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Conservation laws of the generalized nonlocal nonlinear Schrödinger equation*

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The derivations of several conservation laws of the generalized nonlocal nonlinear Schrödinger equation are presented. These invariants are the number of particles, the momentum, the angular momentum and the Hamiltonian in the quantum mechanical analogy. The Lagrangian is also presented.

Keywords: nonlocal nonlinear Schrödinger equation, conservation law, Lagrangian
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1. Introduction

Solitary waves/solitons are found in a variety of nonlinear systems.[1−9] For the integrable systems, e.g. the Korteweg de Vries (KdV) equation and the standard nonlinear Schrödinger equation, the inverse scattering method is a powerful tool, and there exist infinite conservation laws.[5] However, the number of conservation laws for the non-integrable systems decreases greatly. An important issue relevant to experimentally realizing solitons in non-integrable systems is about the soliton stability, which intrinsically depends on the structures of the conservation laws of the nonlinear systems.[6] In the present work we investigate several important conservation laws of the generalized nonlocal nonlinear Schrödinger equation (NNLSE). For a special case, such a generalized NNLSE will reduce to the standard nonlocal nonlinear Schrödinger equation that has been successfully applied to some nonlinear systems, e.g. the nematic liquid crystal.[7] The details of the derivations of such conservation laws, i.e. the number of particles, the momentum, the angular momentum and the Hamiltonian, are presented. Correlatively, the Lagrangian constantly applied to analysing the evolutions of the waves/beams[8] is presented.

Structurally, the remainder of the present work is laid out as follows: in Section 2 we outline the conservation laws and the Lagrangian of the generalized NNLSE; the details of the derivations of conservation laws of the number of the particles, the momentum and the angular momentum are presented in Section 3 and the derivation of the conservation law of Hamiltonian is described in Section 4; some discussion on the Lagrangian is presented in Section 5; finally a brief summary is given in Section 6.

2. The conservation laws and the Lagrangian

We consider the following generalized nonlocal nonlinear Schrödinger equation(NNLSE):

$$\mathrm{i}\frac{\partial\psi}{\partial t}=\hat{\mathcal{H}}(\psi)\psi, \tag{1}$$

where

$$\begin{aligned}\hat{\mathcal{H}}(\psi) &= -\frac{1}{2}\nabla_\perp^2-f(|\psi|^2)-g(|\psi|^2)\\ &\quad\times\int R(|\boldsymbol{r}-\boldsymbol{\rho}|)G(|\psi(\boldsymbol{\rho},t)|^2)\mathrm{d}^2\boldsymbol{\rho},\end{aligned} \tag{2}$$

and $\boldsymbol{r}=x\boldsymbol{e}_x+y\boldsymbol{e}_y$, $\boldsymbol{\rho}=\xi\boldsymbol{e}_x+\eta\boldsymbol{e}_y$, $\nabla_\perp=\boldsymbol{e}_x\frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}}+\boldsymbol{e}_y\frac{\boldsymbol{\partial}}{\boldsymbol{\partial y}}$, $R(|\boldsymbol{r}-\boldsymbol{\rho}|)$ is the real nonlocal response function and $f(x)$, $g(x)$, $G(x)$ are continuous real functions. In the present work we assume

$$\frac{\mathrm{d}G(x)}{\mathrm{d}x}=g(x),\quad g(0)=0,\quad G(0)=0, \tag{3}$$

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and

$$f(0)=0. \tag{4}$$

It is worth noting that when $g(x)=G(x)=0$, the NNLSE (1) will reduce to the generalized nonlinear Schrödinger equation,[9] and specially when $f(x)=x$, the generalized nonlinear Schrödinger equation will turn into the standard nonlinear Schrödinger equation (NLSE).[5] On the other hand if $f(x)=0$, $g(x)=1$ and $G(x)=x$, the NNLSE(1) will become the standard nonlocal nonlinear Schrödinger equation.[7]

