the nematic liquid crystal are strongly nonlocal solitons (SNSs)[2] and the characteristic nonlocal length of the nematic liquid crystal can be changed by employing different voltage bias.[3] Some properties of the SNSs and their interaction are greatly different from those in the local case, e.g. two coherent SNSs with a π phase difference attract rather than repel each other,[4] the phase shift of the SNS can be very large as compared with the local soliton with the same beam width,[5] and the phase shift of a probe beam can be modulated by a pump beam in the strongly nonlocal case.[6]

As previously noted,[7] the power and phase constant of the SNS are both in inverse proportion to the 3rd power of its beam width for the nonlocal case of an exponential-decay type nonlocal response $R(x-\xi) = (1/w)\exp(-|x-\xi|/w)$, and are both in inverse proportion to the 4th power of its beam width for the nonlocal case of a Gaussian function type nonlocal response $R(x-\xi) = (1/w)\exp[-(x-\xi)^2/w^2]$. It will be shown in the present paper that these two nonlocal response functions are two special cases of a general family of nonlocal response functions. By Ritz's variational method we present the approximate solutions of the fundamental and second order SNSs for such a family of nonlocal response functions. It is indicated that for a specific type of nonlocal response, the solutions of the SNSs with the same beam width but different degrees of nonlocality are identical except for an amplitude factor. For a nonlocal case where the nonlocal response function decays in direct proportion to the mth power of the distance near the source point, the power and the phase constant of the SNS are in inverse proportion to the $(m+2)$th power of its beam width.

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2. The fundamental strongly nonlocal soliton

Now we consider the (1+1)-D dimensionless nonlocal nonlinear Schrödinger equation (NNLSE)[1−7]

$$\mathrm{i}\frac{\partial u}{\partial z}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+u\int_{-\infty}^{+\infty}R(x-\xi)|u(\xi,z)|^2\mathrm{d}\xi=0, \quad (1)$$

where $u(x,z)$ is the complex amplitude envelop of the light beam, x and z are the transverse and longitude coordinates respectively, $R(x-\xi)=R(\xi-x)>0$ is the real symmetric nonlocal response function, and $|u(\xi,z)|^2$ is the source that induces a perturbed refractive index of $R(x-\xi)|u(\xi,z)|^2\mathrm{d}\xi$ at a point x.

For a spatial soliton, we have $|u(x,z)|=|u(x,0)|$ and $|u(-x,z)|=|u(x,z)|$. Let

$$V(x)=-\int_{-\infty}^{+\infty}R(x-\xi)|u(\xi,z)|^2\mathrm{d}\xi, \quad (2)$$

then we will have $V(x)=V(-x)$. Taking the Taylor's expansion of $V(x)$ at $x=0$, we obtain

$$V(x)=V_0+\frac{1}{2\mu^4}x^2+\alpha x^4+\beta x^6+\cdots, \quad (3)$$

where

$$V_0=V(0), \quad (4a)$$
$$\frac{1}{\mu^4}=V^{(2)}(0), \quad (4b)$$
$$\alpha=\frac{1}{4!}V^{(4)}(0), \quad (4c)$$
$$\beta=\frac{1}{6!}V^{(6)}(0). \quad (4d)$$

Since $V(x)$ depends on $u(x,z)$, the parameters V_0,μ,α,β all depend on $u(x,z)$ too. As previously noted,[7] in the strongly nonlocal case the parameter μ can be viewed as the beam width of the soliton, and when $x<\mu$, the terms αx^4 and βx^6 are one and two orders of the magnitude smaller than the term $x^2/(2\mu^4)$ respectively and the x^8 power term and other higher power terms of the Taylor series of $V(x)$ are negligible. For convenience, we simply adopt

$$V(x)=V_0+\frac{1}{2\mu^4}x^2+\alpha x^4+\beta x^6. \quad (5)$$

By taking a transformation

$$u(x,z)=\psi(x)\mathrm{e}^{-\mathrm{i}(E+V_0)z}, \quad (6)$$

the NNLSE(1) reduces to

$$\hat{H}(x,\psi)\psi=E\psi, \quad (7)$$

where

$$\hat{H}(x,\psi)=-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\left(\frac{1}{2\mu^4}x^2+\alpha x^4+\beta x^6\right). \quad (8)$$