In the present paper we prove that if $\psi(\boldsymbol{r},t)$ is a vector of Hilbert space (which implies that when $|\boldsymbol{r}|\to\infty$, we have $\psi(\boldsymbol{r},t)=0$ and $\nabla_\perp\psi(\boldsymbol{r},t)=0$), there exist several important invariants with respect to the coordinate t, which in the quantum mechanical analogy are the number of particles

$$N=\int|\psi(\boldsymbol{r},t)|^2\mathrm{d}^2\boldsymbol{r}, \tag{5}$$

the momentum

$$\boldsymbol{P}=\int\psi^*(-\mathrm{i}\nabla_\perp)\psi\mathrm{d}^2\boldsymbol{r}, \tag{6}$$

the angular momentum

$$L_z=\int\psi^*\left[-\mathrm{i}\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)\right]\psi\mathrm{d}^2\boldsymbol{r}, \tag{7}$$

and the Hamiltonian

$$H=\int\left[-\frac{\psi^*\nabla_\perp^2\psi}{2}-F(|\psi|^2)\right]\mathrm{d}^2\boldsymbol{r}-\iint\frac{R(|\boldsymbol{r}-\boldsymbol{\rho}|)}{2}G(|\psi(\boldsymbol{r},t)|^2)G(|\psi(\boldsymbol{\rho},t)|^2)\mathrm{d}^2\boldsymbol{r}\mathrm{d}^2\boldsymbol{\rho}, \tag{8}$$

where

$$\frac{\mathrm{d}F(x)}{\mathrm{d}x}=f(x),\quad \text{and}\quad F(0)=0. \tag{9}$$

Correlatively, the Lagrangian for the generalized NNLSE(1) is

$$\mathcal{L}=\frac{\mathrm{i}}{2}\left(\psi^*\frac{\partial\psi}{\partial t}-\psi\frac{\partial\psi^*}{\partial t}\right)-\frac{1}{2}|\nabla_\perp\psi|^2+F(|\psi|^2)+\frac{1}{2}G(|\psi|^2)\int R(|\boldsymbol{r}-\boldsymbol{\rho}|)G(|\psi(\boldsymbol{\rho},t)|^2)\mathrm{d}^2\boldsymbol{\rho}. \tag{10}$$

As will be shown, the variational problem

$$\delta\iint\mathcal{L}\mathrm{d}^2\boldsymbol{r}\mathrm{d}t=0 \tag{11}$$

will lead to the generalized NNLSE (1) again.

3. The derivations of the conservation laws of N, P and L_z

It is easy to prove that if $\psi(\boldsymbol{r},t)$ satisfies the generalized NNLSE(1) and $\varepsilon,\alpha,\beta,\varphi$ are real numbers, the following three functions:

$$\psi_1(\boldsymbol{r},t)=\psi(\boldsymbol{r},t)\mathrm{e}^{\mathrm{i}\varepsilon}=\hat{P}_1(\varepsilon)\psi(\boldsymbol{r},t), \tag{12}$$

$$\psi_2(\boldsymbol{r},t)=\psi(x+\alpha,y+\beta,t)=\hat{P}_2(\alpha,\beta)\psi(\boldsymbol{r},t), \tag{13}$$

$$\psi_3(\boldsymbol{r},t)=\psi(x\cos\varphi-y\sin\varphi,x\sin\varphi+y\cos\varphi,t)=\hat{P}_3(\varphi)\psi(\boldsymbol{r},t) \tag{14}$$

satisfy the generalized NNLSE (1) too. Equations (12), (13) and (14) are three symmetry transformations. Transformation (12) implies that the states with a difference of a phase factor $\mathrm{e}^{\mathrm{i}\varepsilon}$ describe the same state. Transformation (13) is the translational transformation and $\boldsymbol{e}_x\alpha+\boldsymbol{e}_y\beta$ is a translational vector. And transformation (14) is the rotational transformation, and φ is a rotational angle. When $\varepsilon,\alpha,\beta,\varphi\longrightarrow 0$, we have

$$\hat{P}_1=1+\mathrm{i}\varepsilon\hat{I}_1, \tag{15}$$

$$\hat{P}_2=1+\mathrm{i}(\boldsymbol{e}_x\alpha+\boldsymbol{e}_y\beta)\cdot\hat{I}_2, \tag{16}$$

$$\hat{P}_3=1+\mathrm{i}\varphi\hat{I}_3, \tag{17}$$

where

$$\hat{I}_1 = 1, \tag{18}$$

$$\hat{I}_2 = -\mathrm{i}\nabla_\perp, \tag{19}$$

$$\hat{I}_3 = -\mathrm{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right). \tag{20}$$

In the quantum mechanical analogy, $\hat{I}_2$ and $\hat{I}_3$ are the operators of momentum and angular momentum respectively.[10] We will find there exist the following three identities about the three operators:

$$\begin{aligned}&\langle \hat{I}_k\psi, \hat{\mathcal{H}}(\psi)\psi\rangle - \langle \hat{\mathcal{H}}(\psi)\psi, \hat{I}_k\psi\rangle \\ =& \left\langle \psi, \left[\hat{I}_k, \hat{\mathcal{H}}(\psi)\right]\psi\right\rangle \\ =& \ 0, \qquad (k=1,2,3)\end{aligned} \tag{21}$$

where $\langle u(\boldsymbol{r}), v(\boldsymbol{r})\rangle = \int u^*(\boldsymbol{r})v(\boldsymbol{r})\mathrm{d}^2\boldsymbol{r}$ is the inner product of complex functions $u(\boldsymbol{r})$ and $v(\boldsymbol{r})$, and $\left[\hat{A}, \hat{B}\right] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$ is a commutator. We begin the procedure of the proving of (21) with identities $\hat{I}_k^{\dagger} = \hat{I}_k$ which imply $\hat{I}_k$ are Hermitian operators. Another important Hermitian operator in this paper is the operator of Hamiltonian $\hat{\mathcal{H}}(\psi)$ in the quantum mechanical analogy. Now we consider the case of $k=1$. In view of the commutator $\left[\hat{I}_1, \hat{\mathcal{H}}(\psi)\right] = 0$, we have

$$\begin{aligned}&\langle \hat{I}_1\psi, \hat{\mathcal{H}}(\psi)\psi\rangle - \langle \hat{\mathcal{H}}(\psi)\psi, \hat{I}_1\psi\rangle \\ =& \left\langle \psi, \left[\hat{I}_1, \hat{\mathcal{H}}(\psi)\right]\psi\right\rangle \\ =& \ 0.\end{aligned} \tag{22}$$

Now we consider the case of $k=2$. Firstly, since $\left[\hat{I}_2, -\frac{\nabla_\perp^2}{2}\right] = 0$, we have

$$\begin{aligned}&\left\langle \hat{I}_2\psi, -\frac{\nabla_\perp^2}{2}\psi\right\rangle - \left\langle -\frac{\nabla_\perp^2}{2}\psi, \hat{I}_2\psi\right\rangle \\ =& \left\langle \psi, \left[\hat{I}_2, -\frac{\nabla_\perp^2}{2}\right]\psi\right\rangle \\ =& \ 0.\end{aligned} \tag{23}$$

By using Eq.(9), we obtain

$$\begin{aligned}&\langle \hat{I}_2\psi, f(|\psi|^2)\psi\rangle - \langle f(|\psi|^2)\psi, \hat{I}_2\psi\rangle \\ =& \ \mathrm{i}\int f(|\psi|^2)\nabla_\perp|\psi|^2\mathrm{d}^2\boldsymbol{r} \\ =& \ \mathrm{i}\int \nabla_\perp F(|\psi|^2)\mathrm{d}^2\boldsymbol{r} \\ =& \ 0.\end{aligned} \tag{24}$$

On the other hand there exists the following identity:

$$\begin{aligned}&\nabla_{\perp\boldsymbol{r}}R(|\boldsymbol{r}-\boldsymbol{\rho}|) \\ \equiv& \left(e_x\frac{\partial}{\partial x} + e_y\frac{\partial}{\partial y}\right)R(|\boldsymbol{r}-\boldsymbol{\rho}|) \\ =& \ R'\frac{\boldsymbol{r}-\boldsymbol{\rho}}{|\boldsymbol{r}-\boldsymbol{\rho}|} \\ =& -\left(e_x\frac{\partial}{\partial \xi} + e_y\frac{\partial}{\partial \eta}\right)R(|\boldsymbol{\rho}-\boldsymbol{r}|) \\ \equiv& -\nabla_{\perp\boldsymbol{\rho}}R(|\boldsymbol{\rho}-\boldsymbol{r}|).\end{aligned} \tag{25}$$