If $\alpha=0$ and $\beta=0$, Eq.(7) will reduce to the well-known stationary Schrödinger equation for a harmonic oscillator that has Hermitian–Gaussian eigenfunctions.[14] Since in the strongly nonlocal case the terms αx^4 and βx^6 are far smaller than the term $x^4/(2\mu^4)$, we view the terms αx^4 and βx^6 as perturbations in the process of solving Eq.(7) and assume the fundamental SNS to take the following approximate form:

$$\begin{aligned}&\psi_0(A,a,b,c,d;x)\\ \approx\ & A\left(\frac{1}{\pi\mu^2}\right)^{1/4}\mathrm{e}^{-\frac{x^2}{2\mu^2}}\\ &\times\left(1+a\frac{x^2}{\mu^2}+b\frac{x^4}{\mu^4}+c\frac{x^6}{\mu^6}+d\frac{x^8}{\mu^8}\right). \quad (9)\end{aligned}$$

To meet the requirements of the perturbation approximation, we need $a,b,c,d\ll 1$. As will be shown, for a specific type of nonlocal response function the parameters a,b,c and d are all constants that have nothing to do with the beam width or the characteristic nonlocal length provided that the characteristic nonlocal length is much larger than the beam width. Since $a,b,c,d\ll 1$, the beam width of $\psi_0^2(A,a,b,c,d;x)$ is approximately equal to μ apparently and the soliton's power in the first order approximation reads

$$P=\int_{-\infty}^{+\infty}\psi_0^2(A,a,b,c,d;x)\mathrm{d}x\approx A^2(1+a). \quad (10)$$

From Eq.(6), we obtain the phase constant of the fundamental SNS

$$\gamma=-(V_0+E), \quad (11)$$

where

$$E\approx\frac{\langle\psi_0|\hat{H}(x,\psi_0)|\psi_0\rangle}{\langle\psi_0|\psi_0\rangle}\equiv F(\alpha,\beta;a,b,c,d). \quad (12)$$

Since $a,b,c,d\ll 1$, we have $E\sim\dfrac{1}{2\mu^2}$. As will be shown for the SNS, we have $V_0\gg E$ and $\gamma\approx -V_0$. According to Ritz's variational method,[14] in order to make the difference between the approximate solution $\psi_0(A,a,b,c,d;x)$ and the exact solution of the fundamental SNS as small as possible, the parameters a,b,c and d should make $F(\alpha,\beta;a,b,c,d)$ reach its stationary point. So we have

$$\partial_a F(\alpha,\beta;a,b,c,d)=0, \tag{13a}$$

$$\partial_b F(\alpha,\beta;a,b,c,d)=0, \tag{13b}$$

$$\partial_c F(\alpha,\beta;a,b,c,d)=0, \tag{13c}$$

$$\partial_d F(\alpha,\beta;a,b,c,d)=0, \tag{13d}$$

where $\partial_a \equiv \dfrac{\partial}{\partial a}$, etc. For a fixed value of μ, the parameters A,α,β,a,b,c and d can be numerically calculated by solving coupling Eqs.(4) and (13) with a numerical method presented in Ref.[7]. Here and above we have formally presented the main steps to calculate the approximate solution $\psi_0(A,a,b,c,d;x)$ of the fundamental SNS.

Now we consider a type of nonlocal response function $R(x-\xi)$ that decays in direct proportion to the mth power of the distance $|x-\xi|$ near the source point ξ. In general, we have

$$R(x-\xi)\approx\frac{1}{w}\left(1-\frac{|x-\xi|^m}{w^m}\right)\quad\text{for}\quad |x-\xi|\ll w, \tag{14}$$

where w can be viewed as the characteristic nonlocal length of $R(x-\xi)$. Several types of nonlocal response functions can be sorted into this type of nonlocal response function, for example, $R_{1m}(x-\xi)=\dfrac{1}{w}\exp\left(-\dfrac{|x-\xi|^m}{w^m}\right)$ and $R_{2m}(x-\xi)=\dfrac{1}{w}\dfrac{1}{1+\frac{|x-\xi|^m}{w^m}}$. It is worth to note that $R_{11}(x-\xi)$ is the exponential-decay type of nonlocal response function that has been successfully applied to the description of the nonlocal nonlinearity of the nematic liquid crystal[2,3,7] and $R_{12}(x-\xi)$ is the Gaussian-function type of nonlocal response function.[5−7] By introducing

$$\psi_0(A,a,b,c,d;x)=\frac{A}{\sqrt{\mu}}\phi\left(a,b,c,d;\frac{x}{\mu}\right), \tag{15}$$

we obtain

$$\phi\left(a,b,c,d;\frac{x}{\mu}\right)\approx 0\qquad\text{for } x\gg\mu. \tag{16}$$