Keeping in mind this identity and Eq.(3), we obtain

$$\begin{aligned}&\left\langle \hat{I}_2\psi, \left[g(|\psi|^2)\int R(|\boldsymbol{r}-\boldsymbol{\rho}|)G(|\psi(\boldsymbol{\rho},z)|^2)\mathrm{d}^2\boldsymbol{\rho}\right]\psi\right\rangle - \mathrm{C.C.} \\ =& \ \mathrm{i}\iint\left[\nabla_{\perp\boldsymbol{r}}G(|\psi(\boldsymbol{r},t)|^2)\right]R(|\boldsymbol{r}-\boldsymbol{\rho}|)G(|\psi(\boldsymbol{\rho},z)|^2)\mathrm{d}^2\boldsymbol{r}\mathrm{d}^2\boldsymbol{\rho} \\ =& -\mathrm{i}\iint\left[\nabla_{\perp\boldsymbol{r}}R(|\boldsymbol{r}-\boldsymbol{\rho}|)\right]G(|\psi(\boldsymbol{r},t)|^2)G(|\psi(\boldsymbol{\rho},z)|^2)\mathrm{d}^2\boldsymbol{r}\mathrm{d}^2\boldsymbol{\rho} \\ =& \ \mathrm{i}\iint\left[\nabla_{\perp\boldsymbol{\rho}}R(|\boldsymbol{\rho}-\boldsymbol{r}|)\right]G(|\psi(\boldsymbol{r},t)|^2)G(|\psi(\boldsymbol{\rho},z)|^2)\mathrm{d}^2\boldsymbol{r}\mathrm{d}^2\boldsymbol{\rho} \\ =& \ 0,\end{aligned} \tag{26}$$

where C.C. stands for the complex conjugation of its foregoing term. By substituting Eqs.(23), (24) and (26) into identities (21), we have

$$\langle \hat{I}_2\psi, \hat{\mathcal{H}}(\psi)\psi\rangle - \langle \hat{\mathcal{H}}(\psi)\psi, \hat{I}_2\psi\rangle = \left\langle \psi, \left[\hat{I}_2, \hat{\mathcal{H}}(\psi)\right]\psi\right\rangle = 0. \tag{27}$$

Now we consider the case of $k=3$. Firstly we have

$$\left\langle \hat{I}_3\psi, -\frac{\nabla_\perp^2}{2}\psi\right\rangle - \left\langle -\frac{\nabla_\perp^2}{2}\psi, \hat{I}_3\psi\right\rangle = \left\langle \psi, \left[\hat{I}_3, -\frac{\nabla_\perp^2}{2}\right]\psi\right\rangle = 0, \tag{28}$$

$$\langle \hat{I}_3\psi, f(|\psi|^2)\psi\rangle - \langle f(|\psi|^2)\psi, \hat{I}_3\psi\rangle = \mathrm{i}\int\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)F(|\psi|^2)\mathrm{d}^2\boldsymbol{r} = 0. \tag{29}$$

Keeping in mind the following identity

$$\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)R(|\boldsymbol{r}-\boldsymbol{\rho}|)=R'\frac{y\xi-x\eta}{|\boldsymbol{r}-\boldsymbol{\rho}|}=-\left(\xi\frac{\partial}{\partial \eta}-\eta\frac{\partial}{\partial \xi}\right)R(|\boldsymbol{\rho}-\boldsymbol{r}|), \tag{30}$$

we have

$$\begin{aligned}
&\langle \hat{I}_3\psi,\left[g(|\psi|^2)\int R(|\boldsymbol{r}-\boldsymbol{\rho}|)G(|\psi(\boldsymbol{\rho},z)|^2)\mathrm{d}^2\boldsymbol{\rho}\right]\psi > -\mathrm{C.C.}\\
=\ & \mathrm{i}\iint\left[\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)G(|\psi(\boldsymbol{r},t)|^2)\right]R(|\boldsymbol{r}-\boldsymbol{\rho}|)G(|\psi(\boldsymbol{\rho},z)|^2)\mathrm{d}^2\boldsymbol{r}\mathrm{d}^2\boldsymbol{\rho}\\
=\ & -\mathrm{i}\iint\left[\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)R(|\boldsymbol{r}-\boldsymbol{\rho}|)\right]G(|\psi(\boldsymbol{r},t)|^2)G(|\psi(\boldsymbol{\rho},z)|^2)\mathrm{d}^2\boldsymbol{r}\mathrm{d}^2\boldsymbol{\rho}\\
=\ & \mathrm{i}\iint\left[\left(\xi\frac{\partial}{\partial \eta}-\eta\frac{\partial}{\partial \xi}\right)R(|\boldsymbol{\rho}-\boldsymbol{r}|)\right]G(|\psi(\boldsymbol{r},t)|^2)G(|\psi(\boldsymbol{\rho},z)|^2)\mathrm{d}^2\boldsymbol{r}\mathrm{d}^2\boldsymbol{\rho}\\
=\ & 0.
\end{aligned}\tag{31}$$

By substituting Eqs.(28), (29) and (31) into identities (21), we obtain at once

$$\begin{aligned}
&\langle \hat{I}_3\psi,\hat{\mathcal{H}}(\psi)\psi\rangle-\langle\hat{\mathcal{H}}(\psi)\psi,\hat{I}_3\psi\rangle\\
=&\left\langle\psi,\left[\hat{I}_3,\hat{\mathcal{H}}(\psi)\right]\psi\right\rangle\\
=&\ 0.
\end{aligned}\tag{32}$$

Here and above, identities (21) have been completely proven. In quantum mechanics, if an operator is not explicitly time dependent and commutes with the Hamiltonian operator, its mean value in the state is a constant of motion. However, generally speaking, for the generalized NNLSE (1), the commutators $\left[\hat{I}_k,\hat{\mathcal{H}}(\psi)\right]$ are not equal to zero, but their mean values in the state $\left\langle\psi,\left[\hat{I}_k,\hat{\mathcal{H}}(\psi)\right]\psi\right\rangle$ are zero. In this paper, we say that an operator commutes with the Hamiltonian operator in the sense that their commutator's mean value in the state is equal to zero. So in this sense, operators $\hat{I}_k$ all commute with the Hamiltonian operator $\hat{\mathcal{H}}(\psi)$.

Now consider the derivative of the mean value of an Hermitian operator $\hat{F}$ with respect to t as follows:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi,\hat{F}\psi\rangle&=\left\langle\psi,\frac{\partial\hat{F}}{\partial t}\psi\right\rangle+\mathrm{i}\left\langle\mathrm{i}\frac{\partial\psi}{\partial t},\hat{F}\psi\right\rangle-\mathrm{i}\left\langle\psi,\hat{F}i\frac{\partial\psi}{\partial t}\right\rangle\\
&=\left\langle\psi,\frac{\partial\hat{F}}{\partial t}\psi\right\rangle+\mathrm{i}\left\langle\hat{\mathcal{H}}(\psi)\psi,\hat{F}\psi\right\rangle-\mathrm{i}\left\langle\psi,\hat{F}\hat{\mathcal{H}}(\psi)\psi\right\rangle\\
&=\left\langle\psi,\frac{\partial\hat{F}}{\partial t}\psi\right\rangle+\mathrm{i}\left\langle\psi,\left[\hat{\mathcal{H}}(\psi),\hat{F}\right]\psi\right\rangle.
\end{aligned}\tag{33}$$

Specially, when $\hat{F}$ is not explicitly time dependent, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi,\hat{F}\psi\rangle=\mathrm{i}\left\langle\psi,\left[\hat{\mathcal{H}}(\psi),\hat{F}\right]\psi\right\rangle. \tag{34}$$

For example, when $\hat{F}=\boldsymbol{r}$, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi,\boldsymbol{r}\psi\rangle\\
=&\ \mathrm{i}\langle\psi,\left[\hat{\mathcal{H}}(\psi),\boldsymbol{r}\right]\psi\rangle\\
=&\ \mathrm{i}\langle\psi,\left[-\frac{1}{2}\nabla_\perp^2,\boldsymbol{r}\right]\psi\rangle\\
=&\ \langle\psi,-\mathrm{i}\nabla_\perp\psi\rangle.
\end{aligned}\tag{35}$$

That is the Ehrenfest theorem.[10] Now keeping in mind identities (21), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi,\hat{I}_k\psi\rangle=\mathrm{i}\left\langle\psi,\left[\hat{\mathcal{H}}(\psi),\hat{I}_k\right]\psi\right\rangle=0. \tag{36}$$

So $\langle\psi,\hat{I}_k\psi\rangle$ $(k=1,2,3)$ are invariants with respect to t and result in the invariants N in Eq.(5), $\boldsymbol{P}$ in Eq.(6) and L_z in Eq.(7).