Now we consider the strongly nonlocal case where the degree of nonlocality $w/\mu\gg 1$. When $|x|\ll w$, by substituting Eq.(15) into Eq.(2) and keeping in mind Eqs.(14) and (16), we obtain

$$\begin{aligned} V_m(x) &= -\frac{A^2}{\mu}\int_{-\infty}^{+\infty} R(x-\xi)\phi^2\left(\frac{\xi}{\mu}\right)\mathrm{d}\xi \\ &\approx -\int_{-\infty}^{+\infty}\frac{A^2}{\mu w}\left(1-\frac{|x-\xi|^m}{w^m}\right)\phi^2\left(\frac{\xi}{\mu}\right)\mathrm{d}\xi \\ &= -\frac{P}{w}+\frac{A^2\mu^m}{w^{m+1}}\int_{-\infty}^{+\infty}|\eta|^m\phi^2\left(\eta+\frac{x}{\mu}\right)\mathrm{d}\eta \\ &= -\frac{P}{w}+\frac{A^2\mu^m}{w^{m+1}}U_m\left(\frac{x}{\mu}\right), \end{aligned} \tag{17}$$

where

$$U_m\left(\frac{x}{\mu}\right)=\int_{-\infty}^{+\infty}|\eta|^m\phi^2\left(\eta+\frac{x}{\mu}\right)\mathrm{d}\eta. \tag{18}$$

It is worth noting that $U_m\left(\dfrac{x}{\mu}\right)$ is independent of the characteristic nonlocal length w. As will be shown, such a feature of $U_m\left(\dfrac{x}{\mu}\right)$ leads the parameters a,b,c and d to be independent of the characteristic nonlocal length w too in the strongly nonlocal case. From Eqs.(4), (10), and (17), we have

$$A=\sqrt{\frac{w^{m+1}}{U_m^{(2)}(0)\mu^{m+2}}}, \tag{19}$$

$$P=\frac{w^{m+1}(1+a)}{U_m^{(2)}(0)\mu^{m+2}}, \tag{20}$$

$$V_m(0)=-\frac{w^m(1+a)}{U_m^{(2)}(0)\mu^{m+2}}, \tag{21}$$

$$\alpha=\frac{U_m^{(4)}(0)}{4!\mu^6U_m^{(2)}(0)}, \tag{22}$$

$$\beta=\frac{U_m^{(6)}(0)}{6!\mu^8U_m^{(2)}(0)}, \tag{23}$$

where $U_m^{(2)}(0)$, $U_m^{(4)}(0)$ and $U_m^{(6)}(0)$ are all independent of w and μ. By substituting Eqs.(22) and (23) into Eq.(7) and employing transformations $y=x/\mu$ and $\varepsilon=\mu^2E$, we obtain

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\frac{y^2}{2}+\frac{U_m^{(4)}(0)}{4!U_m^{(2)}(0)}y^4+\frac{U_m^{(6)}(0)}{6!U_m^{(2)}(0)}y^6\right]\psi=\varepsilon\psi. \tag{24}$$

Owing to Eq.(24) being independent of w and μ, the parameters a,b,c and d calculated by solving Eqs.(13) are also independent of w and μ. In the strongly nonlocal case, except for an amplitude factor A in Eq.(19), the fundamental soliton solutions $\psi_0(A,a,b,c,d;x)$ with the same beam width μ but different degrees of nonlocality w/μ are identical.

In view of Eq.(21), since $w \gg \mu$, we have $|V_m(0)| \gg E$. In respect that $a, b, c, d \ll 1$, from Eq. (18), in the first order approximation, we have

$$U_m^{(2)}(0) \approx \frac{2m\Gamma\left(\frac{1+m}{2}\right)[1+a(m-1)]}{\sqrt{\pi}}, \quad (25)$$

where $\Gamma(1+x) = x\Gamma(x)$ is the Euler gamma function. By substituting Eq.(25) into Eqs.(19), (20) and (21) we have

$$A \approx \sqrt{\frac{w^{m+1}\sqrt{\pi}}{\mu^{m+2}2m\Gamma\left(\frac{1+m}{2}\right)[1+a(m-1)]}}, \quad (26)$$

$$P \approx \frac{w^{m+1}\sqrt{\pi}(1+a)}{\mu^{m+2}2m\Gamma\left(\frac{1+m}{2}\right)[1+a(m-1)]}, \quad (27)$$

$$\gamma \approx -V_m(0) \approx \frac{w^{m}\sqrt{\pi}(1+a)}{\mu^{m+2}2m\Gamma\left(\frac{1+m}{2}\right)[1+a(m-1)]}. \quad (28)$$