4. The derivations of the conservation law of H

We begin the derivation of the invariant H in Eq.(8) by introducing

$$I_0 = \int_0^1 \Big[\langle\hat{\mathcal{H}}(\lambda\psi)\lambda\psi,\psi\rangle + \langle\psi,\hat{\mathcal{H}}(\lambda\psi)\lambda\psi\rangle\Big]\mathrm{d}\lambda. \tag{37}$$

The derivative of I_0 with respect to the coordinate t leads to

$$\begin{aligned}
\mathrm{i}\frac{\mathrm{d}I_0}{\mathrm{d}t} &= \int_0^1 \bigg[\Big\langle\hat{\mathcal{H}}(\lambda\psi)\lambda\psi,\mathrm{i}\frac{\partial\psi}{\partial t}\Big\rangle - \mathrm{C.C.} + \Big\langle\psi,\hat{\mathcal{H}}(\lambda\psi)\mathrm{i}\frac{\partial\lambda\psi}{\partial t}\Big\rangle - \mathrm{C.C.} + \Big\langle\psi,\mathrm{i}\frac{\partial\hat{\mathcal{H}}(\lambda\psi)}{\partial t}\lambda\psi\Big\rangle - \mathrm{C.C.}\bigg]\mathrm{d}\lambda \\
&= \int_0^1 \frac{\partial}{\partial\lambda}\bigg[\Big\langle\hat{\mathcal{H}}(\lambda\psi)\lambda^2\psi,\hat{\mathcal{H}}(\psi)\psi\Big\rangle - \mathrm{C.C.}\bigg]\mathrm{d}\lambda \\
&= 0.
\end{aligned} \tag{38}$$

So I_0 is an invariant with respect to the coordinate t. Now we directly calculate the I_0,

$$\begin{aligned}
I_0 = &\int_0^1 \Big[\langle\psi,\hat{\mathcal{H}}(\lambda\psi)\lambda\psi\rangle + \mathrm{C.C.}\Big]\mathrm{d}\lambda \\
= &\int_0^1 \bigg\{\int\psi^*\bigg[-\frac{1}{2}\nabla_\perp^2 - f(\lambda^2|\psi|^2) - g(\lambda^2|\psi|^2)\int R(|\boldsymbol{r}-\boldsymbol{\rho}|)G(\lambda^2|\psi(\boldsymbol{\rho},t)|^2)\mathrm{d}^2\boldsymbol{\rho}\bigg]\lambda\psi\mathrm{d}^2\boldsymbol{r} + \mathrm{C.C.}\bigg\}\mathrm{d}\lambda \\
= &\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\lambda}\int\bigg[-\lambda^2\frac{\psi^*\nabla_\perp^2\psi}{2} - F(\lambda^2|\psi|^2)\bigg]\mathrm{d}^2\boldsymbol{r}\mathrm{d}\lambda \\
&-\int_0^1\bigg[\iint R(|\boldsymbol{r}-\boldsymbol{\rho}|)G(\lambda^2|\psi(\boldsymbol{\rho},t)|^2)\frac{\mathrm{d}G(\lambda^2|\psi(\boldsymbol{r},t)|^2)}{\mathrm{d}\lambda}\mathrm{d}^2\boldsymbol{\rho}\ \mathrm{d}^2\boldsymbol{r}\bigg]\mathrm{d}\lambda \\
= &\int\bigg[-\frac{\psi^*\nabla_\perp^2\psi}{2} - F(|\psi|^2)\bigg]\mathrm{d}^2\boldsymbol{r} - \iint R(|\boldsymbol{r}-\boldsymbol{\rho}|)G(|\psi(\boldsymbol{\rho},t)|^2)G(|\psi(\boldsymbol{r},t)|^2)\mathrm{d}^2\boldsymbol{\rho}\ \mathrm{d}^2\boldsymbol{r} \\
&+\int_0^1\bigg[\iint R(|\boldsymbol{r}-\boldsymbol{\rho}|)G(\lambda^2|\psi(\boldsymbol{r},t)|^2)\frac{\mathrm{d}G(\lambda^2|\psi(\boldsymbol{\rho},t)|^2)}{\mathrm{d}\lambda}\mathrm{d}^2\boldsymbol{\rho}\ \mathrm{d}^2\boldsymbol{r}\bigg]\mathrm{d}\lambda \\
= &\int\bigg[-\frac{\psi^*\nabla_\perp^2\psi}{2} - F(|\psi|^2)\bigg]\mathrm{d}^2\boldsymbol{r} - \frac{1}{2}\iint R(|\boldsymbol{r}-\boldsymbol{\rho}|)G(|\psi(\boldsymbol{\rho},t)|^2)G(|\psi(\boldsymbol{r},t)|^2)\mathrm{d}^2\boldsymbol{\rho}\mathrm{d}^2\boldsymbol{r}.
\end{aligned} \tag{39}$$