For a fixed value of the characteristic nonlocal length w, the power P and the phase constant γ of the SNS are both in inverse proportion to the $(m+2)$th power of its beam width μ. The cases of $m = 1, 2$ have been previously studied by the perturbation method,[7] and the functional dependences of the power P and phase constant γ of the SNS on the beam width agree well with Eqs.(27) and (28) respectively. In the present paper we consider the cases of $m = 3, 4$ as another two examples to validate our results. Using $\psi_0^2(A, a, b, c, d; x)$ as an input intensity profile, $R_{1m}(x-\xi)$ or $R_{2m}(x-\xi)$ as a nonlocal response function, and the NNLSE(1) as an evolution equation, we numerically investigate the evolutions of light beams in different nonlocal cases with different degrees of nonlocality by the numerical simulation method. The values of a, b, c and d calculated by Ritz's variational method in the strongly nonlocal case are listed in Table 1.

Table 1. The values of a, b, c and d calculated by Ritz's variational method in the strongly nonlocal case. The values with the superscript '1' are for the fundamental solitons, and those with the superscript '2' are for second order solitons.

m	a	b	c	d
3	-0.050^1	-0.016^1	0.0022^1	-0.000084^1
4	-0.12^1	-0.030^1	0.0057^1	-0.00024^1
3	0.0021^2	0.00091^2	-0.0010^2	0.000047^2
4	-0.065^2	-0.012^2	0.0015^2	-0.000044^2

As shown in Fig.1, when the degree of nonlocality $w_0/\mu > 7$, soliton solution $\psi_0(A, a, b, c, d; x)$ with the values of a, b, c and d listed in Table 1 can describe the fundamental soliton state of the NNLSE (1) exactly but cannot describe it exactly when $w/\mu = 3$ that is beyond the strongly nonlocal case but is classified into the generally nonlocal case.

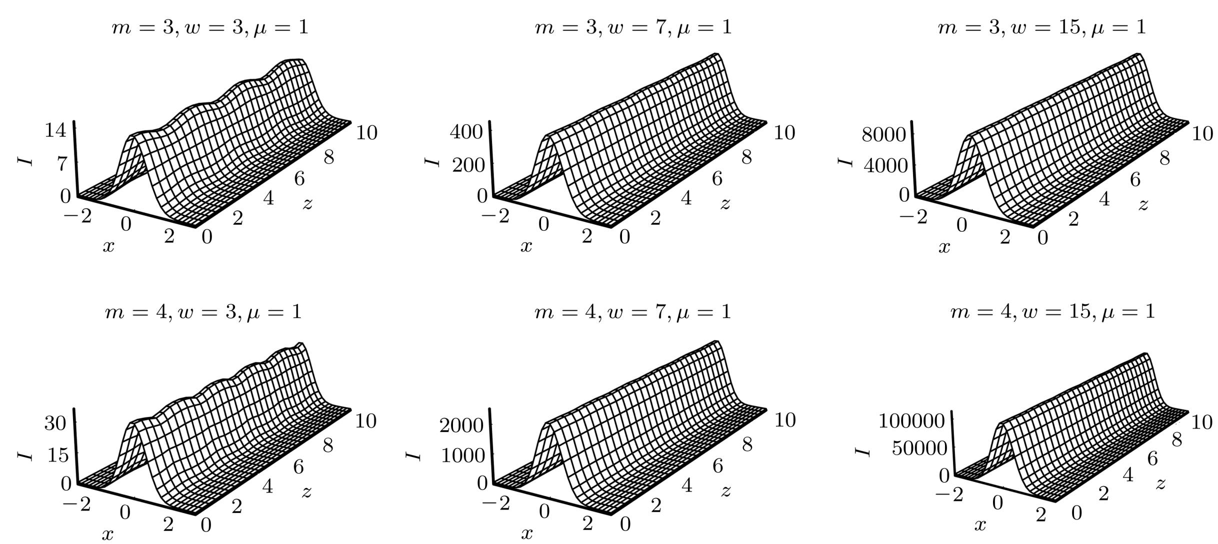

Fig.1. The intensity profiles $|u(x,z)|^2$ of light beams with input intensity profiles described by $|u(x,0)|^2 = \psi_0^2(A, a, b, c, d; x)$. The upper three figures are for the case of $m = 3$ and the lower three are for the case of $m = 4$. From left to right, the degrees of nonlocality w/μ are 3, 7 and 15 respectively. The employed nonlocal response function for numerical simulations can be $R_{1m}(x-\xi)$ or $R_{2m}(x-\xi)$, it makes no difference in the simulation results in the strongly nonlocal case.