That is just the Hamiltonian H in Eq.(8). In quantum mechanics, a Hamiltonian of a system is the mean value of the Hamiltonian operator. However, for the generalized NNLSE (1), since the Hamiltonian operator $\hat{\mathcal{H}}(\psi)$ explicitly depends on the state ψ, the Hamiltonian of a system is no longer the mean value of the Hamiltonian operator.

5. The variational principle problem and the Lagrangian

For convenience, we introduce the notations $\partial_i Q \equiv \dfrac{\partial Q(x_1,x_2,\cdots,x_i,\cdots)}{\partial x_i}$ and $u_x(x,y)\equiv\dfrac{\partial u(x,y)}{\partial x}$ in this section. Now we consider the following variational principle problem:

$$\delta\bigg[\iint J\mathrm{d}^2\boldsymbol{r}\mathrm{d}t + \iiint K\mathrm{d}^2\boldsymbol{\rho}\ \mathrm{d}^2\boldsymbol{r}\mathrm{d}t\bigg] = 0, \tag{40}$$

where

$$J = J\big(t,\boldsymbol{r};\psi(\boldsymbol{r},t),\psi_t(\boldsymbol{r},t),\psi_x(\boldsymbol{r},t),\psi_y(\boldsymbol{r},t);\psi^*(\boldsymbol{r},t),\psi^*_t(\boldsymbol{r},t),\psi^*_x(\boldsymbol{r},t),\psi^*_y(\boldsymbol{r},t)\big), \tag{41}$$

$$K = K\big(t,\boldsymbol{r},\boldsymbol{\rho};\psi(\boldsymbol{r}),\psi_x(\boldsymbol{r}),\psi_y(\boldsymbol{r});\psi^*(\boldsymbol{r}),\psi^*_x(\boldsymbol{r}),\psi^*_y(\boldsymbol{r});\psi(\boldsymbol{\rho}),\psi_\xi(\boldsymbol{\rho}),\psi_\eta(\boldsymbol{\rho});\psi^*(\boldsymbol{\rho}),\psi^*_\xi(\boldsymbol{\rho}),\psi^*_\eta(\boldsymbol{\rho})\big). \tag{42}$$

It can be proved that if $J^*=J$ and $K^*=K$, we will obtain the following generalized Euler–Lagrangian equation:

$$\begin{aligned}&\partial_7 J-\frac{\partial}{\partial t}(\partial_8 J)-\frac{\partial}{\partial x}(\partial_9 J)-\frac{\partial}{\partial y}(\partial_{10} J)\\&+\int\Big[\partial_7 K(t,\boldsymbol{r},\boldsymbol{\rho};\psi(\boldsymbol{r}),\cdots)-\frac{\partial}{\partial x}\big(\partial_8 K(t,\boldsymbol{r},\boldsymbol{\rho};\psi(\boldsymbol{r}),\cdots)\big)-\frac{\partial}{\partial y}\big(\partial_9 K(t,\boldsymbol{r},\boldsymbol{\rho};\psi(\boldsymbol{r}),\cdots)\big)\Big]\,\mathrm{d}^2\boldsymbol{\rho}\\&+\int\Big[\partial_{13} K(t,\boldsymbol{\rho},\boldsymbol{r};\psi(\boldsymbol{\rho}),\cdots)-\frac{\partial}{\partial x}\big(\partial_{14} K(t,\boldsymbol{\rho},\boldsymbol{r};\psi(\boldsymbol{\rho}),\cdots)\big)-\frac{\partial}{\partial y}\big(\partial_{15} K(t,\boldsymbol{\rho},\boldsymbol{r};\psi(\boldsymbol{\rho}),\cdots)\big)\Big]\,\mathrm{d}^2\boldsymbol{\rho}=0. \quad (43)\end{aligned}$$