3. The second order strongly nonlocal soliton

In the strongly nonlocal case, besides the fundamental soliton state, the NNLSE (1) allows other higher order soliton states. As another example, we present the approximate solution of the second order SNS

$$\psi_1(A,a,b,c,d;x) \approx A\left(\frac{1}{\pi\mu^2}\right)^{1/4} \mathrm{e}^{-\frac{x^2}{2\mu^2}} \sqrt{2}\frac{x}{\mu}\left(1+a\frac{x^2}{\mu^2}+b\frac{x^4}{\mu^4}+c\frac{x^6}{\mu^6}+d\frac{x^8}{\mu^8}\right), \tag{29}$$

where

$$A \approx \sqrt{\frac{w^{m+1}\sqrt{\pi}}{\mu^{m+2}2m(m-1)\Gamma\left(\frac{1+m}{2}\right)[1+a(m+1)]}}. \tag{30}$$

It is worth to note that $\psi_1(A,0,0,0,0;x)$ is the second order eigenfunction of the stationary Schrödinger equation for a harmonic oscillator. The values of a, b, c and d calculated in the strongly nonlocal case are listed in Table 1 and the intensity profiles $|u(x,z)|^2$ with $|u(x,0)|^2 = \psi_1^2(A,a,b,c,d;x)$ are shown in Fig.2. It is indicated that when the degree of nonlocality $w/\mu > 10$, the soliton solution $\psi_1(A,a,b,c,d;x)$ can describe the second order soliton state of the NNLSE (1) exactly but cannot describe it exactly when $w/\mu = 5$.

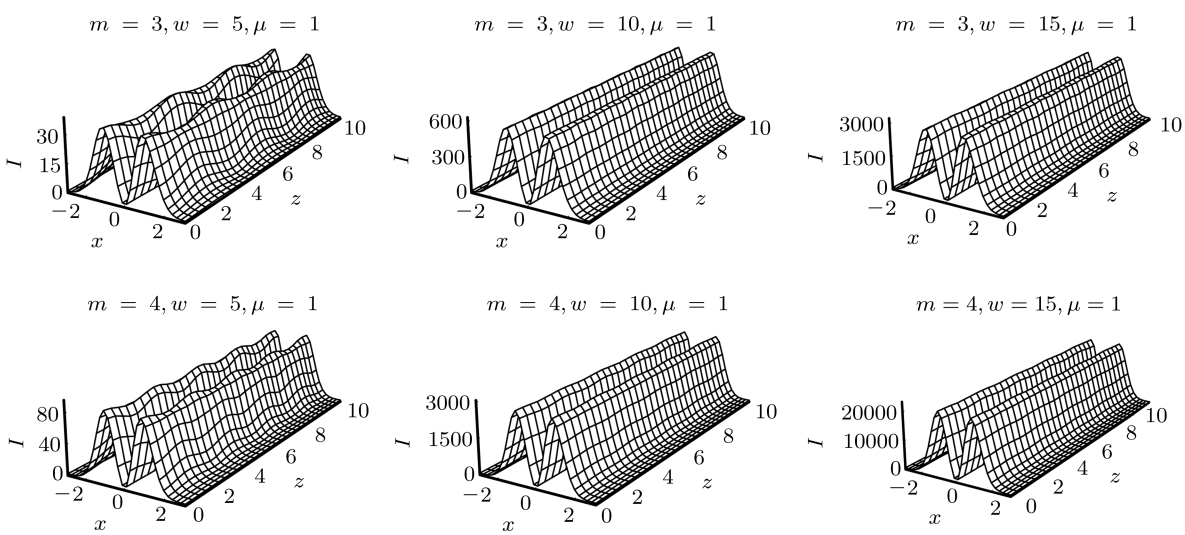

Fig.2. The intensity profiles $|u(x,z)|^2$ with $|u(x,0)|^2 = \psi_1^2(A,a,b,c,d;x)$. The upper three figures are for the case of $m=3$ and the lower three are for the case of $m=4$. From left to right, the degrees of nonlocality w/μ are 5, 10 and 15 respectively.

4. Summary

By Ritz's variational method, we present the approximate solutions of the fundamental and second order SNS of the NNLSE for several types of nonlocal responses. It is indicated that for a specific type of nonlocal response, except for an amplitude factor, the solutions of the strongly nonlocal solitons with the same beam width but different degrees of nonlocality are identical. For a nonlocal case where the nonlocal response function decays in direct proportion to the mth power of the distance near the source point, the power and the phase constant of the SNS are in inverse proportion to the $(m+2)$th power of its beam width.

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