In view of the variational problem (11), we have

$$J=\frac{\mathrm{i}}{2}\left(\psi^*\frac{\partial\psi}{\partial t}-\psi\frac{\partial\psi^*}{\partial t}\right)-\frac{1}{2}|\nabla_\perp\psi|^2+F(|\psi|^2), \quad (44)$$

$$K=\frac{1}{2}R(|\boldsymbol{r}-\boldsymbol{\rho}|)G(|\psi(\boldsymbol{r},t)|^2)G(|\psi(\boldsymbol{\rho},t)|^2). \quad (45)$$

By substituting Eqs.(44) and (45) into Eq.(43), we obtain the generalized NNLSE (1) again. So $\mathcal{L}$ in Eq.(10) is just the Lagrangian of the generalized NNLSE (1). Now we consider the action of the system as follows:

$$\begin{aligned}S=&\iint\mathcal{L}\,\mathrm{d}^2\boldsymbol{r}\mathrm{d}t\\=&\iint\frac{\mathrm{i}}{2}\left(\psi^*\frac{\partial\psi}{\partial t}-\psi\frac{\partial\psi^*}{\partial t}\right)\mathrm{d}^2\boldsymbol{r}\,\mathrm{d}t+\iint\Big[-\frac{1}{2}|\nabla_\perp\psi|^2+F(|\psi|^2)\Big]\,\mathrm{d}^2\boldsymbol{r}\mathrm{d}t\\&+\iiint\frac{1}{2}R(|\boldsymbol{r}-\boldsymbol{\rho}|)G(|\psi(\boldsymbol{r},t)|^2)G(|\psi(\boldsymbol{\rho},t)|^2)\mathrm{d}^2\boldsymbol{\rho}\,\mathrm{d}^2\boldsymbol{r}\mathrm{d}t\\=&\iint\frac{\mathrm{i}}{2}\left(\psi^*\frac{\partial\psi}{\partial t}-\psi\frac{\partial\psi^*}{\partial t}\right)\mathrm{d}^2\boldsymbol{r}\mathrm{d}t+\iint\Big[\frac{\psi^*\nabla_\perp^2\psi}{2}+F(|\psi|^2)\Big]\,\mathrm{d}^2\boldsymbol{r}\,\mathrm{d}t\\&+\iiint\frac{1}{2}R(|\boldsymbol{r}-\boldsymbol{\rho}|)G(|\psi(\boldsymbol{r},t)|^2)G(|\psi(\boldsymbol{\rho},t)|^2)\mathrm{d}^2\boldsymbol{\rho}\,\mathrm{d}^2\boldsymbol{r}\mathrm{d}t\\=&\iint\frac{\mathrm{i}}{2}\left(\psi^*\frac{\partial\psi}{\partial t}-\psi\frac{\partial\psi^*}{\partial t}\right)\mathrm{d}^2\boldsymbol{r}\mathrm{d}t-\int H\mathrm{d}t. \quad (46)\end{aligned}$$

Equation (46) constructs relations between the action S and the Lagrangian $\mathcal{L}$ and Hamiltonian H.

6. Summary

The derivations of several important conservation laws of the generalized nonlocal nonlinear Schrödinger equation are presented. These invariants are the number of particles, the momentum, the angular momentum and the Hamiltonian in the quantum mechanical analogy. The mean value of a commutator of a momentum operator with a Hamiltonian operator and that of an angular momentum operator with a Hamiltonian operator both are equal to zero, which results in the conservation laws of the momentum and the angular momentum. The number of the particles is always conserved provided that the Hamiltonian operator is an Hermitian operator. The Ehrenfest theorem applies to the generalized nonlocal nonlinear Schrödinger equation too. Differing from the quantum mechanics, a Hamiltonian of a system is no longer the direct mean value of the Hamiltonian operator. The relation between the action and Lagrangian and Hamiltonian is also presented.

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